SIAM J. NUMER. ANAL.
Vol. 52, No. 2, pp. 1076–1101


# OPTIMIZATION OF TRANSMISSION CONDITIONS IN WAVEFORM RELAXATION TECHNIQUES FOR RC CIRCUITS*

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**Abstract.** Waveform relaxation techniques have become increasingly important with the wide availability of parallel computers with a large number of processors. A limiting factor for classical waveform relaxation, however, is the convergence speed for an important class of problems, especially if long time windows are considered. In contrast, the optimized waveform relaxation algorithm discussed in this paper is well suited to address this problem. Today several numerical analyses have shown that optimized waveform relaxation algorithms can overcome slow convergence over long time windows. However, the optimized waveform relaxation techniques require the determination of optimized parameters. In this paper, we present a theoretical foundation for the determination of the optimized parameters for an important class of RC circuits.

**Key words.** waveform relaxation, transmission conditions, circuit simulation

**AMS subject classifications.** 65L05, 65L20, 65Y05

**DOI.** 10.1137/110854187

## 1. Introduction.

Waveform relaxation (WR) methods are based on partitioning large circuits into subcircuits which then are solved separately for multiple time steps. Recently, WR techniques have increased in importance with the wide availability of parallel computers with a potentially large number of processors. The exchange of waveforms for multiple time steps increases the efficiency for the iterative approach and it reduces the importance of processor communication latency.

The classical WR approach was conceived in the early 1980s for circuit solver applications [24]. The approach was applied to a multitude of problems in the circuit theory area [33, 23, 28, 22, 4] and for a more general class of problems [31, 7, 8]. It also has a close relationship to the classical Picard–Lindelöf iteration, as was shown in [25], which led to a complete analysis of the convergence behavior of WR; see [26, 27]. WR techniques have also been extended to partial differential equations of evolution type, see [5, 21, 6, 17, 18, 19] and references therein.

The limiting factor for classical WR is its rather slow convergence for several important classes of problems, especially when long time windows are employed, and there were early attempts to overcome this [29, 30, 11, 10]. In order to systematically address this issue, an important step forward was taken with the so-called optimized WR (oWR); see [14, 15, 12, 16] and the cited references therein. The oWR algorithm was motivated by work at the PDE level; see [13, 3]. Today, as one can see from the references, the oWR algorithms have the potential for solving the convergence issue. However, the oWR techniques require the determination of optimization parameters. The approach is based on optimized *transmission* conditions which transfer the information between the subsystems more efficiently than in the classical WR. WR methods are very much related to domain decomposition techniques. In this

*Received by the editors November 7, 2011; accepted for publication (in revised form) February 18, 2014; published electronically April 29, 2014.
http://www.siam.org/journals/sinum/52-2/85418.html
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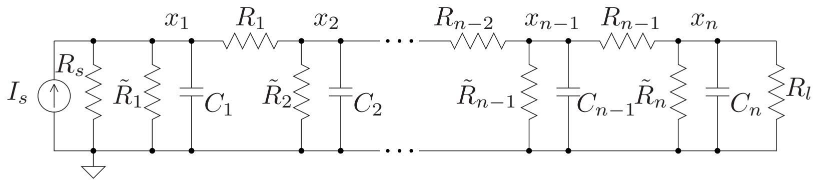


FIG. 1. *Our model RC circuit.*

paper, we present a theoretical foundation for the determination of the optimization parameter for the important class of diffusive RC circuits.

Circuit equations are often specified in terms of the modified nodal analysis (MNA) formulation [20], usually in the form $\mathbf{C}\dot{\boldsymbol{x}}(t) + \mathbf{G}\boldsymbol{x}(t) = \mathbf{B}\boldsymbol{r}(t)$, where $\mathbf{C}$ contains the reactive elements and $\mathbf{G}$ the other elements, $\mathbf{B}$ is the input selector matrix, $\boldsymbol{r}(t)$ are the excitation or forcing functions, and $\boldsymbol{x}(t)$ is the solution vector which consists in general of nodal voltages and currents. The model problem of an RC circuit in Figure 1 is of practical interest and is suitable for our analysis. To simplify the notation, we rewrite the MNA circuit equations in tridiagonal form,

$$\dot{\boldsymbol{x}} = \begin{bmatrix} b_1 & c_1 & & \\ a_1 & b_2 & \ddots & \\ & \ddots & \ddots & c_{n-1} \\ & & a_{n-1} & b_n \end{bmatrix} \boldsymbol{x} + \boldsymbol{f}, \tag{1.1}$$

where the entries in the tridiagonal matrix are

$$\begin{cases} a_i = \frac{1}{R_i C_{i+1}}, \\ c_i = \frac{1}{R_i C_i}, \end{cases} \quad i = 1, 2, \dots n-1, \qquad b_i = \begin{cases} -(\frac{1}{R_s} + \frac{1}{\tilde{R}_1} + \frac{1}{R_1})\frac{1}{C_1}, & i = 1, \\ -(\frac{1}{R_{i-1}} + \frac{1}{\tilde{R}_i} + \frac{1}{R_i})\frac{1}{C_i}, & i = 2, 3, \dots, n-1, \\ -(\frac{1}{R_{i-1}} + \frac{1}{\tilde{R}_i} + \frac{1}{R_l})\frac{1}{C_i}, & i = n, \end{cases}$$

and the resistor values $R_i$, $\tilde{R}_i$, $R_s$, and $R_l$ and the capacitors $C_i$ are strictly positive constants.

Note that if the resistor values $\tilde{R}_i$ go to infinity, they can be omitted from the RC circuit in Figure 1, and the entries in the tridiagonal matrix become

$$\begin{cases} a_i = \frac{1}{R_i C_{i+1}}, \\ c_i = \frac{1}{R_i C_i}, \end{cases} \quad i = 1, 2, \dots n-1, \qquad b_i = \begin{cases} -(\frac{1}{R_s} + \frac{1}{R_1})\frac{1}{C_1}, & i = 1, \\ -(\frac{1}{R_{i-1}} + \frac{1}{R_i})\frac{1}{C_i}, & i = 2, 3, \dots, n-1, \\ -(\frac{1}{R_{i-1}} + \frac{1}{R_l})\frac{1}{C_i}, & i = n. \end{cases}$$

The source term on the right-hand side is given by the $n \times 1$ vector $\boldsymbol{f} = (I_s(t)/C_1, 0, 0, 0, \dots, 0)^T$, for some source function $I_s(t)$, the solution vector $\boldsymbol{x}(t)$ consists of nodal voltages, and we are also given the initial voltage values $\boldsymbol{x}(0) = (v_1^0, v_2^0, v_3^0, v_4^0, \dots, v_n^0)^T$ at the time $t = 0$.

**2. WR algorithms.** We assume that $n$ is an even number in this paper, $n = 2j$, $j = 1, 2, \dots,$ and the case $n$ odd can be handled similarly. We partition the system (1.1) at row $j$ into two subsystems and obtain the classical WR algorithm

$$
\begin{aligned}
\dot{\boldsymbol{u}}^{k+1} &= \begin{bmatrix} b_1 & c_1 & & \\ a_1 & b_2 & \ddots & \\ & \ddots & \ddots & c_{j-1} \\ & & a_{j-1} & b_j \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_j \end{pmatrix}^{k+1} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ c_j u_{j+1}^{k+1} \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_j \end{pmatrix}, \\
\dot{\boldsymbol{w}}^{k+1} &= \begin{bmatrix} b_{j+1} & c_{j+1} & & \\ a_{j+1} & b_{j+2} & \ddots & \\ & \ddots & \ddots & c_{2j-1} \\ & & a_{2j-1} & b_{2j} \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_j \end{pmatrix}^{k+1} + \begin{pmatrix} a_j w_0^{k+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} f_{j+1} \\ f_{j+2} \\ \vdots \\ f_{2j} \end{pmatrix},
\end{aligned} \tag{2.1}
$$

where the unknown voltage values $u_{j+1}^{k+1}$ and $w_0^{k+1}$ are determined by relaxation through the classical transmission conditions

$$
u_{j+1}^{k+1} = w_1^k, \quad w_0^{k+1} = u_j^k. \tag{2.2}
$$

The subsystems are given the initial voltage values $\boldsymbol{u}(0) = (v_1^0, v_2^0, \ldots, v_j^0)^T$ and $\boldsymbol{w}(0) = (v_{j+1}^0, v_{j+2}^0, \ldots, v_{2j}^0)^T$, and to start the WR algorithm, we need to specify two initial waveforms $u_j^0(t)$ and $w_1^0(t)$ for $t \in [0, T]$.

In [15], new transmission conditions for WR algorithms were proposed. These new transmission conditions are given by

$$
(u_{j+1}^{k+1} - u_j^{k+1}) + \alpha u_{j+1}^{k+1} = (w_1^k - w_0^k) + \alpha w_1^k, \quad (w_1^{k+1} - w_0^{k+1}) + \beta w_0^{k+1} = (u_{j+1}^k - u_j^k) + \beta u_j^k. \tag{2.3}
$$

The new transmission conditions (2.3) also exchange the voltages $u_{j+1}$ and $w_0$, but they are multiplied with weighting factors $\alpha$ and $\beta$, respectively, which could even be operators in time. The voltage differences between the nodal voltages $(u_{j+1} - u_j)$ and $(w_1 - w_0)$ ensure that the currents are also taken into account in the transmission conditions since we could write the currents as $\alpha^{-1}(u_{j+1} - u_j)$ and $\beta^{-1}(w_1 - w_0)$, where $\alpha$ and $\beta$ can be viewed as resistors. Therefore, the new transmission conditions attempt to transmit voltages as well as currents at the interfaces between the subsystems during the iteration, instead of only voltage values like the classical transmission conditions. Gander and Ruehli proved in [15] that the converged solution of the new WR algorithm with transmission conditions (2.3) is identical to the converged solution of the classical WR algorithm with transmission conditions (2.2) if $\beta^{-1} \neq 1 + \alpha^{-1}$.

Using the new transmission conditions, the optimizable WR algorithm is given by

$$
\begin{aligned}
\dot{\boldsymbol{u}}^{k+1} &= \begin{bmatrix} b_1 & c_1 & & \\ a_1 & b_2 & \ddots & \\ & \ddots & \ddots & c_{j-1} \\ & & a_{j-1} & b_j + \frac{c_j}{\alpha+1} \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_j \end{pmatrix}^{k+1} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ c_j w_1^k - \frac{c_j}{\alpha+1} w_0^k \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_j \end{pmatrix}, \\
\dot{\boldsymbol{w}}^{k+1} &= \begin{bmatrix} b_{j+1} - \frac{a_j}{\beta-1} & c_{j+1} & & \\ a_{j+1} & b_{j+2} & \ddots & \\ & \ddots & \ddots & c_{2j-1} \\ & & a_{2j-1} & b_{2j} \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_j \end{pmatrix}^{k+1} + \begin{pmatrix} a_j u_j^k + \frac{a_j}{\beta-1} u_{j+1}^k \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} f_{j+1} \\ f_{j+2} \\ \vdots \\ f_{2j} \end{pmatrix},
\end{aligned} \tag{2.4}
$$

where we start with initial waveforms $u_j^0(t)$, $u_{j+1}^0(t)$, $w_1^0(t)$, and $w_0^0(t)$ for $t \in [0, T]$, which must satisfy the initial conditions, and for the next iterations, the values $u_{j+1}^k$ and $w_0^k$ are determined by the transmission conditions (2.3).

**3. Convergence analysis.** In order to keep the analysis and the optimization process we are solving simpler, we consider here the simplifying assumptions

$$c_i = a_i = a \text{ for } i = 1, 2, \ldots, n-1, \quad b_i = b \text{ for } i = 1, 2, \ldots, n. \tag{3.1}$$

Indeed, this is a justified choice since we often have circuits where the subsystems or subcircuits have very similar electrical properties on both sides of the partitioning boundary.

By linearity, we analyze the homogeneous problem and we use the Laplace transform for the convergence study of the linear circuits considered here. The Laplace transform for $s = \eta + i\omega \in \mathbb{C}$ of (2.1) at convergence, with the simplifying assumptions (3.1), is given by

$$\begin{aligned} s\hat{\boldsymbol{u}} &= \begin{bmatrix} b & a & & \\ a & b & \ddots & \\ & \ddots & \ddots & a \\ & & a & b \end{bmatrix} \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_j \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ a\hat{u}_{j+1} \end{pmatrix}, \\ s\hat{\boldsymbol{w}} &= \begin{bmatrix} b & a & & \\ a & b & \ddots & \\ & \ddots & \ddots & a \\ & & a & b \end{bmatrix} \begin{pmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \vdots \\ \hat{w}_j \end{pmatrix} + \begin{pmatrix} a\hat{w}_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \end{aligned} \tag{3.2}$$

where we dropped the iteration index to simplify the notation. We first note that the Laplace transform allows us to easily obtain explicit formulas for the solutions. Then, depending on which transmission conditions we use, we obtain for our algorithms relations of the form $\hat{u}_j^{2k} = (\rho(s))^k \hat{u}_j^0$ and $\hat{w}_1^{2k} = (\rho(s))^k \hat{w}_1^0$. The quantity $\rho(s)$ is called the convergence factor for the corresponding WR algorithm in the Laplace transform variable $s$. Now, from these relations, for $\eta \geq 0$, and the well-known Parseval–Plancherel identity we get

$$\begin{aligned} \|e^{-\eta t} u_j^{2k}(t)\|_{L^2} &\leq \left( \sup_{\omega \in \mathbb{R}} |\rho(s)| \right)^k \|e^{-\eta t} u_j^0(t)\|_{L^2}, \\ \|e^{-\eta t} w_1^{2k}(t)\|_{L^2} &\leq \left( \sup_{\omega \in \mathbb{R}} |\rho(s)| \right)^k \|e^{-\eta t} w_1^0(t)\|_{L^2}, \end{aligned}$$

which, with the weighted norm $\|\cdot\|_\eta := \|e^{-\eta t} \cdot\|_{L^2}$, implies

$$\|u_j^{2k}(t)\|_\eta \leq \left( \sup_{\omega \in \mathbb{R}} |\rho(s)| \right)^k \|u_j^0(t)\|_\eta, \quad \|w_1^{2k}(t)\|_\eta \leq \left( \sup_{\omega \in \mathbb{R}} |\rho(s)| \right)^k \|w_1^0(t)\|_\eta.$$

Therefore, convergence in the frequency domain $\omega$ implies convergence in the time domain $t$, and using Laplace transforms, we can show convergence in the exponentially weighted norm for $\eta > 0$, or in the $L^2$ norm if $\eta = 0$. The following technical lemmas are needed to determine the convergence factors of the classical and optimizable WR algorithms in closed form.

LEMMA 3.1. *Let $S_1$, $S_2$, and $S_3$ be given by*

$$S_1 := (s-b)\sum_{r=1}^{\lfloor\frac{q+2}{2}\rfloor}(-1)^{r+1}\binom{q-r+1}{r-1}(s-b)^{q-2r+2}a^{2r-2},$$
$$S_2 := -a^2\sum_{r=1}^{\lfloor\frac{q+1}{2}\rfloor}(-1)^{r+1}\binom{q-r}{r-1}(s-b)^{q-2r+1}a^{2r-2},$$
$$S_3 := \sum_{r=1}^{\lfloor\frac{q+3}{2}\rfloor}(-1)^{r+1}\binom{q-r+2}{r-1}(s-b)^{q-2r+3}a^{2r-2},$$

*where $q$ is any integer greater than or equal to 1, and for any real number $t$, we have denoted above $\lfloor t\rfloor = l$, where $l$ is the unique integer such that $l \le t < l+1$. Then $S_1 + S_2 = S_3$.*

*Proof.* If $q$ is even, then $q = 2\ell,\ \ell = 1, 2, \dots,$ and $\lfloor\frac{q+3}{2}\rfloor = \ell+1$, $\lfloor\frac{q+2}{2}\rfloor = \ell+1$, and $\lfloor\frac{q+1}{2}\rfloor = \ell$, and we write $S_1$, $S_2$, and $S_3$ as

$$\begin{aligned} S_1 &:= (s-b)^{q+1} + \sum_{r=2}^{\ell+1}(-1)^{r+1}\binom{q-r+1}{r-1}(s-b)^{q-2r+3}a^{2r-2},\\ S_2 &:= -\sum_{r=1}^{\ell}(-1)^{r+1}\binom{q-r}{r-1}(s-b)^{q-2r+1}a^{2r},\\ S_3 &:= (s-b)^{q+1} + \sum_{r=2}^{\ell+1}(-1)^{r+1}\binom{q-r+2}{r-1}(s-b)^{q-2r+3}a^{2r-2}. \end{aligned} \tag{3.3}$$

For any $r = j$, where $1 < j \le \ell+1$, the summands in $S_1$ and $S_3$ are, respectively,

$$(-1)^{j+1}\binom{q-j+1}{j-1}(s-b)^{q-2j+3}a^{2j-2}, \qquad (-1)^{j+1}\binom{q-j+2}{j-1}(s-b)^{q-2j+3}a^{2j-2},$$

and for any $r = j-1,\ 1 < j \le \ell+1$, the summand in $S_2$ is

$$(-1)^{j+1}\binom{q-j+1}{j-2}(s-b)^{q-2j+3}a^{2j-2}.$$

Therefore, we have the same powers, and we only need to show that the sum of the coefficients in the summands in $S_2$ for $r = j-1$, and in $S_1$ for $r = j,\ 1 < j \le \ell+1$, is equal to the coefficient in the summand in $S_3$ for $r = j$. This holds due to the property of the binomial coefficients, $\binom{n}{q+1} + \binom{n}{q} = \binom{n+1}{q+1}$, which implies $\binom{q-j+1}{j-1} + \binom{q-j+1}{j-2} = \binom{q-j+2}{j-1}$, for $1 < j \le \ell+1$. The summand in $S_2$ for $r = 1$, or $j = 2$, is already considered above, since for $S_2$ we considered $r = j-1$ and $1 < j \le \ell+1$, so for $r = 1$ we only have the summand in $S_1$ and in $S_3$. From (3.3), for $r = 1$, the summand in $S_1$ is $(s-b)^{q+1}$, and it is the same expression we obtain from the summand in $S_3$ for $r = 1$, and this finishes the proof for $q$ even.

Now, if $q$ is odd, then $q = 2\ell-1,\ \ell = 1, 2, \dots,$ and $\lfloor\frac{q+3}{2}\rfloor = \ell+1$, $\lfloor\frac{q+2}{2}\rfloor = \ell$, and $\lfloor\frac{q+1}{2}\rfloor = \ell$, and we write $S_1$, $S_2$, and $S_3$ as

$$\begin{aligned} S_1 &:= (s-b)^{q+1} + \sum_{r=2}^{\ell}(-1)^{r+1}\binom{q-r+1}{r-1}(s-b)^{q-2r+3}a^{2r-2},\\ S_2 &:= -\sum_{r=1}^{\ell-1}(-1)^{r+1}\binom{q-r}{r-1}(s-b)^{q-2r+1}a^{2r} + (-1)^{\ell+2}a^{2\ell},\\ S_3 &:= (s-b)^{q+1} + \sum_{r=2}^{\ell}(-1)^{r+1}\binom{q-r+2}{r-1}(s-b)^{q-2r+3}a^{2r-2} + (-1)^{\ell+2}a^{2\ell}. \end{aligned} \tag{3.4}$$

This shows that $S_1 + S_2 = S_3$ is similar to the case where $q$ is even, but here we have $1 < j < \ell + 1$ instead of $1 < j \le \ell + 1$, since the last value for $r$ in $S_1$ is $r = \ell$, whereas it was $\ell + 1$ for $q$ even. Therefore, we only need to show equality between the summands in $S_3$ for $r = j = \ell + 1$ and in $S_2$ for $r = j - 1 = \ell$. As is evident from (3.4), the summand in $S_3$ for $r = \ell + 1$ is equal to the summand in $S_2$ for $r = \ell$. ∎

LEMMA 3.2. *For the systems in* (3.2), *for any* $1 \le m \le j$, $j = 1, 2, \ldots$, $\hat{u}_m$ *and* $\hat{w}_{j-m+1}$ *are given by*

$$\hat{u}_m = C(s, m, a, b)\hat{u}_{m+1}, \quad \hat{w}_{j-m+1} = C(s, m, a, b)\hat{w}_{j-m}, \tag{3.5}$$

*where*

$$C(s, m, a, b) := \frac{a \sum_{r=1}^{\lfloor \frac{m+1}{2} \rfloor} (-1)^{r+1} \binom{m-r}{r-1} (s-b)^{m-2r+1} a^{2r-2}}{\sum_{r=1}^{\lfloor \frac{m+2}{2} \rfloor} (-1)^{r+1} \binom{m-r+1}{r-1} (s-b)^{m-2r+2} a^{2r-2}}, \tag{3.6}$$

*and* $\lfloor . \rfloor$ *is the integer defined in Lemma* 3.1.

*Proof.* We show the proof by induction in detail for the first relation in (3.5), and the proof for the second one is similar. For $m = 1$, from the first subsystem in (3.2), we have $\hat{u}_1 = \frac{a}{s-b}\hat{u}_2$, which is the same as we obtain by using (3.5) for $m = 1$, so relation (3.5) holds for $m = 1$. We thus assume that (3.5) holds for $m = q$, i.e., $\hat{u}_q = C(s, q, a, b)\hat{u}_{q+1}$. Now we need to show that (3.5) also holds for $m = q + 1$. The equation for $m = q+1$ from the first subsystem in (3.2) is $s\hat{u}_{q+1} = a\hat{u}_q + b\hat{u}_{q+1} + a\hat{u}_{q+2}$, which implies, after substituting $\hat{u}_q$ from (3.5) for $m = q$, and simplifying, $X\hat{u}_{q+1} = Y\hat{u}_{q+2}$, where

$$\begin{aligned} X &:= (s-b) \sum_{r=1}^{\lfloor \frac{q+2}{2} \rfloor} (-1)^{r+1} \binom{q-r+1}{r-1} (s-b)^{q-2r+2} a^{2r-2} \\ &\quad - a^2 \sum_{r=1}^{\lfloor \frac{q+1}{2} \rfloor} (-1)^{r+1} \binom{q-r}{r-1} (s-b)^{q-2r+1} a^{2r-2}, \\ Y &:= a \sum_{r=1}^{\lfloor \frac{q+2}{2} \rfloor} (-1)^{r+1} \binom{q-r+1}{r-1} (s-b)^{q-2r+2} a^{2r-2}. \end{aligned}$$

Hence, $\hat{u}_{q+1} = \frac{Y}{X}\hat{u}_{q+2}$. The numerator of the expression in (3.5) for $m = q + 1$ is equal to $Y$, so we only need to show that the denominator of (3.5) for $m = q + 1$ is equal to $X$, and this is proved in Lemma 3.1. Therefore, relation (3.5) holds for $m = q + 1$, and the proof by induction is complete. ∎

THEOREM 3.3 (convergence factor of classical WR). *The convergence factor* $\rho_{cla(j)}$ *of the classical WR algorithm* (2.1), *with* $n = 2j$, $j = 1, 2, \ldots$, *and the simplifying assumptions* (3.1), *is given by*

$$\rho_{cla(j)}(s, a, b) = \left( \frac{1}{\lambda_j(s, a, b)} \right)^2, \quad \lambda_j(s, a, b) = \frac{1}{C(s, j, a, b)}, \tag{3.7}$$

*where* $C(s, j, a, b)$ *is given in* (3.6).

*Proof.* For $j > 1$, the last equation in the first subsystem in (2.1) is given by $s\hat{u}_j^{k+1} = a\hat{u}_{j-1}^{k+1} + b\hat{u}_j^{k+1} + a\hat{w}_1^k$. Using Lemma 3.2 to substitute for $\hat{u}_{j-1}^{k+1}$, and simplifying, we get

$$\hat{u}_j^{k+1} = \frac{1}{\lambda_j}\hat{w}_1^k, \tag{3.8}$$

where $\lambda_j = \frac{N_j}{D_j}$, and $N_j$, $D_j$ are given by

$$N_j = (s-b)\sum_{r=1}^{\lfloor\frac{j+1}{2}\rfloor}(-1)^{r+1}\binom{j-r}{r-1}(s-b)^{j-2r+1}a^{2r-2} - a^2\sum_{r=1}^{\lfloor\frac{j}{2}\rfloor}(-1)^{r+1}\binom{j-r-1}{r-1}(s-b)^{j-2r}a^{2r-2},$$

$$D_j = a\sum_{r=1}^{\lfloor\frac{j+1}{2}\rfloor}(-1)^{r+1}\binom{j-r}{r-1}(s-b)^{j-2r+1}a^{2r-2},$$

and using Lemma 3.1, we obtain

$$\lambda_j = \frac{N_j}{D_j} = \frac{\sum_{r=1}^{\lfloor\frac{j+2}{2}\rfloor}(-1)^{r+1}\binom{j-r+1}{r-1}(s-b)^{j-2r+2}a^{2r-2}}{a\sum_{r=1}^{\lfloor\frac{j+1}{2}\rfloor}(-1)^{r+1}\binom{j-r}{r-1}(s-b)^{j-2r+1}a^{2r-2}},$$

as given in (3.7). Note also that $N_j = (s-b)N_{j-1} - aD_{j-1}$, and $D_j = aN_{j-1}$, which we will use later. Similarly, for $j = 1$, one finds the same equation (3.8), and analogously, for the second subsystem

$$\hat{w}_1^{k+1} = \frac{1}{\lambda_j}\hat{u}_j^k. \tag{3.9}$$

Inserting (3.9) at step $k$ into (3.8), we get $\hat{u}_j^{k+1} = \rho_{cla(j)}(s,a,b)\hat{u}_j^{k-1}$ with convergence factor $\rho_{cla(j)}$ of the classical WR algorithm given in (3.7). The same result holds for $\hat{w}_1^{k+1}$, and we find $\hat{u}_j^{2k} = (\rho_{cla(j)})^k\hat{u}_j^0$, and $\hat{w}_1^{2k} = (\rho_{cla(j)})^k\hat{w}_1^0$, which shows that $\rho_{cla(j)}$ is indeed the convergence factor of the algorithm. ∎

For convergence in the frequency domain, we need that $|\rho_{cla(j)}(s,a,b)| < 1$ for $\Re(s) \geq 0$, and for fast convergence, the modulus of $\rho_{cla(j)}$ should be much smaller than 1, $|\rho_{cla(j)}| \ll 1$. However, the convergence factor $|\rho_{cla(j)}|$ is a fixed function of the circuit parameters in the classical WR algorithm, and thus the algorithm does not have any adjustable parameters like the optimizable WR algorithm. We can only analyze for the classical WR algorithm if the convergence test $|\rho_{cla(j)}| < 1$ is satisfied for a given $\eta \geq 0$ which then gives convergence in the corresponding exponentially weighted $L^2$ norm in time. Examples of the classical convergence factor as a function of $\omega$, i.e., $\eta = 0$ in $s = \eta + i\omega$ corresponding to the unweighted $L^2$ norm in time, for different values of $j$, with $|b| = 2a$, where the resistor values $\tilde{R}_i$ are omitted, and with $|b| > 2a$, are given in Figures 2 and 3, respectively. We choose $a = \frac{200}{63}$ and $b = -2a$ in Figure 2, and in Figure 3 we choose $a = \frac{200}{63}$ and $b = -(2a + \frac{1}{5}\cdot\frac{100}{63})$ as typical RC circuit parameters. We observe from Figures 2 and 3 that the modulus of the convergence factor for finite $j$ is less than one, and it becomes bigger and bigger around $\omega = 0$ as we increase the size of the circuit. For the infinitely large circuit considered in theory in section 5, the convergence factor is one at $\omega = 0$ for the case $|b| = 2a$ (see Figure 2), and it is less than one for the case $|b| > 2a$ (see Figure 3). Note also that the case $|b| > 2a$ has a better classical convergence factor than the case $b = 2a$.

Theorem 3.4 (convergence factor of optimizable WR). *The convergence factor of the optimizable WR algorithm* (2.4) *with* $n = 2j$, $j = 1, 2, \ldots,$ *and the simplifying assumptions* (3.1) *is given by*

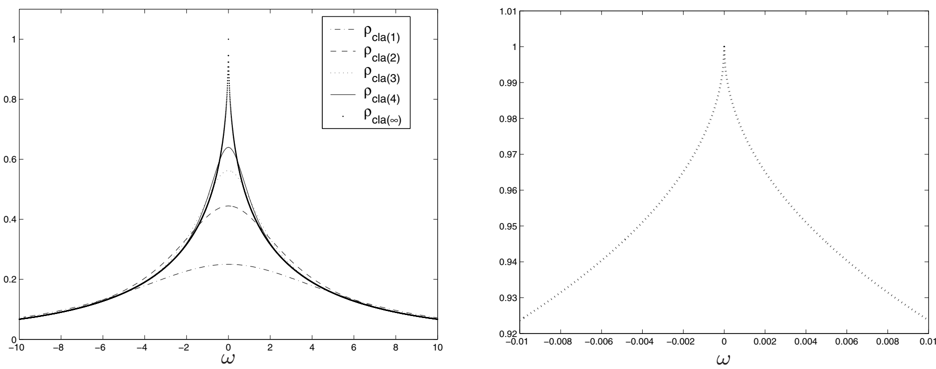


FIG. 2. *Convergence factor* $|\rho_{cla(j)}|$ *for different values of* $j$ *with* $|b| = 2a$ *on the left, and on the right, a zoom around the maximum of* $|\rho_{cla(\infty)}|$.

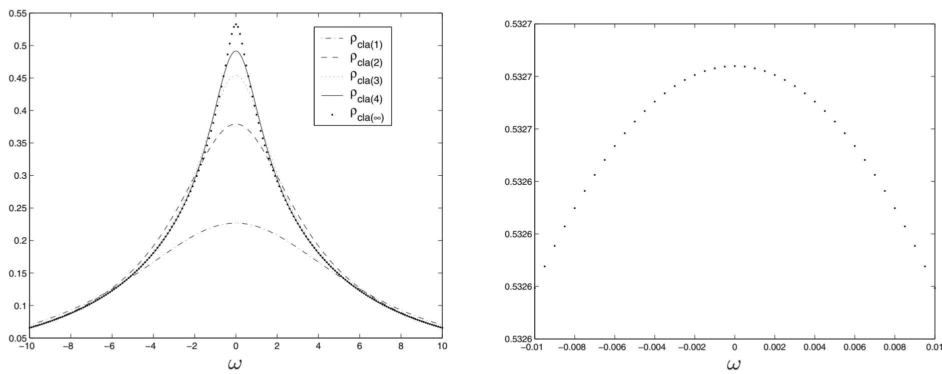


FIG. 3. *Convergence factor* $|\rho_{cla(j)}|$ *for different values of* $j$ *with* $|b| > 2a$ *on the left, and on the right, a zoom around the maximum of* $|\rho_{cla(\infty)}|$.

$$\rho_{opt(j)}(s,a,b,\alpha,\beta) = \frac{(\alpha+1)-\lambda_j}{(\alpha+1)\lambda_j - 1} \cdot \frac{(\beta-1)+\lambda_j}{(\beta-1)\lambda_j+1}, \tag{3.10}$$

*where* $\lambda_j$ *is given in* (3.7).

*Proof.* The proof is similar to the proof of Theorem 3.3. For $j > 1$, the last equation in the first subsystem in (3.2) is given by $s\hat{u}_j = a\hat{u}_{j-1} + b\hat{u}_j + a\hat{u}_{j+1}$, and using Lemmas 3.2 and 3.1, we get

$$\frac{\sum_{r=1}^{\lfloor \frac{j+2}{2} \rfloor} (-1)^{r+1}\binom{j-r+1}{r-1}(s-b)^{j-2r+2}a^{2r-2}}{\sum_{r=1}^{\lfloor \frac{j+1}{2} \rfloor} (-1)^{r+1}\binom{j-r}{r-1}(s-b)^{j-2r+1}a^{2r-2}} \hat{u}_j = a\hat{u}_{j+1}. \tag{3.11}$$

Now inserting the iterations, and substituting $\hat{u}_{j+1}^{k+1}$ from the first transmission condition in (2.3) into (3.11), which is basically the last equation in the first subsystem in (2.4) after taking Laplace transform and considering the homogeneous problem, we get

$$\left(X - \frac{a}{\alpha+1}\right)\hat{u}_j^{k+1} = \frac{a}{\alpha+1}(\hat{w}_1^k - \hat{w}_0^k + \alpha\hat{w}_1^k),$$

where

$$X := \frac{\sum_{r=1}^{\lfloor \frac{j+2}{2} \rfloor} (-1)^{r+1} \binom{j-r+1}{r-1} (s-b)^{j-2r+2} a^{2r-2}}{\sum_{r=1}^{\lfloor \frac{j+1}{2} \rfloor} (-1)^{r+1} \binom{j-r}{r-1} (s-b)^{j-2r+1} a^{2r-2}},$$

and after simplifying, we get

$$\hat{u}_j^{k+1} = F_1(\hat{w}_1^k(\alpha+1) - \hat{w}_0^k), \quad F_1 := \frac{aX_1}{(\alpha+1)X_2 - aX_1}, \tag{3.12}$$

where

$$X_1 := \sum_{r=1}^{\lfloor \frac{j+1}{2} \rfloor} (-1)^{r+1} \binom{j-r}{r-1} (s-b)^{j-2r+1} a^{2r-2},$$
$$X_2 := \sum_{r=1}^{\lfloor \frac{j+2}{2} \rfloor} (-1)^{r+1} \binom{j-r+1}{r-1} (s-b)^{j-2r+2} a^{2r-2}.$$

Similarly, from the second subsystem, we obtain

$$\hat{w}_1^{k+1} = F_2(\hat{u}_j^k(\beta-1) + \hat{u}_{j+1}^k), \quad F_2 := \frac{aX_1}{(\beta-1)X_2 + aX_1}, \tag{3.13}$$

where $X_1$ and $X_2$ are as given above. Again, we need to derive a relation between $\hat{u}_j^{k+1}$ and $\hat{w}_1^k$ from (3.12) and similarly a relation between $\hat{w}_1^{k+1}$ and $\hat{u}_j^k$ from (3.13) to obtain the convergence factor for the optimizable WR algorithm in closed form. Using the second transmission condition in (2.3), together with (3.13), we find

$$\hat{w}_0^{k+1} = \left( \frac{1}{(\beta-1)F_2} - \frac{1}{\beta-1} \right) \hat{w}_1^{k+1},$$

and using this result at step $k$ in (3.12) we find for the first subsystem

$$\hat{u}_j^{k+1} = F_1 \left( \alpha + 1 - \frac{1}{(\beta-1)F_2} + \frac{1}{\beta-1} \right) \hat{w}_1^k. \tag{3.14}$$

With a similar manipulation for the second subsystem, we obtain

$$\hat{w}_1^{k+1} = F_2 \left( \frac{1}{(\alpha+1)F_1} + \frac{1}{\alpha+1} + \beta - 1 \right) \hat{u}_j^k. \tag{3.15}$$

Finally, by inserting (3.15) at iteration $k$ into (3.14), we get a relation over two iteration steps of the optimizable WR algorithm,

$$\hat{u}_j^{k+1} = F_1 F_2 \left( \alpha + 1 - \frac{1}{(\beta-1)F_2} + \frac{1}{\beta-1} \right) \left( \frac{1}{(\alpha+1)F_1} + \frac{1}{\alpha+1} + \beta - 1 \right) \hat{u}_j^{k-1},$$

and after simplifying, $\hat{u}_j^{k+1} = \rho_{opt(j)}(s, a, b, \alpha, \beta)\hat{u}_j^{k-1}$, where the convergence factor $\rho_{opt(j)}$ is given by (3.10). The same result also holds for the second subsystem and we find, as before, $\hat{u}_j^{2k} = (\rho_{opt(j)})^k \hat{u}_j^0$, and $\hat{w}_1^{2k} = (\rho_{opt(j)})^k \hat{w}_1^0$. Similarly, one can find the same equations for $j = 1$. ∎

**4. Optimization of the transmission condition.** From the convergence factor (3.10), we can see what the optimal choice would be of the parameters $\alpha$ and $\beta$ in the Laplace transform domain.

COROLLARY 4.1 (optimal convergence). *The new WR algorithm* (2.4) *converges in two iterations, independently of the initial waveforms* $\hat{u}_j^0$ *and* $\hat{w}_1^0$*, if*

$$\alpha := \alpha_{opt(j)} = \lambda_j - 1, \quad \beta := \beta_{opt(j)} = 1 - \lambda_j,\ j = 1, 2, \ldots, \tag{4.1}$$

*and hence* $\beta_{opt(j)} = -\alpha_{opt(j)}$.

*Proof.* The convergence factor vanishes if we insert (4.1) into $\rho_{opt(j)}$ given by (3.10). Hence, $\hat{u}_j^2$ and $\hat{w}_1^2$ are identically zero, independently of $\hat{u}_j^0$ and $\hat{w}_1^0$. ∎

This convergence result is optimal, since the resulting waveforms in each subsystem depend in general also on the source term $f_j$ in the other subsystem. Therefore, the minimum number of iterations needed for convergence for any WR algorithm with two subsystems is two: a first iteration where each subsystem incorporates the information of its source term $f_j$ into its waveforms and then transmits this information to the neighboring subsystem, and a second iteration to incorporate this transmitted information about $f_j$ from the neighboring subsystem into its own waveforms.

We observe that the optimal choice (4.1) is not just a parameter but the Laplace transform of a linear operator in time, since it depends on $s$. Since we have a rational function in $s$, the optimal transmission conditions correspond to nonlocal operators in time. They require integral operators which cannot be avoided in general and are expensive to use, since they would require convolutions in the transmission conditions. In the present case, one could multiply through by the denominators and thus obtain local transmission conditions, which would, however, contain higher order derivatives in time along the interface, again inconvenient to implement and prone to stability issues. We therefore propose approximations of these best transmission conditions, which are easy to use and inexpensive, in what follows.

We first assume for simplicity that $\beta = -\alpha$ motivated by the optimal choice (4.1). This leads to the convergence factor

$$\rho_{opt(j)}(s, a, b, \alpha) = \left( \frac{(\alpha + 1) - \lambda_j}{(\alpha + 1)\lambda_j - 1} \right)^2, \tag{4.2}$$

and $\lambda_j$ is given in (3.7).

Assuming that the optimal choice for $\alpha$ given in (4.1) is approximated by a constant $\alpha_{(j)}$, the simplest way to obtain a constant approximation is to use a Taylor expansion about $s = 0$, which corresponds to a low-frequency approximation. Note that for low frequencies the classical convergence factor behaves worst, as one can observe from Figures 2 and 3. The low-frequency constant approximation for the optimal choice given in (4.1) is

$$\alpha_{(j)T} = \frac{\sum_{r=1}^{\lfloor \frac{j+2}{2} \rfloor} (-1)^{r+1} \binom{j-r+1}{r-1} (-b)^{j-2r+2} a^{2r-2}}{a \sum_{r=1}^{\lfloor \frac{j+1}{2} \rfloor} (-1)^{r+1} \binom{j-r}{r-1} (-b)^{j-2r+1} a^{2r-2}} - 1, \quad \beta_{(j)T} = -\alpha_{(j)T} \tag{4.3}$$

(we use the index $T$ to denote Taylor). As an example, for $j = 1$, $j = 2$, $j = 3$, and $j = 4$ we get

$$\alpha_{(1)T} = \tfrac{-b}{a} - 1,\ \alpha_{(2)T} = \tfrac{a^2-b^2}{ab} - 1,\ \alpha_{(3)T} = \tfrac{b(2a^2-b^2)}{a(b^2-a^2)} - 1,\ \alpha_{(4)T} = \tfrac{a^2(3b^2-a^2)-b^4}{ab(b^2-2a^2)} - 1.$$

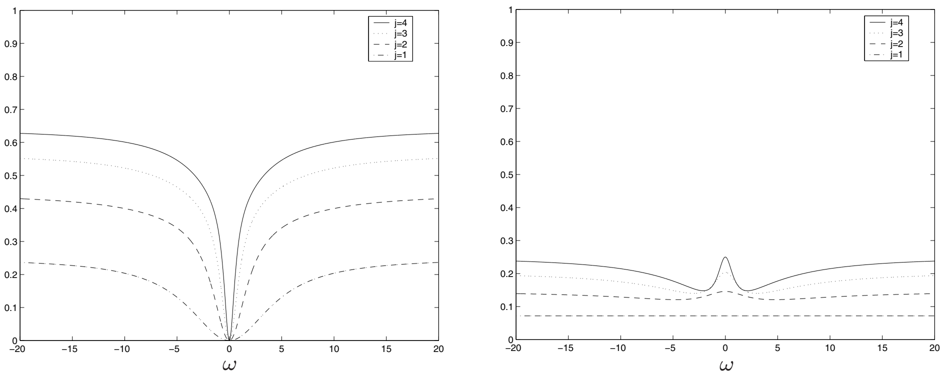


FIG. 4. *The case* $|b| = 2a$. *On the left, convergence factors* $|\rho_{opt(j)}(\omega, \alpha_{(j)T})|$ *obtained with low-frequency approximations, and on the right, optimized convergence factors* $|\rho_{opt(j)}(\omega, \alpha^*_{(j)})|$, $j = 1, \ldots, 4$.

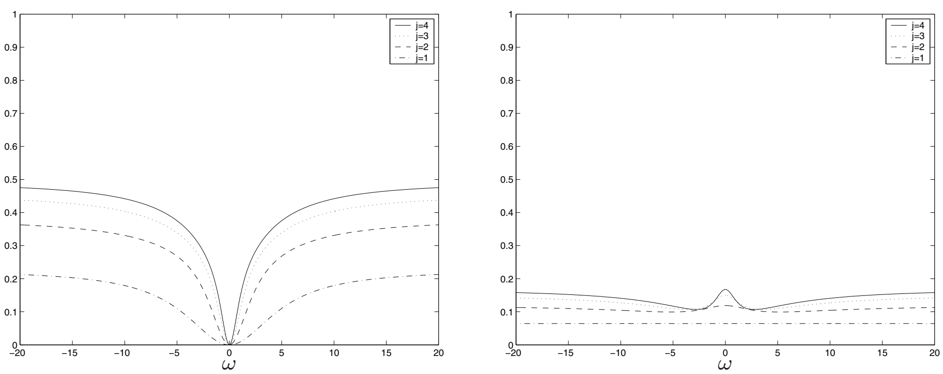


FIG. 5. *The case* $|b| > 2a$. *On the left, convergence factors* $|\rho_{opt(j)}(\omega, \alpha_{(j)T})|$ *obtained with low-frequency approximations, and on the right, optimized convergence factors* $|\rho_{opt(j)}(\omega, \alpha^*_{(j)})|$, $j = 1, \ldots, 4$.

We show the corresponding convergence factors as a function of $\omega$ on the left in Figures 4 and 5. In Figure 4 we choose $a = \frac{200}{63}$ and $b = -2a$, and in Figure 5 we choose $a = \frac{200}{63}$ and $b = -(2a + \frac{1}{5} \cdot \frac{100}{63})$ as typical RC circuit parameters. This leads to $\alpha_{(1)T} = 1$, $\alpha_{(2)T} = 0.5$, $\alpha_{(3)T} = 0.3333$, and $\alpha_{(4)T} = 0.25$ for the case $|b| = 2a$, and for the case $|b| > 2a$, this leads to $\alpha_{(1)T} = 1.1$, $\alpha_{(2)T} = 0.6238$, $\alpha_{(3)T} = 0.4842$, and $\alpha_{(4)T} = 0.4262$. To find a better constant approximation, assuming $\beta_{(j)} = -\alpha_{(j)}$, we need to solve the min-max problem

$$\min_{\alpha_{(j)}} \left( \max_{\Re(s) \geq 0} |\rho_{opt(j)}(s, a, b, \alpha_{(j)})| \right), \tag{4.4}$$

where $\alpha_{(j)}$ is the only optimization parameter left. Solving this problem numerically for $\eta = 0$, i.e., the unweighted $L^2$ norm in time, we obtain $\alpha^*_{(1)} = 2.732$, $\alpha^*_{(2)} = 1.618$, $\alpha^*_{(3)} = 1.215$, and $\alpha^*_{(4)} = 1$ for the case $|b| = 2a$, and $\alpha^*_{(1)} = 2.947$, $\alpha^*_{(2)} = 1.903$, $\alpha^*_{(3)} = 1.581$, and $\alpha^*_{(4)} = 1.443$ for the case $|b| > 2a$, which leads to the convergence factors shown on the right in Figures 4 and 5. Comparing these convergence factors with the ones obtained from a Taylor expansion on the left in the same figures and with the ones of the classical WR for the same circuit in Figures 2 and 3, we see that the optimization of the transmission condition leads to much faster algorithms. We also see in these figures that at the optimum, the convergence factor for $\omega = 0$ and $\omega \to \infty$ (where the convergence factor stays below a constant which is strictly less than 1) are equal, and we therefore propose to use the equioscillation equation

$$|\rho_{opt(j)}(0, \alpha^*_{(j)})| = \lim_{\omega\to\infty} |\rho_{opt(j)}(\omega, \alpha^*_{(j)})| \quad (4.5)$$

to determine the optimized choice of the parameter $\alpha^*_{(j)}$. This equation can be solved in closed form: denoting by $\lambda_{j0}$ the value of $\lambda_j$, $j = 1, 2, \ldots$, in (3.7), with $s = 0$, which is a real value since $\omega = 0$, we have

$$|\rho_{opt(j)}(0, \alpha_{(j)})| = \left|\frac{\alpha_{(j)} + 1 - \lambda_{j0}}{(\alpha_{(j)} + 1)\lambda_{j0} - 1}\right|^2 = \left(\frac{\alpha_{(j)} + 1 - \lambda_{j0}}{(\alpha_{(j)} + 1)\lambda_{j0} - 1}\right)^2 := R_{j0}.$$

Now, since the numerator in $\lambda_j$ is of a degree higher than the denominator by one, we have

$$|\lim_{\omega\to\infty} \rho_{opt(j)}(i\omega, \alpha_{(j)})| = \left(\frac{1}{\alpha_{(j)} + 1}\right)^2 := R_{j\infty}.$$

By solving the equation $R_{j0} = R_{j\infty}$, we get the solutions $\alpha_{(j)} = 0, \ -2, \ \lambda_{j0} \pm \sqrt{\lambda_{j0}^2 - 1} - 1$, and thus the following

*Conjecture* 4.2 (optimal constant approximation for arbitrary size circuit). The solution of the min-max problem (4.4) is

$$\alpha^*_{(j)} := \lambda_{j0} + \sqrt{\lambda_{j0}^2 - 1} - 1, \ j = 1, 2, \ldots. \quad (4.6)$$

For the case $j = 1, 2$ of small circuits, this conjecture is proved in [1]; see also [2]. We investigate in the next section the special case of a very large circuit.

**5. Optimality for a very large RC circuit.** We first need to study $\lambda_j$ in (3.7) in more detail. As we have seen in the proof of Theorem 3.3, $\lambda_1 = \frac{s-b}{a}$, and for $j > 1$, $N_j = (s - b)N_{j-1} - aD_{j-1}$, and $D_j = aN_{j-1}$, and

$$\lambda_j = \frac{N_j}{D_j} = \frac{(s-b)N_{j-1} - aD_{j-1}}{aN_{j-1}} = \frac{s-b}{a} - \frac{D_{j-1}}{N_{j-1}} = \frac{s-b}{a} - \frac{1}{\lambda_{j-1}}.$$

Hence, we get the recurrence relation

$$\lambda_1 = \frac{s-b}{a}, \quad \lambda_{j+1} = \lambda_1 - \frac{1}{\lambda_j}, \ j \geq 1. \quad (5.1)$$

In the following lemma we prove convergence of the sequence (5.1).

LEMMA 5.1. *For $s$ in the right half of the complex plane, $s = \eta + i\omega$, $\eta > 0$, and $|b| \geq 2a$, the recurrence relation* (5.1) *converges to the limit*

$$\lambda_+ = \frac{s - b + \sqrt{(s-b)^2 - 4a^2}}{2a}, \quad (5.2)$$

*as $j$ goes to infinity.*

*Proof.* Let $\lambda_j = \frac{y_j(\lambda_1)}{z_j(\lambda_1)}$. Then $\lambda_{j+1} = \lambda_1 - \frac{z_j(\lambda_1)}{y_j(\lambda_1)} = \frac{\lambda_1 y_j(\lambda_1) - z_j(\lambda_1)}{y_j(\lambda_1)}$, and hence we have

$$\begin{aligned} z_{j+1}(\lambda_1) &= y_j(\lambda_1), \\ y_{j+1}(\lambda_1) &= \lambda_1 y_j(\lambda_1) - z_j(\lambda_1) = \lambda_1 y_j(\lambda_1) - y_{j-1}(\lambda_1). \end{aligned}$$

Now, we write the above equations in the system

$$\begin{pmatrix} y_{j+1}(\lambda_1) \\ y_j(\lambda_1) \end{pmatrix} = \begin{pmatrix} \lambda_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_j(\lambda_1) \\ y_{j-1}(\lambda_1) \end{pmatrix} = \begin{pmatrix} \lambda_1 & -1 \\ 1 & 0 \end{pmatrix}^j \begin{pmatrix} y_1 \\ y_0 \end{pmatrix}.$$

The eigenvalues of the matrix

$$\begin{pmatrix} \lambda_1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \lambda_1 = \frac{s-b}{a}, \tag{5.3}$$

in the system above are given by

$$\lambda_\pm = \frac{s - b \pm \sqrt{(s-b)^2 - 4a^2}}{2a},$$

where $|\lambda_+| > 1$ and $|\lambda_-| < 1$ in the right half of the complex plane, $s = \eta + i\omega$, $\eta > 0$, for $|b| \geq 2a$; see [15]. Therefore, we have two distinct eigenvalues, and hence the matrix in (5.3) is diagonalizable and can be written as

$$\begin{pmatrix} \lambda_1 & -1 \\ 1 & 0 \end{pmatrix} = P \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} P^{-1},$$

where $P$ is an invertible matrix, which has the corresponding eigenvectors as its column vectors. By forming the matrix $P$, and using $y_1 = s - b$ and $y_0 = a$, we can find $\sigma_1$ and $\sigma_2$ by direct calculations, such that

$$\begin{pmatrix} y_{j+1}(\lambda_1) \\ y_j(\lambda_1) \end{pmatrix} = P \begin{pmatrix} \lambda_+^j & 0 \\ 0 & \lambda_-^j \end{pmatrix} P^{-1} \begin{pmatrix} y_1 \\ y_0 \end{pmatrix} = \begin{pmatrix} \sigma_1 \lambda_+^{j+1} + \sigma_2 \lambda_-^{j+1} \\ \sigma_1 \lambda_+^j + \sigma_2 \lambda_-^j \end{pmatrix}.$$

Hence,

$$\lambda_{j+1} = \frac{y_{j+1}}{z_{j+1}} = \frac{y_{j+1}}{y_j} = \frac{\sigma_1 \lambda_+^{j+1} + \sigma_2 \lambda_-^{j+1}}{\sigma_1 \lambda_+^j + \sigma_2 \lambda_-^j},$$

and since $|\lambda_+| > 1$ and $|\lambda_-| < 1$, we have, as $j$ goes to infinity,

$$\lim_{j \to \infty} \lambda_{j+1} = \lambda_+.$$

Note that the exact values for $\sigma_1$ and $\sigma_2$ are not really needed for the result above. $\square$

Using this result, we can formally pass to the limit in the convergence factor of the classical WR algorithm and obtain

$$\rho_{claL}(s, a, b) = \left( \frac{1}{\lambda_+} \right)^2, \tag{5.4}$$

where $\lambda_+$ is given in (5.2). This result was directly shown in [15] by analyzing a circuit of infinite size. We also have $|\lambda_+| > 1$, for $s := \eta + i\omega$, $\eta > 0$, and $|b| \geq 2a$.

The convergence factor depends as before on $s \in \mathbb{C}$, the parameter in the Laplace transform. The classical WR, as is evident from (5.4), always converges for a large number of iterations since $|\lambda_+| > 1$ for $\eta > 0$, but convergence might be very slow. Also, the convergence factor is analytic for $s = \eta + i\omega$, $\eta > 0$, under the condition $|b| \geq 2a$, and in addition, if we let $s = re^{i\theta}$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then the limit as $r$ goes to infinity is zero, and therefore, using the maximum principle for analytic functions, the maximum of $\rho_{claL}$ is attained on the boundary of the right half of the complex plane, at $\eta = 0$. Taking the limit on the boundary as $\omega$ goes to zero implies

$$\lim_{\omega\to 0} |\rho_{claL}(i\omega, a, b)| = \frac{|b|-\sqrt{b^2-4a^2}}{|b|+\sqrt{b^2-4a^2}} \quad \begin{array}{l} = 1 \text{ if } |b| = 2a, \\ < 1 \text{ if } |b| > 2a, \end{array}$$

where $|b| = 2a$ is most often the case for RC type circuits, or diffusion type problems. Therefore, the convergence will be very slow for low frequencies $\omega$ and the mode $\omega = 0$ will not converge for the important case $|b| = 2a$; see also Figures 2 and 3.

When passing to the limit as $n$ goes to infinity in the optimizable WR algorithm, we obtain the convergence factor

$$\rho_{optL}(s, a, b, \alpha, \beta) = \frac{(\alpha+1) - \lambda_+}{(\alpha+1)\lambda_+ - 1} \cdot \frac{(\beta-1) + \lambda_+}{(\beta-1)\lambda_+ + 1}. \tag{5.5}$$

Therefore the algorithm also converges in two iterations for the optimal choice of parameters

$$\alpha := \lambda_+ - 1, \quad \beta := 1 - \lambda_+, \tag{5.6}$$

independently of the guess for the initial waveforms, which is formally proved in [15]. Since this choice, however, also involves symbols depending on $s$ in the Laplace transformed domain, we analyze in the next subsections approximations by constant and also first order approximations.

**5.1. An optimized WR algorithm with constant transmission conditions.** In this subsection, we assume that the parameters are just constants, and we again have to solve a min-max problem. The analyticity of $\rho_{optL}$ in the right half of the complex plane was shown in [15] for $a > 0$, $b < 0$, $|b| \geq 2a$, and $\alpha > 0$, $\beta < 0$. By the maximum principle for analytic functions, the maximum of $\rho_{optL}$ for $s = \eta + i\omega$, $\eta > 0$, is attained on the boundary. The limit of $\rho_{optL}$ for $s = re^{i\theta}$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, as $r$ goes to infinity is one limit in all directions, which is equal to $(\frac{-1}{(\alpha+1)(\beta-1)})$, and therefore, the maximum of $\rho_{optL}$ is attained on the boundary $\eta = \text{const}$.

Based on the optimal choice (5.6), we again take $\beta = -\alpha$, and hence the convergence factor $\rho_{optL}$ in (5.5) with constant approximation is

$$\rho_{optL0}(i\omega, a, b, \alpha) = \left(\frac{\alpha + 1 - \lambda_+}{(\alpha+1)\lambda_+ - 1}\right)^2. \tag{5.7}$$

Let us now consider $\lambda_+$ in (5.2) and the $L^2$ norm, i.e., $s = i\omega$. Let $\lambda_+ := x + iy = \Re(\lambda_+) + i\Im(\lambda_+)$. Then the real part $x$ is given by $x := X(\omega) = \frac{-b}{2a} + \psi(\omega)$, where

$$\psi(\omega) = \frac{\sqrt{2\sqrt{\omega^4 + 2\omega^2 b^2 + 8\omega^2 a^2 + b^4 - 8b^2 a^2 + 16a^4} - 2\omega^2 + 2b^2 - 8a^2}}{4a},$$

and the imaginary part $y$ is given by $y := Y(\omega) = \frac{\omega}{2a} - \frac{2\omega b}{|2b\omega|}\varphi(\omega)$, where

$$\varphi(\omega) = \frac{\sqrt{2\sqrt{\omega^4 + 2\omega^2 b^2 + 8\omega^2 a^2 + b^4 - 8b^2 a^2 + 16a^4} + 2\omega^2 - 2b^2 + 8a^2}}{4a}.$$

For any $\omega > 0$, we have $Y(\omega) = \frac{\omega}{2a} + \varphi(\omega)$, and for $\omega < 0$, we have $Y(\omega) = -\frac{|\omega|}{2a} - \varphi(\omega) = -(\frac{|\omega|}{2a} + \varphi(\omega))$ since $b < 0$, and hence $(Y(|\omega|))^2 = (Y(-|\omega|))^2$, since $\varphi(\omega)$ depends only on $\omega^2$. This implies that the modulus $|\rho_{optL0}(\omega)|$ satisfies $|\rho_{optL0}(|\omega|)| = |\rho_{optL0}(-|\omega|)|$, since the real part $x := X(\omega)$ depends only on $\omega^2$, and the imaginary part $y := Y(\omega)$, using the fact that $b < 0$, satisfies $(Y(|\omega|))^2 = (Y(-|\omega|))^2$. Therefore, it suffices to optimize for positive frequencies, $\omega > 0$. Furthermore, for any $\omega \neq 0$, we have $x > 1$, since $|b| \geq 2a$, and $\frac{-b}{2a}$ is added to a positive quantity, and hence $|\lambda_+| > 1$. Now if $\omega = 0$, then we get $\lambda_+ = x = \frac{-b}{2a} + \frac{\sqrt{b^2-4a^2}}{2a}$, so $|\lambda_+| > 1$ for the case when $-b > 2a$, and $\lambda_+ = 1$ if and only if $-b = 2a$. We thus cannot use the idea of equilibrating $\rho_{optL0}$ at $\omega = 0$ and for $\omega$ large which led to the conjecture for the finite size RC circuit here in the important case where $-b = 2a$, but we could use it for the case $|b| > 2a$. Note also that the modulus of $\rho_{optL0}$ in (5.7) satisfies

$$\begin{aligned}
|\rho_{optL0}| \leq 1 &\iff |\alpha + 1 - \lambda_+| \leq |(\alpha+1)\lambda_+ - 1| \\
&\iff |\alpha + 1 - x - iy| \leq |(\alpha+1)x - 1 + i(\alpha+1)y| \\
&\iff (\alpha + 1 - x)^2 + y^2 \leq ((\alpha+1)x - 1)^2 + (\alpha+1)^2 y^2 \\
&\iff (\alpha+1)^2 + x^2 + y^2 \leq (\alpha+1)^2 x^2 + 1 + (\alpha+1)^2 y^2 \\
&\iff (\alpha+1)^2 - 1 \leq ((\alpha+1)^2 - 1)x^2 + ((\alpha+1)^2 - 1)y^2 \\
&\iff 1 \leq x^2 + y^2 \quad \iff \quad 1 \leq |\lambda_+|.
\end{aligned}$$

In a realistic transient analysis, however, estimates for the maximum and minimum frequencies are considered (see, for example, [15, 13]). This is due to the fact that a partition with step-size $\Delta t$ cannot carry arbitrary high frequencies, i.e., $\omega$ will vary from $-\omega_{max}$ to $\omega_{max}$. We use the estimate $\omega_{max} = \frac{\pi}{\Delta t}$, which is the highest possible oscillation on a grid with spacing $\Delta t$. The estimate for the lowest frequency occurring in the transient analysis depends on the length of the time interval $[0, T]$. As in [15], we expand the signal in a sine series $sin(\frac{k\pi t}{T})$ for $k = 1, 2, \ldots$. This leads to the estimate $\omega_{min} = \frac{\pi}{T}$ for the lowest relevant frequency. We can therefore consider the min-max problem

$$\min_{\alpha > 0}\left(\max_{\omega_{min} \leq \omega \leq \omega_{\max}} |\rho_{optL0}(i\omega, a, b, \alpha)|\right), \; -b \geq 2a. \tag{5.8}$$

We introduce a change of variables based on the real part of $\lambda_+$, and a time scaling as well, where $a$ and $b$ are scaled to $\tilde{a} = 1$ and $\tilde{b} = \frac{b}{a} = -2c^2$, where $c = \sqrt{\frac{-b}{2a}} \geq 1$, and then everywhere in the analysis, $a$ can be replaced by 1 and $b$ by $-2c^2$. The real part $x$ of $\lambda_+$ is now given by

$$x := X(\omega, c) = c^2 + \frac{1}{4}\sqrt{2\sqrt{\omega^4 + 8\omega^2 c^4 + 8\omega^2 + 16c^8 - 32c^4 + 16} - 2\omega^2 + 8c^4 - 8}, \tag{5.9}$$

and $x \in [x_{min}, x_{max}]$, where

(5.10)
$$
\begin{aligned}
x_{min} &= c^2 + \tfrac{\sqrt{2}}{4}\sqrt{\sqrt{\chi_{min}} - \tilde{\omega}_{min}^2 + 4c^4 - 4} > x_0 = \lim_{\omega\to 0} X(\omega, c) = c^2 + \sqrt{c^4-1},\\
\chi_{min} &= \tilde{\omega}_{min}^4 + 8\tilde{\omega}_{min}^2 c^4 + 8\tilde{\omega}_{min}^2 + 16c^8 - 32c^4 + 16,\\
x_{max} &= c^2 + \tfrac{\sqrt{2}}{4}\sqrt{\sqrt{\chi_{max}} - \tilde{\omega}_{max}^2 + 4c^4 - 4} < x_\infty = \lim_{\omega\to \infty} X(\omega, c) = 2c^2,\\
\chi_{max} &= \tilde{\omega}_{max}^4 + 8\tilde{\omega}_{max}^2 c^4 + 8\tilde{\omega}_{max}^2 + 16c^8 - 32c^4 + 16,
\end{aligned}
$$

and $\tilde{\omega}_{min} = \frac{\omega_{min}}{a}$, $\tilde{\omega}_{max} = \frac{\omega_{max}}{a}$. The modulus of $\rho_{optL0}$ is given by

$$R_0(x,c,\alpha) = |\rho_{optL0}| = \frac{2(\alpha+1)^2c^2 - (\alpha+1)^2x - 4(\alpha+1)xc^2 + 2(\alpha+1)x^2 + x}{-4(\alpha+1)xc^2 + 2(\alpha+1)x^2 + 2c^2 - x + (\alpha+1)^2x},$$

where $\alpha > 0$ and $c \geq 1$. If we assume $\gamma = \alpha + 1$, then we find $\gamma$ such that the convergence factor

$$R_0(x,c,\gamma) = \frac{2\gamma^2c^2 - \gamma^2x - 4\gamma xc^2 + 2\gamma x^2 + x}{-4\gamma xc^2 + 2\gamma x^2 + 2c^2 - x + \gamma^2 x}, \quad x \in [x_{min}, x_{max}], \tag{5.11}$$

where $\gamma > 1$, is minimized over all $x \in [x_{min}, x_{max}]$. Hence the optimal choice for $\gamma$ is the solution of the min-max problem

$$\min_{\gamma>1}\left(\max_{x_{min}\leq x\leq x_{max}} R_0(x,c,\gamma)\right). \tag{5.12}$$

LEMMA 5.2. *The function $x \mapsto R_0(x,c,\gamma)$ defined in* (5.11) *has a unique local minimum at*

$$\underline{x}(c,\gamma) = c^2 + \frac{c\sqrt{1+\gamma^2-2\gamma c^2}}{\sqrt{2\gamma}}$$

*in $(c^2+\sqrt{c^4-1}, 2c^2)$ if $\gamma > c^2 + \sqrt{c^4-1}$. For any other value of $\gamma > 1$, $R_0$ has no extrema in $x \in (c^2+\sqrt{c^4-1}, 2c^2)$.*

*Proof.* A partial derivative of $R_0$ with respect to $x$ shows that the zeros of the polynomial

$$P(x) = (4\gamma^3 - 4\gamma)x^2 + (8\gamma c^2 - 8\gamma^3c^2)x + 2c^2 - 8\gamma c^4 - 2\gamma^4c^2 + 8\gamma^3c^4$$

determine the extrema of $R_0$. The polynomial $P(x)$ has two real zeros $\overline{x}$ and $\underline{x}$ given by

$$\overline{x}(c,\gamma) = c^2 + \frac{c\sqrt{1+\gamma^2-2\gamma c^2}}{\sqrt{2\gamma}}, \qquad \underline{x}(c,\gamma) = c^2 - \frac{c\sqrt{1+\gamma^2-2\gamma c^2}}{\sqrt{2\gamma}}.$$

The argument under the square root, $1+\gamma^2-2\gamma c^2$, is greater than zero for $\gamma > c^2+\sqrt{c^4-1}$, and hence $\overline{x}$ and $\underline{x}$ are real for $\gamma > c^2+\sqrt{c^4-1}$. Since $\overline{x}$ is not in $(c^2+\sqrt{c^4-1}, 2c^2)$, it can be discarded, and by studying the sign of $\frac{\partial R_0}{\partial x}$, $\underline{x}$ is a minimum. If the argument under the square root is negative, then $\overline{x}$ and $\underline{x}$ are complex, and if it equals zero, then $\overline{x} = \underline{x} = c^2$, which is not in $(c^2+\sqrt{c^4-1}, 2c^2)$, and in both cases $R_0$ has no local extrema. ∎

LEMMA 5.3. *For fixed $x \in (c^2+\sqrt{c^4-1}, 2c^2)$ and $\gamma > c^2+\sqrt{c^4-1}$, we have $\frac{\partial R_0(x,c,\gamma)}{\partial x}(\gamma - \underline{\gamma}(x,c)) \geq 0$, where $\underline{\gamma}(x,c)$ is given by*

$$\underline{\gamma}(x,c) = \frac{2c^2 + 2\sqrt{(2xc^2 - x^2 + c^2)(x^2 - 2xc^2 + c^2)}}{2(2xc^2 - x^2)}. \tag{5.13}$$

*Proof.* A partial derivative of $R_0(x,c,\gamma)$ with respect to $\gamma$ shows that the zeros of the polynomial

$$Q(\gamma) = (12x^2c^2 - 8xc^4 - 4x^3)\gamma^2 + (8c^4 - 8xc^2)\gamma - 8xc^4 + 12x^2c^2 - 4x^3$$

determine the extrema of $R_0$. The polynomial $Q(\gamma)$ has two zeros which are

$$\underline{\gamma}(x,c) = \frac{2c^2+2\sqrt{(2xc^2-x^2+c^2)(x^2-2xc^2+c^2)}}{2(2xc^2-x^2)}, \quad \overline{\gamma}(x,c) = \frac{2c^2-2\sqrt{(2xc^2-x^2+c^2)(x^2-2xc^2+c^2)}}{2(2xc^2-x^2)}.$$

Since $0 < 2xc^2 - x^2 < 1$ for $x \in (c^2 + \sqrt{c^4-1}, 2c^2)$, the argument under the square root is positive and hence $\underline{\gamma}$ and $\overline{\gamma}$ are real. Some algebra shows that $\overline{\gamma} < 1$ and therefore it can be discarded. From the sign of $\frac{\partial R_0}{\partial \gamma}$, $\underline{\gamma}$ is a minimum. Therefore, for $c^2 + \sqrt{c^4-1} < \gamma < \underline{\gamma}$, increasing $\gamma$ decreases $R_0$, whereas for $\gamma > \underline{\gamma}$, increasing $\gamma$ increases $R_0$. ∎

THEOREM 5.4. *The solution $\gamma^*$ of the min-max problem* (5.12) *is partially characterized by the equioscillation of $R_0$ given in* (5.11)*: for $\tilde{\gamma}$ defined by the equation $R_0(x_{min},c,\tilde{\gamma}) = R_0(x_{max},c,\tilde{\gamma})$, i.e., $\tilde{\gamma} = \frac{x_{min}x_{max}+2c^4-x_{min}c^2-x_{max}c^2}{c^2} + r$ with $r = \frac{\sqrt{(x_{min}x_{max}-x_{min}c^2-c^2+2c^4-x_{max}c^2)(x_{min}x_{max}-x_{min}c^2+c^2+2c^4-x_{max}c^2)}}{c^2}$, we have*

$$\gamma^* = \begin{cases} \underline{\gamma}(x_{min}) & \text{if} \quad \tilde{\gamma} \le \underline{\gamma}(x_{min}), \\ \tilde{\gamma} & \text{if} \quad \underline{\gamma}(x_{min}) < \tilde{\gamma} < \underline{\gamma}(x_{max}), \\ \underline{\gamma}(x_{max}) & \text{if} \quad \tilde{\gamma} \ge \underline{\gamma}(x_{max}), \end{cases}$$

*where $x_{max}$, $x_{min}$ are given in* (5.10), *$\underline{\gamma}(x,c)$ is given in* (5.13), *and $c = \sqrt{\frac{-b}{2a}} \ge 1$.*

*Proof.* By Lemma 5.3, the optimal $\gamma^*$ must lie in the interval $[\underline{\gamma}(x_{min}), \underline{\gamma}(x_{max})]$, since with $\gamma$ outside this interval, $R_0$ can be uniformly decreased for all $x_{min} \le x \le x_{max}$ by moving $\gamma$ toward this interval. Note that $\underline{\gamma}(x_{min}) < \underline{\gamma}(x_{max})$ since $\underline{\gamma}(x)$ is increasing for all $x \in (x_0, x_\infty)$. Now, Lemma 5.2 implies that the maximum of $R_0$ in the min-max problem can only be attained on the boundaries, at $x = x_{min}$ and at $x = x_{max}$, since $R_0$ has no interior maxima. For $\gamma = \underline{\gamma}(x_{min})$, we have $R_0(x_{min}, c, \underline{\gamma}(x_{min})) > 0$, and the smaller $x_{min}$ is, the closer $R_0(x_{min}, c, \underline{\gamma}(x_{min}))$ to zero is, and increasing $\gamma$ increases $R_0(x_{min}, c, \gamma)$ monotonically, by Lemma 5.3. On the other hand, for $\gamma = \underline{\gamma}(x_{min})$, we have $R_0(x_{max}, c, \underline{\gamma}(x_{min})) > 0$, and increasing $\gamma$ decreases $R_0(x_{max}, c, \gamma)$. By continuity, there are three possibilities: either $\tilde{\gamma}$, the solution of the equation $R_0(x_{min}, c, \tilde{\gamma}) = R_0(x_{max}, c, \tilde{\gamma})$, is less than $\underline{\gamma}(x_{min})$, which means $R_0(x_{min}, c, \gamma) > R_0(x_{max}, c, \gamma)$ for $[\underline{\gamma}(x_{min}), \underline{\gamma}(x_{max})]$, the interval in which $R_0(x_{min}, c, \gamma)$ is always increasing and $R_0(x_{max}, c, \gamma)$ is always decreasing, and hence the solution of the min-max problem (5.12) is $\gamma^* = \underline{\gamma}(x_{min})$, or the functions $R_0(x_{min}, c, \gamma)$ and $R_0(x_{max}, c, \gamma)$ intersect at $\tilde{\gamma}$ that lies in the interval $(\underline{\gamma}(x_{min}, c), \underline{\gamma}(x_{max}, c))$, which means $R_0(x_{min}, c, \gamma) < R_0(x_{max}, c, \gamma)$ for $\gamma \in [\underline{\gamma}(x_{min}), \tilde{\gamma})$, and $R_0(x_{min}, c, \gamma) > R_0(x_{max}, c, \gamma)$ for $\gamma \in (\tilde{\gamma}, \underline{\gamma}(x_{max})]$, and therefore the solution of the min-max problem is $\gamma^* = \tilde{\gamma}$. The last possibility is that the functions intersect at $\tilde{\gamma}$ that is greater than $\underline{\gamma}(x_{max})$, which means $R_0(x_{min}, c, \gamma) < R_0(x_{max}, c, \gamma)$ for $[\underline{\gamma}(x_{min}), \underline{\gamma}(x_{max})]$, and hence the solution of the min-max problem (5.12) is $\gamma^* = \underline{\gamma}(x_{max})$. ∎

Using this theorem and $\alpha = \gamma - 1$, one can find the optimal choice for $\alpha$.

**5.2. An optimized WR algorithm with first order transmission conditions.** In this subsection, we introduce a first order approximation for the optimal parameter $\alpha$ given in (5.6),

$$\alpha := \alpha_0 + \alpha_1 s, \quad s := \eta + i\omega \in \mathbb{C},$$

for some constants $\alpha_0$ and $\alpha_1 \neq 0$, since otherwise we get the constant approximation. We thus obtain the convergence factor

$$\rho_{optL1}(s, a, b, \alpha_0, \alpha_1) = \left( \frac{(\alpha_0 + \alpha_1 s + 1) - \lambda_+}{\lambda_+(\alpha_0 + \alpha_1 s + 1) - 1} \right)^2, \tag{5.14}$$

where $\lambda_+$ is given in (5.2). The convergence factor $\rho_{optL1}$ is an analytic function in the right half of the complex plane, $s = \eta + i\omega$, $\eta > 0$, under the conditions $b < 0$, $a > 0$, $|b| \geq 2a$, $\alpha_0 \geq 0$, and $\alpha_1 > 0$, as one can check by direct calculations. Therefore, using the maximum principle for analytic functions, the maximum of $|\rho_{optL1}|$ is attained on the boundary. Now, since for $s = re^{i\theta}$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, the limit of $\rho_{optL1}$ equals zero as $r$ goes to infinity, the maximum of $|\rho_{optL1}|$ is attained at $\eta = \text{const.}$

In order to determine the optimal choice of $\alpha_0$ and $\alpha_1$, it suffices as in subsection 5.1 to optimize for positive frequencies, $\omega > 0$, since $|\rho_{optL1}|$ depends only on $\omega^2$, because we have the same $\lambda_+$ as in (5.2). This yields the optimization problem

$$\min_{\alpha_0 \geq 0, \alpha_1 > 0} \left( \max_{\omega_{\min} \leq \omega < \infty} |\rho_{optL1}(i\omega, a, b, \alpha_0, \alpha_1)| \right), \ |b| \geq 2a, \tag{5.15}$$

where we do not need to truncate with $\omega_{max}$ since $\rho_{optL1} \to 0$ as $\omega \to \infty$. To further analyze the convergence factor, we use a new change of variables that simplifies the computations for finding the solution of the min-max problem (5.15), and it is based on the real part of

$$z := s + \sqrt{(s-b)^2 - 4a^2} = x + iy, \ s = i\omega, \ \omega \geq \omega_{min} > 0,$$

which appears in $\lambda_+$, where we have now

$$\lambda_+ = \frac{-b}{2a} + \frac{z}{2a}. \tag{5.16}$$

The optimal parameter $\alpha$ in (5.6) is given by $\alpha := \lambda_+ - 1$, and hence a first order approximation is

$$\alpha = \alpha_0 + \alpha_1 s = \frac{-b}{2a} - 1 + \frac{p}{2a} + \frac{q}{2a} s, \tag{5.17}$$

where $p$ and $q$ are new parameters.

In the new variable $x$, and using the first order approximation (5.17), the convergence factor $\rho_{optL1}$ in modulus becomes, after factorizing $a^5$ from the denominator and numerator to eliminate one parameter,

$$|\rho_{optL1}(x, a, b, p, q)| := \frac{Q_1(x, a, b, p, q)}{Q_2(x, a, b, p, q)},$$

where

$$
\begin{aligned}
Q_1(x,a,b,p,q) &= \left(-2q+q^2\right)\left(\tfrac{x}{a}\right)^4 + \left(2\tfrac{p}{a}-2\tfrac{b}{a}+2\tfrac{b}{a}q\right)\left(\tfrac{x}{a}\right)^3\\
&+\left(-\left(\tfrac{b}{a}\right)^2 q^2 + 2q\left(\tfrac{b}{a}\right)^2 - 8q + 4q^2 + \left(\tfrac{b}{a}\right)^2 - \left(\tfrac{p}{a}\right)^2 + 4\right)\left(\tfrac{x}{a}\right)^2\\
&+\left(-8\tfrac{b}{a} + 2\left(\tfrac{b}{a}\right)^3 - 2\left(\tfrac{b}{a}\right)^3 q + 8\tfrac{b}{a}q - 2\left(\tfrac{b}{a}\right)^2\tfrac{p}{a}\right)\tfrac{x}{a}\\
&+\left(\left(\tfrac{b}{a}\right)^2\left(\tfrac{p}{a}\right)^2 - \left(\tfrac{b}{a}\right)^4 + 4\left(\tfrac{b}{a}\right)^2\right)\left(\tfrac{b}{a}+\tfrac{x}{a}\right),\\
Q_2(x,a,b,p,q) &= \left(2q-q^2\right)\left(\tfrac{x}{a}\right)^5 + \left(q^2\tfrac{b}{a}+2\tfrac{p}{a}-2\tfrac{b}{a}\right)\left(\tfrac{x}{a}\right)^4\\
&+\left(-2\tfrac{b}{a}\tfrac{p}{a} - 4 + 8q + \left(\tfrac{b}{a}\right)^2 + \left(\tfrac{b}{a}\right)^2 q^2 - 4q^2 - 4q\left(\tfrac{b}{a}\right)^2 + \left(\tfrac{p}{a}\right)^2\right)\left(\tfrac{x}{a}\right)^3\\
&+\left(-\tfrac{b}{a}\left(\tfrac{p}{a}\right)^2 - 2\left(\tfrac{b}{a}\right)^2\tfrac{p}{a} - \left(\tfrac{b}{a}\right)^3 q^2 + 3\left(\tfrac{b}{a}\right)^3 - 4\tfrac{b}{a} + 4\tfrac{b}{a}q^2\right)\left(\tfrac{x}{a}\right)^2\\
&+\left(-\left(\tfrac{b}{a}\right)^4 + 2\left(\tfrac{b}{a}\right)^3\tfrac{p}{a} + 2q\left(\tfrac{b}{a}\right)^4 - 8q\left(\tfrac{b}{a}\right)^2 - \left(\tfrac{b}{a}\right)^2\left(\tfrac{p}{a}\right)^2 + 4\left(\tfrac{b}{a}\right)^2\right)\tfrac{x}{a}\\
&+\left(\tfrac{p}{a}\right)^2\left(\tfrac{b}{a}\right)^3 - \left(\tfrac{b}{a}\right)^5 + 4\left(\tfrac{b}{a}\right)^3.
\end{aligned}
$$

By setting $\tilde{p} = \frac{p}{a}$, and as before, $\frac{b}{a} = -2c^2,\ c \geq 1$, and $\tilde{x} = \frac{x}{a}$, the modulus of the convergence factor $\rho_{optL1}$ is

$$
R_1(\tilde{x},c,\tilde{p},q) := \frac{P_1(\tilde{x},c,\tilde{p},q)}{P_2(\tilde{x},c,\tilde{p},q)}, \tag{5.18}
$$

where

$$
\begin{aligned}
P_1(\tilde{x},c,\tilde{p},q) &= ((-2q+q^2)\tilde{x}^4 + (2\tilde{p}+4c^2-4c^2q)\tilde{x}^3\\
&+(-4q^2c^4+8qc^4-8q+4q^2+4c^4-\tilde{p}^2+4)\tilde{x}^2\\
&+(16c^2-16c^6+16c^6q-16c^2q-8\tilde{p}c^4)\tilde{x}+4c^4\tilde{p}^2-16c^8+16c^4)(\tilde{x}-2c^2),\\
P_2(\tilde{x},c,\tilde{p},q) &= ((2q-q^2)\tilde{x}^5 + (4c^2-2q^2c^2+2\tilde{p})\tilde{x}^4\\
&+(4c^2\tilde{p}-4+8q+4c^4+4q^2c^4-4q^2-16qc^4+\tilde{p}^2)\tilde{x}^3\\
&+(8c^2-8\tilde{p}c^4+2c^2\tilde{p}^2+8q^2c^6-24c^6-8q^2c^2)\tilde{x}^2\\
&+(-16c^8-16c^6\tilde{p}+32qc^8-32qc^4-4c^4\tilde{p}^2+16c^4)\tilde{x}\\
&-8\tilde{p}^2c^6+32c^{10}-32c^6),
\end{aligned}
$$

and $\tilde{x} \in [\tilde{x}_{min}, 2c^2)$, where $\tilde{x}_{min} = \frac{x_{min}}{a}$ is given in terms of $\tilde{\omega}_{min} = \frac{\omega_{min}}{a}$ and $c$, and goes to $\tilde{x}_0 = \frac{x_0}{a} = 2\sqrt{c^4-1}$, as $\tilde{\omega}_{min}$ goes to zero.

The optimized parameters are given by $\alpha_0 = \frac{-b}{2a} - 1 + \frac{p}{2a} = c^2 - 1 + \frac{\tilde{p}}{2}$ and $\alpha_1 = \frac{q}{2a}$, where $\tilde{p}$ and $q$ will be determined using $R_1$ in (5.18). Since for analyticity in the right half of the complex plane, we need $\alpha_0 \geq 0$ and $\alpha_1 > 0$, we require $\tilde{p} \geq 2(1-c^2)$ and $q > 0$.

The new min-max problem which we need to solve is in the new variables given by

$$
\min_{\tilde{p}\geq 2(1-c^2),\, q>0}\left(\max_{\tilde{x}_{min}\leq\tilde{x}<2c^2} R_1(\tilde{x},c,\tilde{p},q)\right) = \max_{\tilde{x}_{min}\leq\tilde{x}<2c^2} R_1(\tilde{x},c,\tilde{p}^*,q^*),\ c \geq 1. \tag{5.19}
$$

Theorem 5.5. *If in the optimized WR algorithm with first order transmission conditions the free parameters are chosen to be*

$$
\alpha_0 = \alpha_0^* = c^2 - 1 + \frac{\tilde{p}^*}{2} \text{ and } \alpha_1 = \alpha_1^* = \frac{q^*}{2a},
$$

*where* $c = \sqrt{\frac{-b}{2a}} \geq 1$, *and* $a$, $b$ *are the entries of the matrices, and* $\tilde{p}^*$ *and* $q^*$ *are defined by the systems of nonlinear equations*

(5.20)
$$R_1(\tilde{x}_{min}, c, \tilde{p}^*, q^*) = R_1(\bar{\tilde{x}}_1, c, \tilde{p}^*, q^*) = R_1(\bar{\tilde{x}}_2, c, \tilde{p}^*, q^*) \text{ if } c = 1 \text{ and } \tilde{\omega}_{min} > 0$$

*and*

(5.21) $$R_1(\tilde{x}_0, c, \tilde{p}^*, q^*) = R_1(\bar{\tilde{x}}_1, c, \tilde{p}^*, q^*) = R_1(\bar{\tilde{x}}_2, c, \tilde{p}^*, q^*) \text{ if } c > 1 \text{ and } \tilde{\omega}_{min} = 0,$$

*where* $R_1(\tilde{x}, c, \tilde{p}, q)$ *is given in* (5.18) *and* $\bar{\tilde{x}}_1$, $\bar{\tilde{x}}_2$ *are given by the positive roots of the polynomial* $P(\tilde{x})$ *giving the maxima of* $R_1$, *then for the case* $c = 1$ *and* $\tilde{\omega}_{min} > 0$, $R_1(\tilde{x}, c, \tilde{p}^*, q^*) \leq R_1(\tilde{x}_{min}, c, \tilde{p}^*, q^*) =: \bar{R}_{O1}$ *for all* $\tilde{x} \in [\tilde{x}_{min}, 2c^2)$, *and for the case* $c > 1$ *and* $\tilde{\omega}_{min} = 0$, $R_1(\tilde{x}, c, \tilde{p}^*, q^*) \leq R_1(\tilde{x}_0, c, \tilde{p}^*, q^*) =: \bar{\bar{R}}_{O1}$ *for all* $\tilde{x} \in [\tilde{x}_0, 2c^2)$. *For* $c = 1$, $\tilde{\omega}_{min} = \epsilon_\omega > 0$, $\epsilon_\omega$ *small, we have the asymptotic result*

(5.22) $$\tilde{p}^* = 2^{\frac{2}{5}}(\tilde{\omega}_{min})^{\frac{4}{10}}, \qquad q^* = 2^{\frac{4}{5}}(\tilde{\omega}_{min})^{\frac{-2}{10}}, \qquad \bar{R}_{O1} \approx 1 - 4(2^{\frac{1}{10}})(\tilde{\omega}_{min})^{\frac{1}{10}},$$

*and for* $\tilde{\omega}_{min} = 0$, $c = \sqrt{1 + \epsilon_c}$, $\epsilon_c$ *small, we have the asymptotic result*

(5.23)
$$\tilde{p}^* = 2(2^{\frac{1}{5}})(c^2 - 1)^{\frac{4}{10}}, \qquad q^* = 2^{\frac{2}{5}}(c^2 - 1)^{\frac{-2}{10}}, \qquad \bar{\bar{R}}_{O1} \approx 1 - 4(2^{\frac{3}{10}})(c^2 - 1)^{\frac{1}{10}}.$$

*Proof.* A partial derivative of $R_1$ with respect to $\tilde{x}$ shows that the roots of $P(\tilde{x})$ determine the extrema of $R_1$. Since $P(\tilde{x})$ is a bi-quartic in $\tilde{x}$ with real coefficients, it has at most four real positive roots, and hence, for $\tilde{x}_{min} \leq \tilde{x} < 2c^2$ with $c = 1$, $\tilde{\omega}_{min} > 0$, and for $\tilde{x}_0 \leq \tilde{x} < 2c^2$ with $c > 1$, $\tilde{\omega}_{min} = 0$, $R_1$ can have at most two interior maxima. Since $R_1$ goes to zero as $\tilde{x}$ goes to $2c^2$, which is the limit as $\omega \longrightarrow \infty$, the maximum in the min-max problem (5.19) can be attained either on the boundary, for the case $c = 1$ and $\tilde{\omega}_{min} > 0$ at $\tilde{x} = \tilde{x}_{min}$ and for the case $c > 1$ and $\tilde{\omega}_{min} = 0$ at $\tilde{x} = \tilde{x}_0$, or at either of the two local maxima, which we denote by $\bar{\tilde{x}}_1$ and $\bar{\tilde{x}}_2$. Balancing the value of $R_1$ at all three locations as stated in (5.20) for the first case and in (5.21) for the second one guarantees then that $R_1$ is uniformly bounded by $R_1$ at $\tilde{x}_{min}$ for the case $c = 1$ and $\tilde{\omega}_{min} > 0$ and at $\tilde{x}_0$ for the case $c > 1$ and $\tilde{\omega}_{min} = 0$. To see that there is indeed such a solution for (5.20) where we have $c = 1$, for $\tilde{\omega}_{min} = \epsilon_\omega$ small, we use the ansatz $\tilde{p} = C_p \epsilon_\omega^{\gamma_1}$, $q = C_q \epsilon_\omega^{\gamma_2}$, $\bar{\tilde{x}}_1 = C_1 \epsilon_\omega^{\delta_1}$, and $\bar{\tilde{x}}_2 = C_2 \epsilon_\omega^{\delta_2}$ and determine the leading asymptotic terms as $\epsilon_\omega$ goes to zero of the two roots of the polynomial $P(\tilde{x})$, which leads to

$$P(\bar{\tilde{x}}_1) = 512 C_1^2 C_p \epsilon_\omega^{\gamma_1 + 2\delta_1} - 256 C_1^4 C_q \epsilon_\omega^{\gamma_2 + 4\delta_1} + \cdots,$$
$$P(\bar{\tilde{x}}_2) = 32 C_2^6 C_q^3 \epsilon_\omega^{3\gamma_2 + 6\delta_2} - 4 C_2^8 C_q^4 \epsilon_\omega^{4\gamma_2 + 8\delta_2} + \cdots.$$

Similarly, expanding (5.20) for $\epsilon_\omega$ small, we find the leading terms

$$\begin{aligned} 1 - \tfrac{4\sqrt{2}}{C_p} \epsilon_\omega^{\frac{1}{2} - \gamma_1} + \cdots \\ = 1 - \tfrac{2C_p}{C_1} \epsilon_\omega^{\gamma_1 - \delta_1} - C_1 C_q \epsilon_\omega^{\gamma_2 + \delta_1} + \cdots \\ = 1 - \tfrac{8}{C_q C_2} \epsilon_\omega^{-(\gamma_2 + \delta_2)} - C_2 \epsilon_\omega^{\delta_2} + \cdots. \end{aligned}$$

Equating the exponents in these four equations leads to $\gamma_1 = \frac{4}{10}$, $\gamma_2 = -\frac{2}{10}$, $\delta_1 = \frac{3}{10}$, and $\delta_2 = \frac{1}{10}$, and since the constants need to match as well, we obtain

$$C_p = 2^{\frac{2}{5}}, \qquad C_q = 2^{\frac{4}{5}}, \qquad C_1 = 2^{\frac{3}{10}}, \qquad C_2 = 2(2^{\frac{1}{10}}).$$

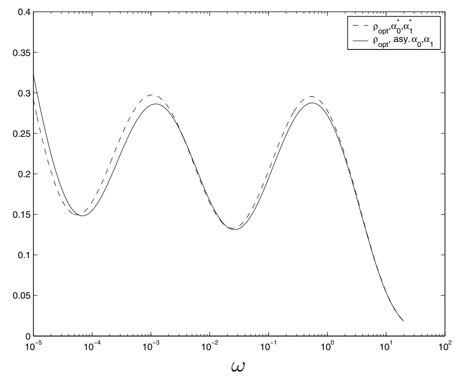


FIG. 6. *Convergence factor of the optimized WR algorithm with the optimized first order approximation* $|\rho_{optL1}(\omega,\alpha_0^*,\alpha_1^*)|$ *versus the one with the asymptotically optimized values.*

Now using these results in $R_1$ and expanding $\bar{R}_{O1} = R_1(\tilde{x}_{min}, c, \tilde{p}^*, q^*)$ for $\tilde{\omega}_{min} = \epsilon_\omega$ small, we find the asymptotic results (5.22).

Similarly for the second case, to see that there is a solution for (5.21) where we now have $\tilde{\omega}_{min} = 0$, for $c = \sqrt{1+\epsilon_c}$, $\epsilon_c$ small, we use the ansatz $\tilde{p} = C_p\epsilon_c^{\gamma_1}$, $q = C_q\epsilon_c^{\gamma_2}$, $\bar{\tilde{x}}_1 = C_1\epsilon_c^{\delta_1}$, and $\bar{\tilde{x}}_2 = C_2\epsilon_c^{\delta_2}$ and determine the leading asymptotic terms as $\epsilon_c$ goes to zero of the two roots of the polynomial $P(\tilde{x})$, which leads to

$$\begin{aligned}P(\bar{\tilde{x}}_1) &= 512C_1^2C_p\epsilon_c^{\gamma_1+2\delta_1} - 256C_1^4C_q\epsilon_c^{\gamma_2+4\delta_1} + \cdots,\\ P(\bar{\tilde{x}}_2) &= 32C_2^6C_q^3\epsilon_c^{3\gamma_2+6\delta_2} - 4C_2^8C_q^4\epsilon_c^{4\gamma_2+8\delta_2} + \cdots.\end{aligned}$$

The leading terms we find by expanding (5.21) for $\epsilon_c$ small are

$$\begin{aligned}&1 - \tfrac{8\sqrt{2}}{C_p}\epsilon_c^{\frac{1}{2}-\gamma_1} + \cdots\\ &\quad = 1 - \tfrac{2C_p}{C_1}\epsilon_c^{\gamma_1-\delta_1} - C_1C_q\epsilon_c^{\gamma_2+\delta_1} + \cdots\\ &\quad = 1 - \tfrac{8}{C_qC_2}\epsilon_c^{-(\gamma_2+\delta_2)} - C_2\epsilon_c^{\delta_2} + \cdots.\end{aligned}$$

Equating the exponents in these four equations leads to $\gamma_1 = \frac{4}{10}$, $\gamma_2 = -\frac{2}{10}$, $\delta_1 = \frac{3}{10}$, and $\delta_2 = \frac{1}{10}$, and since the constants need to match as well, we obtain

$$C_p = 2(2^{\frac{1}{5}}), \qquad C_q = 2^{\frac{2}{5}}, \qquad C_1 = 2^{\frac{9}{10}}, \qquad C_2 = 2(2^{\frac{3}{10}}).$$

Now using these results in $R_1$ and expanding $\bar{\bar{R}}_{O1} = R_1(\tilde{x}_0, c, \tilde{p}^*, q^*)$ for $\epsilon_c$ small, we find the asymptotic results (5.23). Note that the expansions we obtain for the two cases have the same exponents; they are different only by a constant.

Since $\alpha_0 = c^2 - 1 + \frac{\tilde{p}}{2}$, and $\alpha_1 = \frac{q}{2a}$, we have

$$\alpha_0^* = c^2 - 1 + \frac{\tilde{p}^*}{2}, \qquad \alpha_1^* = \frac{q^*}{2a}. \qquad \square$$

**6. Numerical experiments.** We first present a set of numerical experiments illustrating the asymptotic results obtained in the previous section. We show in Figure 6 the result of the optimization with respect to $\alpha_0$ and $\alpha_1$ with the typical values $a = \frac{200}{63}$ and $b = -2a$ of an RC circuit, and we use $\omega_{min} = 0.00001$. The solution of the min-max problem occurs when the convergence factor at $\omega = \omega_{min}$,

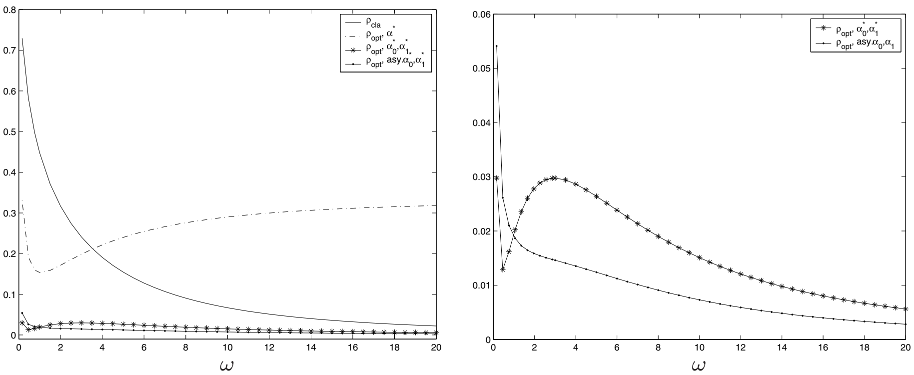


FIG. 7. *On the left: classical convergence factor* $|\rho_{claL}(\omega)|$ *versus convergence factor of the optimized WR algorithm with the optimized constant approximation* $|\rho_{optL0}(\omega, \alpha^*)|$, *and the one with the first order approximation* $|\rho_{optL1}(\omega, \alpha_0^*, \alpha_1^*)|$ *using the numerically and asymptotically optimized values. On the right:* $|\rho_{optL1}(\omega, \alpha_0^*, \alpha_1^*)|$ *using the numerically optimized values versus the one using the asymptotically optimized values.*

TABLE 1
*Comparison of the optimized* $\alpha_0^*, \alpha_1^*$ *from Theorem* 5.5 *and their asymptotic approximation.*

| $\omega_{min}$ | 0.01 | 0.001 | 0.0001 | 0.00001 | 0.000001 |
|---|---|---|---|---|---|
| opt.$\alpha_0^*, \alpha_1^*$ | 0.049,1.095 | 0.021,1.558 | 0.009,2.338 | 0.0038,3.600 | 0.0016,5.6134 |
| asy.$\alpha_0^*, \alpha_1^*$ | 0.066,0.868 | 0.026,1.375 | 0.010,2.180 | 0.0042,3.455 | 0.0017,5.4757 |

| $c^2$ | 1.01 | 1.001 | 1.0001 | 1.00001 | 1.000001 |
|---|---|---|---|---|---|
| opt.$\alpha_0^*, \alpha_1^*$ | 0.209,0.549 | 0.076,0.841 | 0.0294,1.319 | 0.0116,2.083 | 0.0046,3.297 |
| asy.$\alpha_0^*, \alpha_1^*$ | 0.192,0.522 | 0.074,0.827 | 0.0290,1.311 | 0.0115,2.078 | 0.0046,3.294 |

$\omega = \bar{\omega}_1$, and $\omega = \bar{\omega}_2$ is balanced, where $\bar{\omega}_1,\ \bar{\omega}_2 > 0$ are the interior maxima of the modulus of the convergence factor. We observe that the convergence factor with the optimized values $\alpha_0^*$ and $\alpha_1^*$ is close to the one with the asymptotically optimized values.

We now use $\omega_{min} = \frac{\pi}{20}$ to compute the numerically and the asymptotically optimized $\alpha_0^*$ and $\alpha_1^*$ as well as the optimized constant $\alpha^*$, and we show the convergence factors as a function of $\omega$ in Figure 7. On the left of Figure 7, we observe the better convergence factor we get by using the first order approximation over the constant approximation and compared to the classical convergence factor. On the right, we show the convergence factor of the optimized WR algorithm with the first order approximation using the numerically optimized $\alpha_0^*$ and $\alpha_1^*$ and using the asymptotically optimized values. Note that the minimal frequency we choose, $\omega_{min} = \frac{\pi}{20}$, is not small enough to be smaller than the two maxima which we assure their existence for small $\omega_{min}$, and thus we have only one maximum for $\rho_{optL1}$ using the numerically optimized values, which is bigger than $\omega_{min}$, and the other one is smaller, and for the one with the asymptotically optimized values there are no interior maxima in this case. Note also that $\rho_{optL1}$ with the asymptotically optimized values is better than $\rho_{optL1}$ with the numerically optimized values for high frequencies. Table 1 gives a comparison of the optimized $\alpha_0^*$ and $\alpha_1^*$ from (5.20) with the asymptotic approximation (5.22) using the circuit parameters in subsection 6 and gives a comparison of the optimized $\alpha_0^*$ and $\alpha_1^*$ from (5.21) with the asymptotic approximation (5.23). One can see from the first

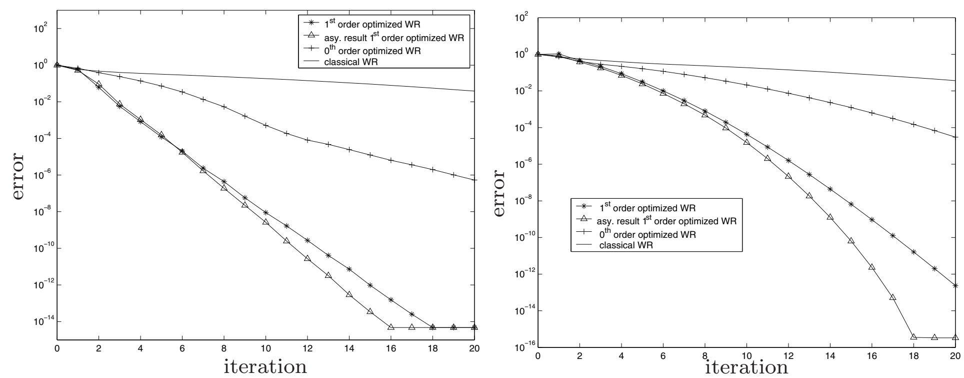


Fig. 8. *The case $|b| = 2a$. Convergence behavior of classical versus optimized WR algorithms for a large RC circuit.*

part that the asymptotic result for $\alpha_0^*$ and $\alpha_1^*$ is close to the optimized $\alpha_0^*$ and $\alpha_1^*$ for small $\omega_{min}$, and from the second, one can see that the values are close for $c$ close to one. Furthermore, for larger values of $\omega_{min}$ and $c$, the asymptotic approximation can be used as a good initial guess for the nonlinear equation solver to find the optimized $\alpha_0^*$ and $\alpha_1^*$ from (5.20) and (5.21), respectively.

We next solve a model RC circuit with 100 nodes with the same typical parameters we used earlier, i.e., $R_s = R_i = R_l = \frac{1}{2}$ Ohm, $i = 1, \dots, 99$, $C_i = \frac{63}{100}$ pF, $i = 1, \dots, 100$. We use the backward Euler method, and our transient analysis time is $t \in [0, 20]$ with a time step of $\Delta t = 1/20$. We start with random initial waveforms and use an input step function with an amplitude of $I_s = 1$ and a rise time of 1 time unit. We consider $\omega_{min} = \frac{\pi}{20}$, and we use the optimized value $\alpha^* = 0.7346$ in the optimized WR algorithm with constant transmission conditions and the numerically optimized values $\alpha_0^* = 0.1756$, $\alpha_1^* = 0.6556$, as well as the asymptotic values $\alpha_0^* = 0.1982$, $\alpha_1^* = 0.5003$ in the optimized WR algorithm with first order transmission conditions. On the left-hand side of Figure 8 we show the error (difference between the numerical solution of the entire circuit and the WR iterates in the maximum norm) as a function of the WR iterations. One can see the remarkable improvement of the optimized WR algorithm over the classical one. Furthermore, the optimized WR algorithm with first order transmission conditions converges faster than the one with constant transmission conditions. Note that the optimized WR algorithm with the asymptotic values $\alpha_0^*$ and $\alpha_1^*$ from Theorem 5.5 converges even a bit faster than the one with the numerically optimized values $\alpha_0^*$ and $\alpha_1^*$ for $\omega_{min} = \frac{\pi}{20}$. This is, however, just a coincidence: as we have seen on the right in Figure 7, the convergence factor with the asymptotic values does not yet equioscillate and can therefore theoretically not yet be optimal; it is, however, better than the one with the numerically optimized values for high frequencies. Since the estimate for $\omega_{min}$ is conservative, and the numerically optimized version needs to equioscillate, the error modes around $\omega = 4$ slow down the numerically optimized version a little in this particular experiment.

On the right-hand side of Figure 8 we also plot the error as a function of the WR iterations but now with different transient analysis time, where we choose $t \in [0, 2]$. We consider now the low frequency $\omega_{min} = \frac{\pi}{2}$ and the backward Euler method with the same time step and same circuit elements as in the previous example. The optimized values that we use are $\alpha^* = 1.5363$ in the optimized WR algorithm with constant transmission conditions and the numerically optimized values $\alpha_0^* = 0.4980$,

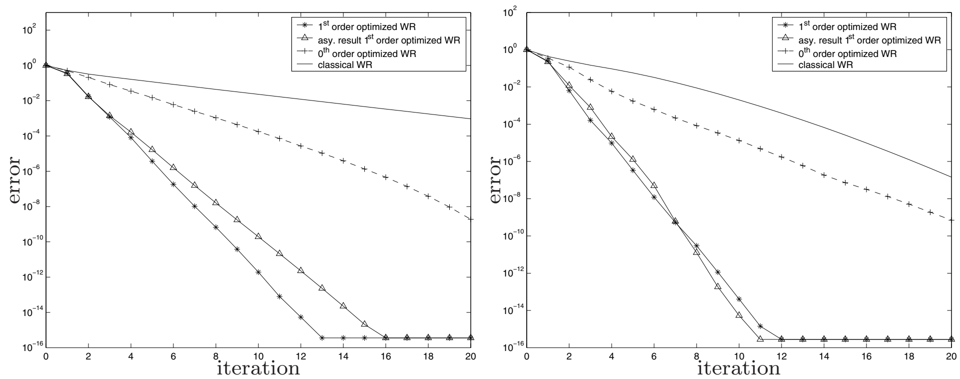


FIG. 9. *The case $|b| > 2a$. Convergence behavior of classical versus optimized WR algorithms for a large RC circuit.*

$\alpha_1^* = 0.4205$, as well as the asymptotic values $\alpha_0^* = 0.4979$, $\alpha_1^* = 0.3157$ in the optimized WR algorithm with first order transmission conditions. We observe similar behavior as for the previous case where the transient analysis time was $[0, 20]$, except that now the algorithm is in its superlinear convergence regime. For a first analysis of optimization parameters in this regime, see [9].

We finally repeat the last experiment with the same typical parameters we used earlier but now without omitting the resistor values $\tilde{R}_i$, where we choose the typical parameters $\tilde{R}_i$ to be $\tilde{R}_i = 5$ Ohm, $i = 1, \ldots, 100$. We plot the error as a function of the WR iterations in Figure 9: on the left for the transient analysis time $t \in [0, 20]$ and on the right for the transient analysis time $t \in [0, 2]$. We consider $\omega_{min} = 0$ but not $\omega_{min} = \frac{\pi}{T}$, since we have here $|b| > 2a$. For $T = 20$ and $T = 2$, we use the optimized value $\alpha^* = 1.3028$ in the optimized WR algorithm with constant transmission conditions and the numerically optimized values $\alpha_0^* = 0.4389$, $\alpha_1^* = 0.4240$, as well as the asymptotic values $\alpha_0^* = 0.3966$, $\alpha_1^* = 0.3783$, using the asymptotic approximation given in (5.23) with $c = 1.0247 > 1$, in the optimized WR algorithm with first order transmission conditions. We observe similar behavior as for the previous experiment, except that the optimized WR algorithm with the numerically optimized first order values has improved from the previous experiment and converges even better than the one with the asymptotic values for $T = 20$, which is expected since we use here $\omega_{min} = 0$ to find the optimized values which allows the numerically optimized version to equioscillate.

**7. Conclusion.** The effectiveness of classical WR algorithms for circuit simulation depends very much on finding a good partitioning: the engineer needs to find subcircuits such that the coupling between them is weak; then the classical WR algorithm with the corresponding partitioning converges quickly. Finding such a partitioning is, however, not always easy: "In practice one is interested in knowing what subdivisions yield fast convergence for the iterations. . . . The splitting into subsystems is assumed to be given. How to split in such a way that the coupling remain "weak" is an important question" [26].

It is evident from the early work on splitting or partitioning of realistic circuits [29, 32] that a great effort was made to improve the convergence using the classical WR. The optimized WR approach is a mathematical technique to weaken the strength of the coupling between subsystems when it is not possible to find a weak link in a given circuit configuration where a partition has to be made. We achieve this by new transmission conditions that exchange a combination of voltages and currents rather than just voltages or just currents from one subsystem to its neighboring subsystems. Mathematically, one can even derive optimal transmission conditions, which involve nonlocal operators in time, but these are less practical. We therefore propose approximations which are optimized for the local nature of the circuit area where the cut needs to be performed. In this paper, we assumed the circuit to be of diffusive nature and thus gave a complete analysis of any finite size RC circuit. If the circuit behaves locally more like a transmission line, one needs to use transmission conditions for this type of circuit; see [14, 12].

**Acknowledgment.** We are very thankful for the careful comments of the reviewers and are particularly in debt to the reviewer who pointed out that in our earlier version of the manuscript, the model circuit put an apparent limitation on our theoretical results not present in the analysis.

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