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2. Optimized Schwarz Methods

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Introduction

Schwarz methods lead to parallel preconditioners for large linear systems of equations arising in the solution process of partial differential equations [SBG96]. Optimal convergence results for the Schwarz method are known in the sense that the condition number of the preconditioned system is independent of (or only weakly dependent on) the mesh parameter and the number of subdomains. Thus asymptotically Schwarz methods have optimal scalability.

This optimality result contains however constants which remain unknown in the analysis. Thus it does not imply that the current Schwarz methods have optimal performance. It does not guarantee either that Schwarz methods are competitive to other parallel methods. Thus the word "optimal" can be misleading.

We analyze the performance of the classical Schwarz method for two model problems, Laplace's equation and the Helmholtz equation. Our analysis is performed at the continuous level which seems natural for the Schwarz method since the method itself is defined at the continuous level. Our investigation reveals that the convergence rate of the Schwarz methods depends intrinsicly on the transmission conditions employed between subdomains. The classical transmission conditions used by Schwarz are Dirichlet transmission conditions [Sch70]. These transmission conditions lead to convergence rates which are not uniform with respect to frequency: high frequency components converge rapidly whereas low frequency components converge only slowly. Motivated by the analysis of Overlapping Schwarz Waveform Relaxation in [GHN99] we construct optimal transmission conditions for the Laplace and Helmholtz equation in two dimensions. These conditions are global in nature and thus not ideal for implementations. We therefore introduce local approximations of the optimal conditions and optimize them for performance, which leads to the optimized Schwarz methods.

Other people have looked at different transmission conditions before. Generalized Schwarz splittings with Robin transmission conditions have been analyzed by Tang [Tan92] and led to an over-determined Schwarz algorithm in [ST96]. The main difficulty remaining in this approach is the determination of the relaxation parameter in the Robin conditions, like for SOR methods. For Helmholtz problems radiation conditions for overlapping Schwarz have been proposed by [CCEW98]. For non-overlapping versions of the Schwarz algorithms Dirichlet transmission conditions are not effective and Lions proposed to use Robin conditions to obtain a convergent algorithm in [Lio90]. Through the work by Charton, Nataf and Rogier [CNR91], Nataf and Rogier [NR95] and Japhet [Jap98] new types of transmission conditions for convection diffu-

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sion problems have been introduced which are optimal in a physical sense and contain the Robin conditions as a first order approximation. A similar approach was developed for the Helmholtz equation in [DJR92] and [CN98]. An overlapping version for Laplace's equation was analyzed in [EZ98]. The same type of analysis was applied to overlapping Schwarz waveform relaxation algorithms in [GHN99] and led to optimized Schwarz algorithms for evolution problems where one can easily visualize that the optimal transmission conditions are absorbing boundary conditions. The key is that a simple optimization procedure leads to local transmission conditions with optimized performance for the Schwarz algorithm. We derive optimized Schwarz methods for elliptic definite and indefinite problems in this note.

Optimized Schwarz Method for Laplace's Equation

We consider Laplace's equation in the domain $\Omega = \mathbb{R}^2$,

$$\Delta u = f(x,y), \quad x,y \in \Omega, \quad u \text{ bounded at infinity.} \tag{1}$$

We decompose the domain Ω into two overlapping half planes $\Omega_1 = (-\infty, L] \times \mathbb{R}$ and $\Omega_2 = [0,\infty) \times \mathbb{R}$ where $L > 0$ is the overlap parameter. The classical Schwarz method to solve (1) solves iteratively Laplace's equation on Ω_1 and Ω_2 and exchanges Dirichlet values on the interfaces at 0 and L,

$$\begin{array}{rcll} \Delta v^{n+1} &=& f(x,y), & x,y \in \Omega_1, \\ v^{n+1}(L,y) &=& w^n(L,y), & \\ \Delta w^{n+1} &=& f(x,y), & x,y \in \Omega_2, \\ w^{n+1}(L,y) &=& v^n(L,y). & \end{array} \tag{2}$$

To analyze the convergence of the classical Schwarz method, it suffices by linearity to consider the homogeneous problem, $f(x,y) = 0$ in (2), and to analyze convergence to zero.

Fourier Analysis of the Classical Schwarz Method

Our results are based on Fourier analysis. We denote the Fourier transform $\hat{f}(k)$ of $f(x) : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\hat{f}(k) = \mathcal{F}_x(f)(k) := \int_{-\infty}^{\infty} e^{-ikx} f(x)dx$$

and the inverse Fourier transform of $\hat{f}(k)$ by

$$f(x) = \mathcal{F}_x^{-1}(\hat{f})(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{f}(k)dk.$$

Taking a Fourier transform in y of (2) for $f(x,y) = 0$ we obtain

$$\begin{array}{rcll} \hat{v}_{xx}^{n+1}(x,k) - k^2\hat{v}^{n+1}(x,k) &=& 0, & x \in (-\infty, L), k \in \mathbb{R}, \\ \hat{v}^{n+1}(L,k) &=& \hat{w}^n(L,k), & \end{array} \tag{3}$$

$$\begin{array}{rcll} \hat{w}_{xx}^{n+1}(x,k) - k^2\hat{w}^{n+1}(x,k) &=& 0, & x \in (0,\infty), k \in \mathbb{R}, \\ \hat{w}^{n+1}(0,k) &=& \hat{v}^n(0,k) & \end{array} \tag{4}$$

where a subscript x denotes a partial derivative with respect to x. Solving the ordinary differential equation (4) using the boundedness condition at infinity and inserting the result into the boundary condition of (3) we find the solution of (3) at $x = 0$ to be

$$\hat{v}^{n+1}(0,k) = e^{-2|k|L}\hat{v}^{n-1}(0,k).$$

Similarly we obtain for the solution of (4) at $x = L$

$$\hat{w}^{n+1}(L,k) = e^{-2|k|L}\hat{w}^{n-1}(L,k).$$

Defining the convergence rate

$$\rho(k,L) := e^{-2|k|L} \tag{5}$$

we see that the classical Schwarz method converges for all $k \neq 0$ if there is overlap, $L > 0$. The convergence rate is linear and depends on the size of the overlap L as well as the frequency k. High frequency components converge fast, whereas low frequency components converge only slowly. Note that for $|k| \to 0$ the convergence rate ρ tends to 1.

Optimal Transmission Conditions

The preceding analysis shows that the Schwarz method is slowed down by the low frequency components. They are dictating the convergence rate and thus the performance of the Schwarz method. For better performance, one would like to improve the convergence rate for the low frequency components. This can be achieved by changing the transmission conditions to become more transparent for low frequency components. Following the approach in [GHN99] for evolution problems, we introduce new transmission conditions into the classical Schwarz method (2). Instead of using Dirichlet transmission conditions, we impose at the artificial boundaries

$$\begin{aligned} v_x^{n+1}(L,y) + \Lambda_v(v^{n+1}(L,y)) &= w_x^n(L,y) + \Lambda_v(w^n(L,y)) \\ w_x^{n+1}(0,y) + \Lambda_w(w^{n+1}(0,y)) &= v_x^n(0,y) + \Lambda_w(v^n(0,y)), \end{aligned} \tag{6}$$

where the linear operators Λ_v and Λ_w are degrees of freedom we can use to optimize the performance of the algorithm. Note that the Schwarz method itself remains the same, only the transmission conditions have been changed. We have the following

Theorem 1 (Optimal Convergence) *Choosing Λ_v to have the symbol $\lambda_v(k) := |k|$ and Λ_w to have the symbol $\lambda_w(k) := -|k|$ the Schwarz method with transmission conditions (6) converges in two iterations independently of the overlap $L \geq 0$.*

Proof Applying a Fourier transform in y to (3), (4) with transmission conditions (6) we obtain

$$\begin{aligned} \hat{v}_{xx}^{n+1}(x,k) - k^2\hat{v}^{n+1}(x,k) &= 0, \quad x \in (-\infty, L), k \in \mathbb{R}, \quad (7)\\ \hat{v}_x^{n+1}(L,k) + \lambda_v(k)\hat{v}^{n+1}(L,k)) &= \hat{w}_x^n(L,k) + \lambda_v(k)\hat{w}^n(L,k), \\ \hat{\hat{w}}_{xx}^{n+1}(x,k) - k^2\hat{\hat{w}}^{n+1}(x,k) &= 0, \quad x \in (0,\infty), k \in \mathbb{R}, \quad (8)\\ \hat{w}_x^{n+1}(0,k) + \lambda_w(k)\hat{w}^{n+1}(0,k) &= \hat{v}_x^n(0,k) + \lambda_w(k)\hat{v}^n(0,k). \end{aligned}$$

Solving (8) at iteration step n for $\hat{w}^n$ and inserting the result into the transmission conditions of (7) we find for $\hat{v}^{n+1}$ at $x = 0$

$$\hat{v}^{n+1}(0, k) = \rho_l \hat{v}^{n-1}(0, k)$$

and by a similar computation for $\hat{w}^{n+1}$ at $x = L$

$$\hat{w}^{n+1}(L, k) = \rho_l \hat{w}^{n-1}(L, k)$$

where the convergence rate ρ_l is given by

$$\rho_l(k, L) := \frac{-|k| + \lambda_v(k)}{|k| + \lambda_v(k)} \cdot \frac{|k| + \lambda_w(k)}{-|k| + \lambda_w(k)} e^{-2|k|L}. \tag{9}$$

Hence choosing $\lambda_v(k) := |k|$ and $\lambda_w(k) := -|k|$ the convergence rate vanishes, $\rho_l \equiv 0$ and thus, independently of the initial guess, after two steps of the Schwarz iteration the iterates are zero on $x = 0$ and $x = L$ respectively. To see that they vanish identically, it suffices to note that by the boundedness condition at infinity, $\hat{v}^2(x, k) = Ae^{|k|x}$ and $\hat{w}^2(x, k) = Be^{-|k|x}$ for some constants A and B. But $\hat{v}^2(0, k) = 0$ then implies $A = 0$ and $\hat{w}^2(L, k) = 0$ implies $B = 0$ and the result follows. ■

Note that the new convergence rate (9) still contains the exponential factor like the classical one (5), but the new transmission conditions (6) introduced an additional factor with the degrees of freedom $\lambda_v(k)$ and $\lambda_w(k)$. Theorem 1 shows what the optimal choice is for the transmission conditions in theory. One can show that with this choice and N subdomains in strips the Schwarz algorithm converges in N steps, see [NRdS94]. This is an optimal result since the solution of Laplace's equation in one subdomain depends on the source term f in every other subdomain and when only a local mechanism of communication is employed one has to communicate at least N steps to get the information from the left most subdomain across all the other subdomains to the rightmost subdomain.

However to use the algorithm in practice, one either needs to work in Fourier space or one has to back-transform the optimal transmission conditions to the real space. The inverse Fourier transform of $\lambda_{vw} = \pm|k|$ leads to the optimal transmission operators Λ_{vw} which are non local in y and thus harder to implement. Note that the optimal transmission operators correspond to the Dirichlet to Neumann map at the artificial interfaces and thus the optimal transmission conditions are the absorbing boundary conditions as in the case of the evolution problems [GHN99].

Optimized Local Transmission Conditions

For a real implementation of the Schwarz algorithm, it is desirable to have local transmission conditions. We therefore approximate the nonlocal optimal transmission conditions found in the previous subsection by local ones. Local operators are represented by polynomials in Fourier space and we analyze in the sequel the performance of the zeroth and second order approximation of the optimal transmission conditions,

$$\lambda_{vw} = \pm p \quad \text{or} \quad \lambda_{vw} = \pm(p + qk^2). \tag{10}$$

The parameters $p, q > 0$ are free parameters and they can be used to optimize the performance of the new Schwarz method which leads to the optimized Schwarz method.

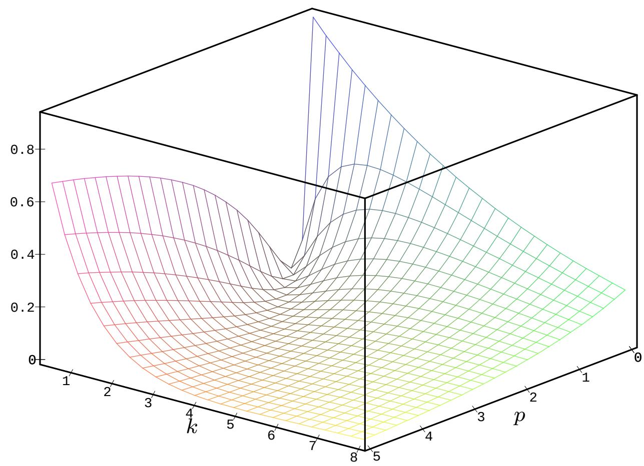

Figure 1: Dependence of the convergence rate on the frequency k and the optimization parameter p for Laplace's equation.

Since real computations are performed on bounded domains and discretized operators, the range of the frequency parameter k is not arbitrary. It is bounded from below by a lowest frequency dependent on the size of the domain in y direction and the boundary conditions imposed, $k^2 > k_{\min}^2$ and from above, k is bounded by the mesh size h in y direction, $k^2 < k_{max}^2 := (\pi/h)^2$. Thus to obtain optimal performance of the Schwarz method, we have to solve the min-max problem

$$\min_{p>0}\left(\max_{k_{\min}<k<k_{\max}} \frac{(|k|-p)^2}{(|k|+p)^2} e^{-2|k|L}\right)$$

in the case of the zeroth order approximation. Figure 1 shows the dependence of the convergence rate on the frequency k and the free parameter p. Note that the convergence rate is symmetric in k and only the part for $k_{\min} < k < k_{\max}$ is shown in the figure. One can clearly identify that for a certain parameter value p the convergence rate will become small for all values of k, $k_{\min} < k < k_{\max}$. For large p however the low frequencies will dominate again the convergence rate and in the limit as p goes to infinity, we recover the classical Schwarz method.

For the second order approximation of the optimal transmission conditions, we find the min-max problem

$$\min_{p,q>0}\left(\max_{k_{\min}<k<k_{\max}} \frac{(|k|-p-qk^2)^2}{(|k|+p+qk^2)^2} e^{-2|k|L}\right).$$

Both min-max problems can be solved analytically and we show in Figure 2 the convergence rates obtained for the classical Schwarz method and the two optimized Schwarz

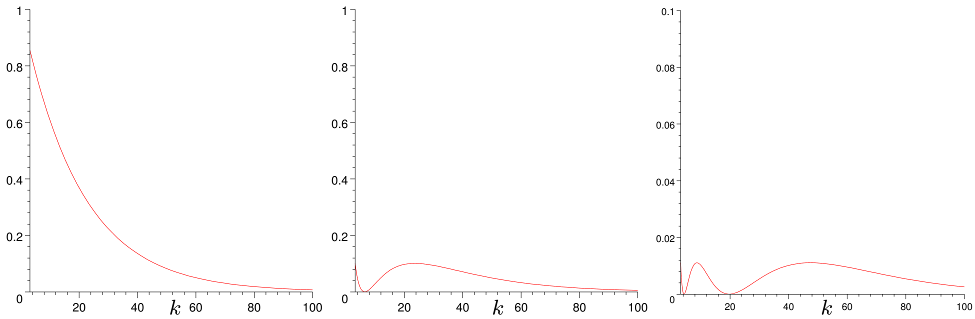

Figure 2: Convergence rates in Fourier space for Laplace's equation. The classical Schwarz method on the left, zeroth order optimized Schwarz method in the middle and second order optimized Schwarz method on the right. Note the *scaling factor of 10* in the right most figure.

methods for the model problem (11) with mesh parameter $h = 1/80$. Note how the zeroth order approximation, which leads to a Robin condition instead of a Dirichlet one in the Schwarz algorithm, reduces the convergence rate already from 0.82 to 0.05 and the second order approximation reduces it further to 0.006. The numerical experiments in the following subsections confirm the enormous improvement of the optimized Schwarz algorithm over the classical one.

Numerical Experiments for Laplace's Equation

We solve Laplace's equation on the rectangular domain $\Omega = [0, 2] \times [0, 1]$,

$$\Delta u = 0, \quad x, y \in \Omega \tag{11}$$

with given Dirichlet boundary conditions. We decompose Ω into two subdomains $\Omega_1 = [0, 1 + \delta] \times [0, 1]$ and $\Omega_2 = [1 - \delta, 2] \times [0, 1]$ and apply the Schwarz algorithm as an iterative solver. Figure 3 shows the performance of the classical Schwarz method compared to the zeroth order optimized one and the second order optimized one for an overlap of $2\delta = 1/40$. Clearly the optimized Schwarz method perform much better than the classical one. The convergence rate improvement due to the new transmission conditions manifests itself in the numerical experiments. While the classical Schwarz method only reduces the error by a few percent in 8 iterations, the zeroth order optimized Schwarz method reduces the error by a factor of 10^5 and the second order optimized Schwarz method reduces the error by a factor of 10^{13}. Note that these contraction rates are comparable to multi-grid, and we have not used a Krylov method yet, just classical Schwarz as an iterative solver.

To accelerate convergence, one usually uses the Schwarz method as a preconditioner, which greatly improves the performance of the classical Schwarz method. Figure 4 shows the decay of the error in the same experiment as above, but now the Schwarz methods are used as preconditioners. Clearly the classical Schwarz method is improved a great deal by the Krylov method, but the optimized Schwarz methods are accelerated as well and still converge much faster than the classical Schwarz method.

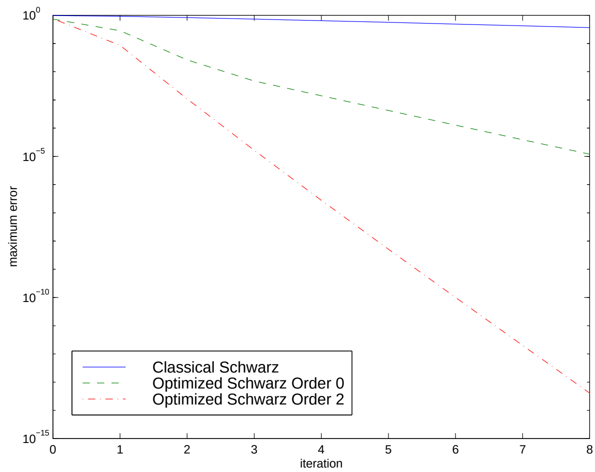

Figure 3: The performance of the optimized Schwarz methods for Laplace's equation compared to the classical Schwarz method as an iterative solver.

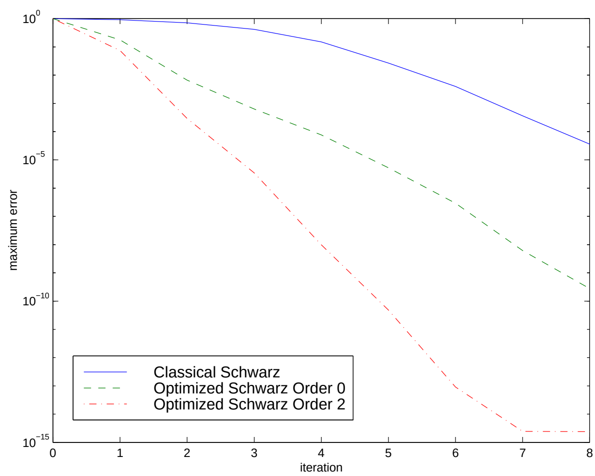

Figure 4: Optimized Schwarz methods used as preconditioners for Laplace's equation.

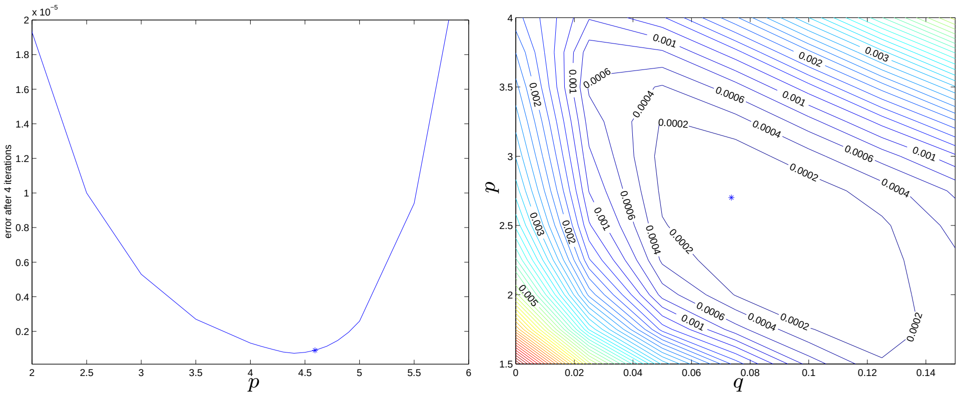

Figure 5: Comparison of the optimal parameters found by Fourier analysis and the best parameters in numerical experiments for Laplace's equation.

Note again that with the second order optimized Schwarz method, we observe a similar phenomenon like with multi grid: the acceleration with the Krylov method is not really necessary, it only brings a small improvement, since the basic iterative solver is already converging at an extremely fast rate.

Finally we investigate how close the optimal parameters obtained by Fourier analysis are to the really optimal parameters we obtained from numerical experiments. Note that the optimal discrete parameters could also be obtained for regular rectangular meshes by a discrete Fourier analysis, but such an analysis would have to be redone for every mesh, whereas our continuous analysis is valid independently of the mesh. It is more important to have results at the continuous level for a method defined at the continuous level, since then these results remain relevant once the problem is solved on a mesh which resolves the continuous properties, independently of the particular mesh. Figure 5 shows on the left the error reduction obtained after 4 iterations of the zeroth order optimized Schwarz method for various parameters p and also indicated by a star the optimal parameter obtained by Fourier analysis. Clearly the Fourier analysis indicates where the discrete optimum lies. On the right we show a level set plot of the error after four iterations for the second order optimized Schwarz method. Again the star indicates the optimum found by the Fourier analysis. This shows that Fourier analysis is a viable tool to compute optimized Schwarz methods and the figures also show that optimized Schwarz methods are rather robust with respect to the optimization parameters.

Optimized Schwarz Method for the Helmholtz Equation

We consider the Helmholtz equation in the domain $\Omega = \mathbb{R}^2$,

$$(\Delta + \omega^2)(u) = f(x, y), \quad x, y \in \Omega \tag{12}$$

with Sommerfeld radiation conditions at infinity. We decompose Ω into two overlapping half planes $\Omega_1 = (-\infty, L] \times \mathbb{R}$ and $\Omega_2 = [0, \infty) \times \mathbb{R}$ where $L > 0$ is the overlap parameter. The classical Schwarz method to for (12) is given by

$$\begin{array}{rcll} (\Delta + \omega^2)(v^{n+1}) & = & f(x,y) & x, y \in \Omega_1, \\ v^{n+1}(L,y) & = & w^n(L,y), & \\ (\Delta + \omega^2)(w^{n+1}) & = & f(x,y) & x, y \in \Omega_2, \\ w^{n+1}(L,y) & = & v^n(L,y). & \end{array} \tag{13}$$

To analyze if the classical Schwarz method converges for the Helmholtz equation, it suffices by linearity to consider again the homogeneous problem, $f(x,y) = 0$ in (13) and to analyze convergence to zero.

Fourier Analysis of the Classical Schwarz Method

Taking a Fourier transform in y of (13) for $f(x,y) = 0$ we obtain

$$\begin{array}{rcll} \hat{v}_{xx}^{n+1}(x,k) + (\omega^2 - k^2)\hat{v}^{n+1}(x,k) & = & 0, & x \in (-\infty, L), k \in \mathbb{R}, \\ \hat{v}^{n+1}(L,k) & = & \hat{w}^n(L,k), & \end{array} \tag{14}$$

$$\begin{array}{rcll} \hat{w}_{xx}^{n+1}(x,k) + (\omega^2 - k^2)\hat{w}^{n+1}(x,k) & = & 0, & x \in (0, \infty), k \in \mathbb{R}, \\ \hat{w}^{n+1}(0,k) & = & \hat{v}^n(0,k). & \end{array} \tag{15}$$

Solving the ordinary differential equation (15) using the radiation condition at infinity and inserting the result into the boundary condition of (14) we find the solution of (14) at $x = 0$ to be

$$\hat{v}^{n+1}(0,k) = e^{-2\sqrt{k^2-\omega^2}L}\hat{v}^{n-1}(0,k)$$

and similarly for (15)

$$\hat{w}^{n+1}(L,k) = e^{-2\sqrt{k^2-\omega^2}L}\hat{w}^{n-1}(L,k).$$

Defining the convergence rate

$$\rho(k, \omega, L) := e^{-2\sqrt{k^2-\omega^2}L} \tag{16}$$

we have now two cases to distinguish: if $k^2 > \omega^2$ then $|\rho(k,\omega,L)| < 1$ and the algorithm converges as in the case of Laplace's equation. If however $k^2 < \omega^2$ then

$$|\rho(k,\omega,L)| = \left| e^{-2i\sqrt{\omega^2-k^2}L} \right| = 1$$

and convergence is lost. Therefore the classical Schwarz algorithm for the Helmholtz equation does not converge in general, the low frequencies in the error are not damped. Often it is precisely the low frequencies which are important in Helmholtz problems, since they correspond to the propagating frequencies. Thus for Helmholtz problems one is obliged to modify the Schwarz algorithm to make it work. In [CW92] a coarse mesh is introduced, fine enough to carry all the propagating modes, and in [CCEW98] the classical radiation conditions of Robin type are employed at the interfaces to obtain damping of the propagating modes. In [DJR92] and [CN98] non-overlapping variants of the Schwarz algorithm are analyzed with approximately absorbing transmission conditions. Following our analysis for Laplace's equation, we first compute the optimal transmission conditions for the Helmholtz case.

Optimal Transmission Conditions

Imposing the new transmission conditions (6) in the Schwarz algorithm for the Helmholtz equation we obtain the analog to Theorem 1 in the case of Laplace's equation:

Theorem 2 (Optimal Convergence) *Choosing Λ_v to have the symbol $\lambda_v(k) := \sqrt{k^2-\omega^2}$ and Λ_w to have the symbol $\lambda_w(k) := -\sqrt{k^2-\omega^2}$ the Schwarz method with transmission conditions (6) for the Helmholtz equation converges in two iterations independently of the overlap $L \geq 0$ and the frequency parameter k.*

Proof A Fourier transform in y and a similar calculation as in the case of Laplace's equation leads to

$$\hat{v}^{n+1}(0,k) = \rho_h \hat{v}^{n-1}(0,k)$$

and similarly for $\hat{w}^{n+1}$

$$\hat{w}^{n+1}(L,k) = \rho_h \hat{w}^{n-1}(L,k)$$

where the convergence rate ρ_h is given by

$$\rho_h(k,L) := \frac{-\sqrt{k^2-\omega^2}+\lambda_v(k)}{\sqrt{k^2-\omega^2}+\lambda_v(k)} \cdot \frac{\sqrt{k^2-\omega^2}+\lambda_w(k)}{-\sqrt{k^2-\omega^2}+\lambda_w(k)} e^{-2\sqrt{k^2-\omega^2}L}. \tag{17}$$

Hence for $\lambda_v = \sqrt{k^2-\omega^2}$ and $\lambda_w = -\sqrt{k^2-\omega^2}$ the convergence rate (17) vanishes, $\rho_h \equiv 0$ and thus, independently of the initial guess, after two steps of the Schwarz iteration the iterates are zero. ∎ Again the optimal transmission conditions involve the Dirichlet to Neumann map, as in the case of Laplace's equation, and to avoid a nonlocal implementation, we propose local approximations of the optimal transmission conditions.

Optimized Local Transmission Conditions

Using a zeroth and second order approximation as given in (10), we are led to the optimization problems

$$\min_{p>0}\left(\max_{k_{\min}<k<k_{\max}}\left|\frac{(\sqrt{k^2-\omega^2}-p)^2}{(\sqrt{k^2-\omega^2}+p)^2}e^{-2\sqrt{k^2-\omega^2}L}\right|\right) \tag{18}$$

in the zeroth order approximation case and to

$$\min_{p,q>0}\left(\max_{k_{\min}<k<k_{\max}}\left|\frac{(\sqrt{k^2-\omega^2}-p-qk^2)^2}{(\sqrt{k^2-\omega^2}+p+qk^2)^2}e^{-2\sqrt{k^2-\omega^2}L}\right|\right) \tag{19}$$

in the second order approximation case. But these optimization problems have an intrinsic difficulty in the Helmholtz case: for $k^2 = \omega^2$ we obtain 1, independently of the choice of the parameter p in (18) and the parameters p and q in (19). Thus there is no hope to minimize the convergence rate uniformly in k and even the optimized Schwarz method might not converge when applied in an iterative way to the Helmholtz problem. When used as a preconditioner however, the Krylov method can easily cope with outliers in the spectrum and thus we optimize the convergence rates for all k relevant to the discrete spectrum except $k = \omega$. This leads to the convergence rates shown in Figure 6 for the model problem (20).

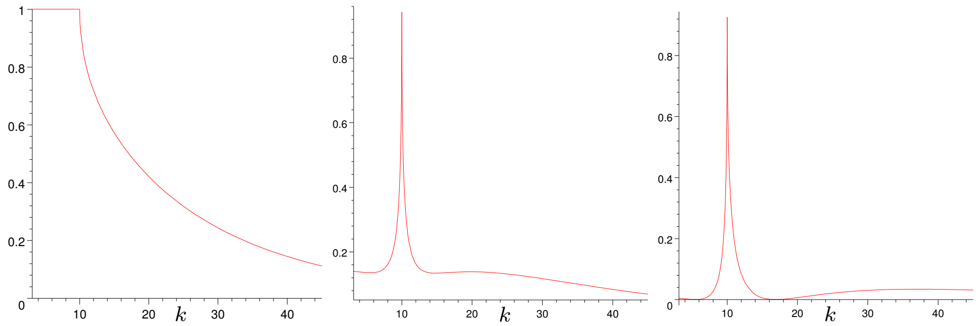

Figure 6: Convergence rates in Fourier space for a Helmholtz problem. The classical Schwarz method on the left, zeroth order optimized Schwarz method in the middle and second order optimized Schwarz method on the right.

Numerical Experiments for the Helmholtz Equation

We solve the Helmholtz equation on a rectangular domain $\Omega = [0,2] \times [0,1]$

$$(\Delta + \omega^2)(u) = 0, \quad x, y \in \Omega, \tag{20}$$

Robin conditions on the left and the right and homogeneous Dirichlet conditions on top and bottom. We decompose Ω into two subdomains $\Omega_1 = [0, 1+\delta] \times [0,1]$ and $\Omega_2 = [1-\delta, 2] \times [0,1]$ and apply the Schwarz algorithm as preconditioner for GMRES. Figure 7 shows the performance of the classical Schwarz method compared to the zeroth order optimized one and the second order optimized one for an overlap of $2\delta = 1/10$ with mesh parameter $h = 1/80$ and $\omega = 10$. Clearly the optimized Schwarz method shows a much better performance than the classical one.

Conclusions

We have introduced a small modification to the classical Schwarz method with a big impact. Exchanging the classical transmission conditions of Dirichlet type with transmission conditions involving local approximations of the Dirichlet to Neumann operator, the Schwarz algorithm converges orders of magnitudes faster, both when used as an iterative solver and as a preconditioner for symmetric definite and indefinite model problems.

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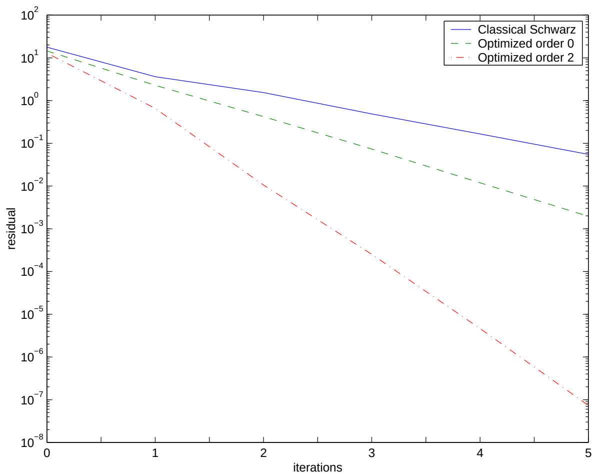

Figure 7: Performance of the classical Schwarz preconditioner compared to the optimized Schwarz preconditioners for a Helmholtz problem.

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