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# 1. Optimized Schwarz Algorithms for Coupling Convection and Convection-Diffusion Problems

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**Introduction** When solving the compressible Navier-Stokes equations in an exterior domain, it is of interest in the computation to select regions where the viscosity is small and to solve the Euler equations instead in these regions, since the Euler equations are less costly computationally. In recent years, fundamental work has been done to study the range of applicability of this approach. Error estimates have been developed for small viscosity, coupled problems have been formulated and more recently iterative algorithms have been developed to solve the coupled problems (see [2], [3]).

For problems in fluid mechanics new domain decomposition methods with optimized transmission conditions based on artificial boundary conditions [4] have been introduced [1, 6]. In particular, it was proposed for the convection-diffusion equation to use transmission conditions such that the rate of convergence can be optimized [5]. These transmission conditions lead to very fast convergence, and the convergence rate is nearly independent of both the physical and the discretization parameters.

Here we extend these transmission conditions to the case of the coupled convection and convection-diffusion problem. We consider the convection-diffusion equation

$$\begin{array}{rcll} \mathcal{L}_{cd}(u) \equiv -\nu\Delta u + \mathrm{div}(\mathbf{a}u) + cu &=& f & \text{in } \Omega, \\ \mathcal{C}(u) &=& g & \text{on } \partial\Omega, \end{array} \tag{0.1}$$

where $\Omega$ is a bounded open set of $\mathbb{R}^2$, and $\mathcal{C}$ is a linear operator such as the identity or the normal derivative. Here $\nu > 0$ is the viscosity, $c > 0$ is a constant and $\mathbf{a} = (a, b) \in (L^\infty(\Omega))^2$ is the velocity field with $\mathrm{div}\mathbf{a} \in L^\infty(\Omega)$ and $\mathrm{div}\mathbf{a} + c \geq \delta > 0$. This ensures that the problem is well-posed, because it can be associated with a continuous and coercive bilinear form.

We suppose that the diffusion process is only physically relevant in a subregion $\Omega_-$ of $\Omega$. Let $\overline{\Omega} = \overline{\Omega}_- \cup \overline{\Omega}_+$ with $\Omega_- \cap \Omega_+ = \emptyset$. We denote by $\Gamma$ the common interface between $\Omega_-$ and $\Omega_+$ and by $\mathbf{n}$ the unit outward normal for $\Omega_-$. To solve the original problem (0.1), we want to use the fact that the diffusion is only relevant in $\Omega_-$. We therefore couple the convection-diffusion equation

$$\mathcal{L}_{cd}(v) \equiv -\nu\Delta v + \mathrm{div}(\mathbf{a}v) + cv = f \quad \text{in } \Omega_-$$

with the convection equation

$$\mathcal{L}_{c}(w) \equiv \mathrm{div}(\mathbf{a}w) + cw = f \quad \text{in } \Omega_+$$

with $\mathcal{C}$ imposed on $\partial\Omega_- \cap \partial\Omega$ and $\partial\Omega_+ \cap \partial\Omega$ and with suitable transmission conditions on $\Gamma$.

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We first present the optimized Schwarz algorithm for $\Omega = \mathbb{R}^2$ to show the link between transmission conditions and artificial boundary conditions. We consider both inflow into the purely convective region, $\mathbf{a} \cdot \mathbf{n} > \mathbf{0}$, and outflow of the purely convective region, $\mathbf{a} \cdot \mathbf{n} < \mathbf{0}$. Then we present the optimized Schwarz algorithm for an arbitrary velocity field. We recall error estimates for small viscosity and compare in numerical experiments the new optimized Schwarz method for coupled problems to an earlier coupling algorithm in [3].

**Inflow into the Purely Convective Region** Let $\Omega = \mathbb{R}^2$, $\Omega_- = \mathbb{R}^- \times \mathbb{R}$, $\Omega_+ = \mathbb{R}^+ \times \mathbb{R}$ and $\Gamma = \{(x,y),\ y \in \mathbb{R},\ x = 0\}$. In the case of inflow into the purely convective region, the coupling on $\Gamma$ needs both a condition on $v$ and a condition on $w$. Let $\Lambda_+$ be the Dirichlet to Neumann operator of the left half plane defined by

$$\Lambda_+(g) = \frac{\partial u}{\partial x} \quad \text{where } u \text{ solves} \quad \left\{ \begin{array}{rcl} \mathcal{L}_{cd}(u) & = & 0 \quad \text{in } \Omega_-, \\ u & = & g \quad \text{on } \Gamma, \\ u & & \text{bounded at infinity.} \end{array} \right.$$

If the coefficients of $\mathcal{L}_{cd}$ are constants, we can compute the symbol of $\Lambda_+$ using a Fourier transform in the $y$ direction. The symbol is given by the root with positive real part of the characteristic polynomial

$$-\nu\lambda^2 + a\lambda + (\nu k^2 + ibk + c) = 0.$$

Then $(\frac{\partial}{\partial x} - \Lambda_+)$ is the transparent operator on $\Gamma$ for the convection diffusion problem in $\Omega_-$ (see [4]). If we consider the Schwarz algorithm

$$\left\{ \begin{array}{ll} \mathcal{L}_{cd}(v^n) = f & \text{in } \Omega_- \\ \mathcal{L}_c(v^n) = \mathcal{L}_c(w^{n-1}) & \text{on } \Gamma \end{array} \right. \qquad \left\{ \begin{array}{ll} \mathcal{L}_c(w^n) = f & \text{in } \Omega_+ \\ \left(\frac{\partial}{\partial x} - \Lambda_+\right)(w^n) = \left(\frac{\partial}{\partial x} - \Lambda_+\right)(v^{n-1}) & \text{on } \Gamma \end{array} \right. \tag{0.2}$$

then, because the transmission operators are the transparent operators for $\Omega_-$ and $\Omega_+$, we have the following optimal convergence result.

THEOREM 0.1. *The algorithm (0.2) converges in 2 iterations to the solution of the coupled problem*

$$\left\{ \begin{array}{rclrcl} \mathcal{L}_{cd}(v) & = & f \quad \text{in } \Omega_-, & v & = & w \quad \text{on } \Gamma, \\ \mathcal{L}_c(w) & = & f \quad \text{on } \Omega_+, & \frac{\partial v}{\partial x} & = & \frac{\partial w}{\partial x} \quad \text{on } \Gamma. \end{array} \right. \tag{0.3}$$

Note that the coupling conditions satisfied at convergence in (0.3) are the coupling conditions satisfied by the original viscous problem in both subdomains. The continuity of both the values and normal derivatives in the coupled solution seems to be important, since we neglect the diffusion term only for computational purposes, not because the diffusion is physically zero in one subdomain. This is an important distinction from the transmission conditions derived in [3] from a mathematical point of view, which led to a coupled solution with jumps in the the normal derivatives across the artificial interface.

The transmission condition for $\Omega_+$ in the optimal Schwarz algorithm (0.2) involves a non-local operator, which requires a convolution along the interface in a numerical implementation. To avoid this, we replace the non-local operator $\Lambda_+$ by a local approximation given by a differential operator in the $y$ variable, which leads to the new transmission condition

$$\mathcal{B}_+ = \frac{\partial}{\partial x} - \alpha - \beta\frac{\partial}{\partial y} - \gamma\frac{\partial^2}{\partial y^2}$$

where $\alpha > 0$, $\gamma \geq 0$ and the coefficients $\alpha$, $\beta$ and $\gamma$ are chosen to optimize the convergence rate of the Schwarz algorithm as it was done for convection-diffusion problems in [5]. The optimized Schwarz algorithm for the coupled problem is therefore given by

$$\left\{\begin{array}{ll} \mathcal{L}_{cd}(v^n) = f & \text{in } \Omega_- \\ \mathcal{L}_c(v^n) = \mathcal{L}_c(w^{n-1}) & \text{on } \Gamma \end{array}\right. \qquad \left\{\begin{array}{ll} \mathcal{L}_c(w^n) = f & \text{in } \Omega_+ \\ \mathcal{B}_+(w^n) = \mathcal{B}_+(v^{n-1}) & \text{on } \Gamma \end{array}\right. \tag{0.4}$$

REMARK 0.1. *Note that on the interface, $\frac{\partial w^n}{\partial x}$ can be replaced by $\frac{1}{a}(f - cw^n - b\frac{\partial w^n}{\partial y})$ using the convection equation in $\Omega_+$.* A priori estimates show the well-posedness as in [3] and [6].

THEOREM 0.2. *Let $H^{1,1}(\Omega_\pm) = \{v \in H^1(\Omega_\pm), v|_\Gamma \in H^1(\Gamma)\}$. Then the algorithm (0.4) has a unique solution $(v^n, w^n)$ in $H^{1,1}(\Omega_-) \times H^{1,1}(\Omega_+)$.* Because the transmission condition for $\Omega_-$ is still transparent for $w^2$ we have

THEOREM 0.3. *The algorithm (0.4) converges in 3 iterations to the solution of the coupled problem (0.3). More precisely we have $v^2 = v$, $w^3 = w$.*

**Outflow of the Purely Convective Region** In this case only one transmission condition can be imposed and we choose here to impose the continuity of the function values, $v = w$, on $\Gamma$. Note that one could also choose continuity of the normal derivatives or a linear combination. The boundary conditions imposed on the purely convective region $\partial\Omega \cap \partial\Omega_+$ are such that $w$ is uniquely determined by

$$\mathcal{L}_c(w) = f \quad \text{in } \Omega_+$$

without information required from the other subdomain (see [3]). With the solution $w$, the solution $v$ on the other subdomain is then defined by

$$\left\{\begin{array}{rcll} \mathcal{L}_{cd}(v) & = & f & \text{in } \Omega_-, \\ v & = & w & \text{on } \Gamma \end{array}\right.$$

and there is no need to iterate.

**The Case of Mixed Inflow and Outflow** We define $\Gamma_{out} = \{x \in \Gamma, \mathbf{a}\cdot\mathbf{n} < \mathbf{0}\}$ and $\Gamma_{in} = \{x \in \Gamma, \mathbf{a}\cdot\mathbf{n} > \mathbf{0}\}$ with $\Gamma_{in} \cap \Gamma_{out} = \emptyset$ and $\overline{\Gamma}_{in} \cup \overline{\Gamma}_{out} = \Gamma$ as shown in Figure 0.1.

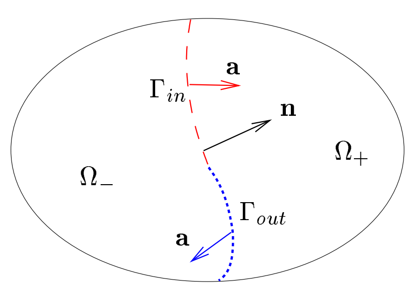


Figure 0.1: A problem with both inflow and outflow along the artificial interface.

We propose the optimized Schwarz algorithm

$$\left\{\begin{array}{rcll} \mathcal{L}_{cd}(v^n) &=& f & \text{in } \Omega_- \\ \mathcal{L}_c(v^n) &=& \mathcal{L}_c(w^{n-1}) & \text{on } \Gamma_{in} \\ v^n &=& w^{n-1} & \text{on } \Gamma_{out} \end{array}\right. \qquad \left\{\begin{array}{rcll} \mathcal{L}_c(w^n) &=& f & \text{in } \Omega_+ \\ \mathcal{B}_+(w^n) &=& \mathcal{B}_+ v^{n-1} & \text{on } \Gamma_{in} \end{array}\right. \tag{0.5}$$

Again, a priori estimates lead to the following

THEOREM 0.4. *The algorithm (0.5) is well-posed in* $H^{1,1}(\Omega_-)\times H^{1,1}(\Omega_+)$. We do not yet have a convergence proof of algorithm (0.5), but numerical experiments show that the iterates $(v^n, w^n)$ of the optimized Schwarz method (0.5) converge to the solution $(v, w)$ of the coupled problem

$$\left\{\begin{array}{rclcrcll} \mathcal{L}_{cd}(v) &=& f & \text{in } \Omega_-, & v &=& w & \text{on } \Gamma, \\ \mathcal{L}_c(w) &=& f & \text{on } \Omega_+, & \frac{\partial v}{\partial n} &=& \frac{\partial w}{\partial n} & \text{on } \Gamma_{in}. \end{array}\right.$$

**Estimates for Small Viscosity** Let $\Omega = \mathbb{R}^2$, $\Omega_- = \mathbb{R}^- \times \mathbb{R}$, $\Omega_+ = \mathbb{R}^+ \times \mathbb{R}$ and $\Gamma = \{(x,y),\ y \in \mathbb{R},\ x = 0\}$. Let $U$ be the solution of the convection-diffusion equation in $\mathbb{R}^2$. Dubach [2] obtained for $\mathbf{a}\cdot\mathbf{n} = a > 0$ and the problem

$$\left\{\begin{array}{rcll} a\partial_x v &=& f, & x<0, \\ a\partial_x w - \nu\Delta w &=& f, & x>0, \\ \left(\partial_x - \frac{a}{\nu}\right) w &=& -\frac{a}{\nu} v, & x = 0, \end{array}\right.$$

the estimates

$$\|(v-U)_x\|_{L^2(\mathbb{R}^2_-)} = 0(\frac{\nu}{a}) \quad \text{and} \quad \|(w-U)_x\|_{L^2(\mathbb{R}^2_+)} = 0((\frac{\nu}{a})^{\frac{3}{2}}).$$

For the problem

$$\left\{\begin{array}{rclcrcll} a\partial_x v - \nu\Delta v &=& f, & x<0, & v &=& w, & \text{at } x=0, \\ a\partial_x w &=& f, & x>0, & \partial_x v &=& \partial_x w, & \text{at } x=0, \end{array}\right.$$

he obtained the estimates

$$\|(v-U)_x\|_{L^2(\mathbb{R}^2_-)} = 0((\frac{\nu}{a})^{\frac{3}{2}}) \quad \text{and} \quad \|(w-U)_x\|_{L^2(\mathbb{R}^2_+)} = 0((\frac{\nu}{a})^{\frac{1}{2}})$$

which will be verified by our numerical experiments.

**Numerical Experiments** We discretize the global convection-diffusion problem and the subproblems in the optimized Schwarz method by upwind finite difference schemes. We call the solution of the global convection-diffusion problem the viscous solution. We use the mesh size $h = 1/200$ and both the viscous solution as well as the subdomain solutions are obtained by a direct solver. We first consider an inflow problem into the purely convective region of the domain and then a rotating velocity. We compare the results obtained with the optimized Schwarz method to the results obtained with the algorithm from [3].

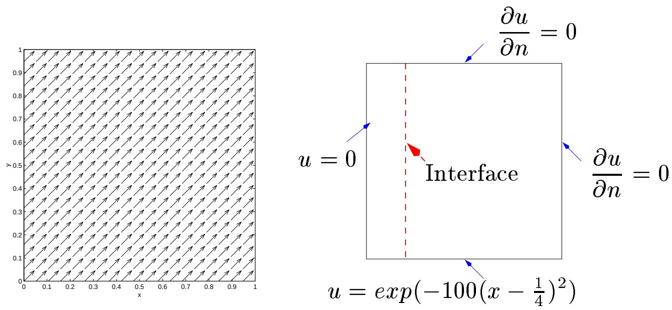


Figure 0.2: Convection field and boundary conditions for inflow into the purely convective region.

**Inflow into the Purely Convective Region** We solve the coupled problem on the unit square using the optimized Schwarz method (0.4) with $\nu = 10^{-3}$, $\mathbf{a} = (0.1, 0.1)$ and $c = 10^{-6}$. The boundary conditions we use are given in Figure 0.2 and the interface is located at $x = 0.2$.

On the left in Figure 0.3 we compare the viscous solution and the solution obtained by the optimized Schwarz method for the coupled problem on the line $y = 0.01$ after 2 iterations. On the right of Figure 0.3 we show the results obtained using the algorithm and transmission conditions from [3] obtained by letting $\nu$ go to zero. They are given by

$$\left\{ \begin{array}{rcll} \left(-\nu \frac{\partial}{\partial \mathbf{n}} + \mathbf{a} \cdot \mathbf{n}\right) v^n & = & -(\mathbf{a} \cdot \mathbf{n}) w^{n-1} & \text{on } \Gamma, \\ w^n & = & v^{n-1} & \text{on } \Gamma_{in} \end{array} \right. \tag{0.6}$$

and do therefore not satisfy continuity of the derivatives across the interface, as one can see in Figure 0.3.

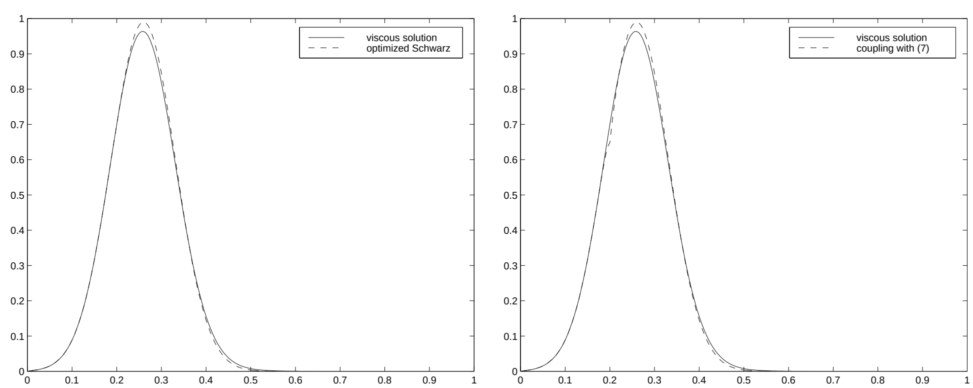


Figure 0.3: Result for constant velocity, the solid line denotes the viscous solution, the dashed line on the left the coupled solution obtained by the optimized Schwarz method and the dashed line on the right the solution from the algorithm proposed in [3]. Note the discontinuity in the derivative on the right at $x = 0.2$.

**A Rotating Velocity** We use the optimized Schwarz method to solve the problem with $\nu = 10^{-2}$, $\mathbf{a} = (0.5 - y, 0.5)$ and $c = 10^{-6}$ and boundary conditions as given in Figure 0.4 on the unit square. The interface is again located at $x = 0.2$. We

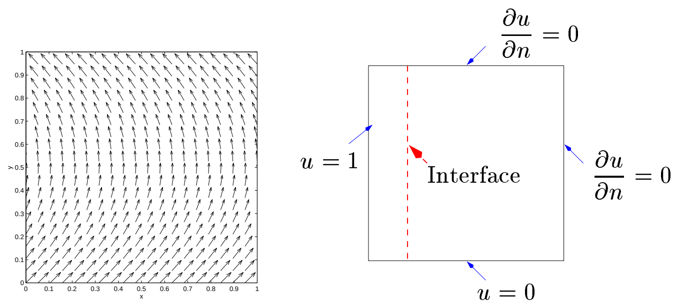


Figure 0.4: Convective field and boundary conditions for the rotation velocity case.

compare the solution obtained by the optimized Schwarz method after 3 iterations to the viscous solution. Figure 0.5 shows both solutions on the line $y = 0.15$ (inflow into

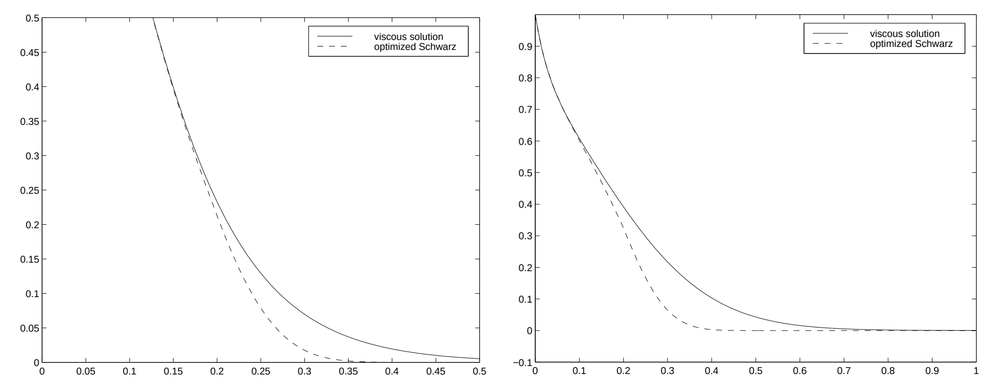


Figure 0.5: Result for rotating velocity on the left at $y = 0.15$ with a zoom on $[0, 0.5] \times [0, 0.5]$, on the right at $y = 0.8$. The solid line denotes the viscous solution, the dashed line the optimized Schwarz solution.

the purely convective region) and on the line $y = 0.8$ (outflow of the purely convective region). The computed solution is continuous on the interface and also its derivative is continuous on $\Gamma_{in}$. On $\Gamma_{out}$ there is a small jump in the normal derivative of the solution because only one condition can be satisfied, as we have seen in the analysis. In Figure 0.6 we show the results obtained with the transmission conditions (0.6) from [3] after 3 iterations. Note that on $\Gamma_{in}$ there is a jump in the normal derivative, whereas on $\Gamma_{out}$ there is a jump in the function value and in the normal derivative. The size of these discontinuities diminishes however with diminishing viscosity. Nevertheless as a physical solution to the original viscous problem, the solutions obtained by the optimized Schwarz methods seem to be preferable.

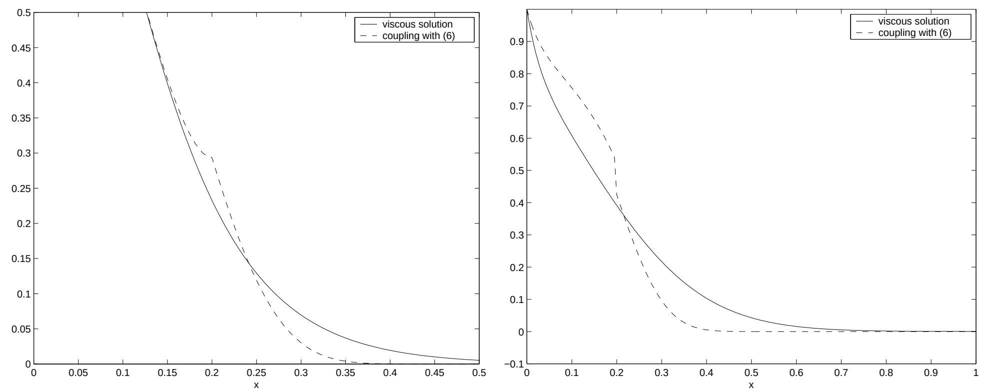


Figure 0.6: Same graphs as in Figure 0.5 but for the algorithm with transmission conditions (0.6) from [3].

Finally we show in Figure 0.7 the error on the interface as a function of decreasing viscosity. The results confirm the asymptotic results from [2].

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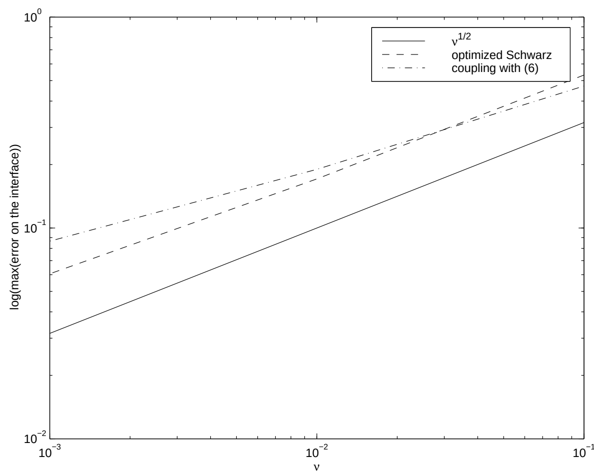


Figure 0.7: Asymptotic results in $\nu$ for the rotating velocity, the solid line is the reference for $\nu^{\frac{1}{2}}$, the dashed line denotes the optimized Schwarz solution, and the dotted line the solution of the algorithm with transmission conditions (0.6).