Optimized Schwarz Methods for Stokes-Darcy flows: the Brinkman equations

Martin J. Gander, Yiying Wang,Yingxiang Xu

1 Introduction

The Brinkman equations model a combination of Darcy's law and the Navier-Stokes equations, see [3]. They describe the incompressible viscous flow of a fluid in complex porous media with a high-contrast permeability coefficient such that the flow is dominated by Darcy in porous media regions and by Stokes in fluid regions, which naturally defines a decomposition of the domain by the physics of the problem, see for example [4, 7].

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded convex domain with Lipschitz boundary $\partial\Omega$; the Brinkman model for the unknown velocity vector function $\mathbf{u} : \overline{\Omega} \to \mathbb{R}^d$, the scalar pressure function $p : \overline{\Omega} \to \mathbb{R}$ and some given force term $\mathbf{f} : \overline{\Omega} \to \mathbb{R}^d$ is

$$\begin{aligned} -\nu\Delta\mathbf{u} + \frac{\nu}{\kappa}\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{g} \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where ν denotes the viscosity and κ is the permeability coefficient of the porous media which occupies the domain Ω.

We present here a new non-overlapping Schwarz method [10, 5] for solving the Brinkman equation (1) with fully-coupled Robin-like transmission conditions [1, 12]. We derive a general expression for the iteration operator, and study the corresponding min-max problems for local approximations to optimize performance

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using asymptotic analysis, which we also illustrate with numerical results. For the sake of simplicity, we will consider the 2-D case with $\mathbf{g} = 0$ and $\nu = 1$ (we can always scale the solution with ν) in two spatial dimensions.

2 Iteration operator of a non-overlapping Schwarz algorithm

We consider (1) on a bounded domain Ω in $\mathbb{R}^2$ formed by two non overlapping subregions: the porous medium Ω_1, the fluid domain Ω_2, separated by an interface Γ. We split the domain Ω into two subdomains determined by the porous media: $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, and the permeability coefficient κ is a corresponding piecewise constant function. On the two subdomains Ω_j $(j = 1, 2)$, we use a parallel Schwarz iteration with generic Robin-like transmission conditions,

$$
\begin{aligned}
-\Delta \mathbf{u}_j^n + \nabla p_j^n + \kappa_j^{-1} \mathbf{u}_j^n &= \mathbf{f} \qquad \text{in } \Omega_j,\\
\nabla \cdot \mathbf{u}_j^n &= 0 \qquad \text{in } \Omega_j,\\
\mathbf{u}_j^n &= \mathbf{g} \qquad \text{on } \partial\Omega_j \setminus \Gamma,\\
\sigma_j^n \cdot \mathbf{n_j} + S_j \mathbf{u}_j^n &= \sigma_{3-j}^{n-1} \cdot \mathbf{n_j} + S_j \mathbf{u}_{3-j}^{n-1} \quad \text{on } \Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2,
\end{aligned}
\tag{2}
$$

where $\sigma_j^n := \nabla \mathbf{u}_j^n - p_j^n I$ is the stress tensor [4, 7, 2] in domain Ω_j (I represents the 2×2 identity matrix), $\mathbf{n}_j$ is the outward normal vector, S_j is a general 2×2-matrix of linear operators, and n is the iteration index of the Schwarz algorithm.

Subdomain Solutions: In order to study solutions of (2), we consider a model problem on the infinite plane $\Omega = \mathbb{R}^2$, with the two subdomains $\Omega_1 = \mathbb{R} \times (-\infty, 0)$, and $\Omega_2 = \mathbb{R} \times (0, \infty)$.

We use the Fourier transform in the x (horizontal) variable for the error equations of (2), i.e. $\mathbf{f} = \mathbf{g} = 0$. In Fourier space, the PDE in Ω_j becomes an ODE in y (for each fixed frequency k),

$$
-\left(-k^2 \hat{\mathbf{e}}_j^n + \frac{d^2 \hat{\mathbf{e}}_j^n}{dy^2}\right) + \begin{pmatrix} ik\hat{\eta}_j^n \\ \frac{d\hat{\eta}_j^n}{dy} \end{pmatrix} + \kappa_j^{-1} \hat{\mathbf{e}}_j^n = 0 \qquad \text{in } \Omega_j, \tag{3}
$$

$$
ik\hat{e}_{j,1}^n + \frac{d\hat{e}_{j,2}^n}{dy} = 0 \qquad \text{in } \Omega_j, \tag{4}
$$

$$
\hat{\mathbf{e}}_j^n \to 0 \qquad \text{when } |y| \to \infty, \tag{5}
$$

$$
\hat{\sigma}_j^n \cdot \mathbf{n}_j + \hat{S}_j \hat{\mathbf{e}}_j^n = \hat{\sigma}_{j'}^{n-1} \cdot \mathbf{n}_j + \hat{S}_j \hat{\mathbf{e}}_{j'}^{n-1} \text{ on } \Gamma,\ j' = 3 - j, \tag{6}
$$

where $\hat{\eta}_j := \hat{p}|_{\Omega_j} - \hat{p}_j^n$ and $\hat{\mathbf{e}}_j^n := \hat{\mathbf{u}}|_{\Omega_j} - \hat{\mathbf{u}}_j^n = (\hat{e}_{j,1}^n, \hat{e}_{j,2}^n)^T$. Here $\hat{e}_{j,1}^n$ and $\hat{e}_{j,2}^n$ denote the horizontal component and the vertical component of $\hat{\mathbf{e}}_j^n$.

As in [8] and [6], we seek solutions using the ansatz

$$\mathbf{E}_j^n := \begin{pmatrix} \hat{\mathbf{e}}_j^n \\ \hat{\eta}_j^n \end{pmatrix}(y) = \Phi_j^n e^{\xi y}.$$

This leads to a system for Φ_j^n, namely

$$\begin{pmatrix} k^2+\kappa_j^{-1}-\xi^2 & 0 & ik \\ 0 & k^2+\kappa_j^{-1}-\xi^2 & \xi \\ ik & \xi & 0 \end{pmatrix}\Phi_j^n = 0. \tag{7}$$

In order to get a non-trivial solution to system (7), a necessary and sufficient condition is that the matrix is singular, which leads to four possible values for ξ, $\xi_1 = |k|,\ \xi_2 = \lambda_1,\ \xi_3 = -|k|,$ and $\xi_4 = -\lambda_2$, with $\lambda_j := \sqrt{k^2+\kappa_j^{-1}}$.

The solutions of (7) are linear combinations of four terms,

$$\mathbf{E}_j^n = \sum_{m=1}^{4} \boldsymbol{\gamma}_{j,m}^n \Phi_m e^{\xi_m y},$$

where $((\Phi_m)_{1\le m\ge 4})$ are the eigenvectors (corresponding to the eigenvalue 0), associated with each of the ξ_m,

$$\Phi_1 = \begin{pmatrix} -ik \\ -|k| \\ \kappa_1^{-1} \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} \lambda_1 \\ -ik \\ 0 \end{pmatrix}, \quad \Phi_3 = \begin{pmatrix} -ik \\ |k| \\ \kappa_2^{-1} \end{pmatrix}, \quad \Phi_4 = \begin{pmatrix} \lambda_2 \\ ik \\ 0 \end{pmatrix}.$$

Due to the condition (5), $\xi_1, \xi_2 \ge 0$ and $\xi_3, \xi_4 \le 0$, only two terms are possible in the expression of $\mathbf{E}_j^n$ in each of the subdomain Ω_j, and we obtain for the subdomain errors

$$\mathbf{E}_1^n(y) = \sum_{m=1}^{2} \boldsymbol{\gamma}_{1,m}^n \Phi_m e^{\xi_m y}, \qquad \mathbf{E}_2^n(y) = \sum_{m=3}^{4} \boldsymbol{\gamma}_{2,m}^n \Phi_m e^{\xi_m y}. \tag{8}$$

Iteration Operator: To obtain the iteration operator, we need to apply the transmission conditions (6) to (8). Using the horizontal component of equation (3), we can simplify the error in the pressure as

$$\hat{\eta}_j^n = \frac{i}{k}\left((k^2+\kappa_j^{-1})\hat{\mathbf{e}}_j^n - \frac{d^2\hat{\mathbf{e}}_j^n}{dy^2}\right)\cdot(1,0)^T. \tag{9}$$

Inserting the gradient of $\hat{\mathbf{e}}_j^n$ and (9) into the transmission condition

$$\nabla\hat{\mathbf{e}}_j^n\cdot\mathbf{n}_j - \hat{\eta}_j^n I\cdot\mathbf{n}_j + \hat{S}_j\hat{\mathbf{e}}_j^n = \nabla\hat{\mathbf{e}}_{j'}^{n-1}\cdot\mathbf{n}_j - \hat{\eta}_{j'}^{n-1} I\cdot\mathbf{n}_j + \hat{S}_j\hat{\mathbf{e}}_{j'}^{n-1},$$

and using that the normal vector $\mathbf{n}_j = (0,(-1)^{j-1})^T$ at the interface, we obtain

$$
\begin{aligned}
M\frac{d^2\hat{\mathbf{e}}_1^n}{dy^2}+\frac{d\hat{\mathbf{e}}_1^n}{dy}+P_1\hat{\mathbf{e}}_1^n+\hat{S}_1\hat{\mathbf{e}}_1^n &= M\frac{d^2\hat{\mathbf{e}}_2^{n-1}}{dy^2}+\frac{d\hat{\mathbf{e}}_2^{n-1}}{dy}+P_2\hat{\mathbf{e}}_2^{n-1}+\hat{S}_1\hat{\mathbf{e}}_2^{n-1},\\
-M\frac{d^2\hat{\mathbf{e}}_2^n}{dy^2}-\frac{d\hat{\mathbf{e}}_2^n}{dy}-P_2\hat{\mathbf{e}}_2^n+\hat{S}_2\hat{\mathbf{e}}_2^n &= -M\frac{d^2\hat{\mathbf{e}}_1^{n-1}}{dy^2}-\frac{d\hat{\mathbf{e}}_1^{n-1}}{dy}-P_1\hat{\mathbf{e}}_1^{n-1}+\hat{S}_1\hat{\mathbf{e}}_1^{n-1},
\end{aligned}
\tag{10}
$$

with $M := \begin{pmatrix} 0 & 0 \\ \frac{i}{k} & 0 \end{pmatrix}$, and $P_j := \begin{pmatrix} 0 & 0 \\ \frac{\lambda_j^2}{ik} & 0 \end{pmatrix}$, $j = 1, 2$. Using (8), we then get

$$
\hat{\mathbf{e}}_1^n(y) = M_{12}e^{\xi_{12}y}\boldsymbol{\gamma}_{12}^n\,, \quad \hat{\mathbf{e}}_2^n(y) = M_{34}e^{\xi_{34}y}\boldsymbol{\gamma}_{34}^n\,, \text{ with}
$$

$$
M_{12} := \begin{pmatrix} -ik & \lambda_1 \\ -|k| & -ik \end{pmatrix}, \quad e^{\xi_{12}y} := \begin{pmatrix} e^{\xi_1 y} & 0 \\ 0 & e^{\xi_2 y} \end{pmatrix}, \quad \boldsymbol{\gamma}_{12}^n := \begin{pmatrix} \gamma_{1,1}^n \\ \gamma_{1,2}^n \end{pmatrix},
$$

$$
M_{34} := \begin{pmatrix} -ik & \lambda_2 \\ |k| & ik \end{pmatrix}, \quad e^{\xi_{34}y} := \begin{pmatrix} e^{\xi_3 y} & 0 \\ 0 & e^{\xi_4 y} \end{pmatrix}, \quad \boldsymbol{\gamma}_{34}^n := \begin{pmatrix} \gamma_{2,3}^n \\ \gamma_{2,4}^n \end{pmatrix}.
$$

Therefore (10) becomes $\hat{H}_{11}\boldsymbol{\gamma}_{12}^n = \hat{H}_{12}\boldsymbol{\gamma}_{34}^{n-1}$, $\hat{H}_{22}\boldsymbol{\gamma}_{34}^n = \hat{H}_{21}\boldsymbol{\gamma}_{12}^{n-1}$, where, with $\frac{dM_{12}e^{\xi_{12}y}\boldsymbol{\gamma}_{12}^n}{dy} = M_{12}\xi_{12}e^{\xi_{12}y}\boldsymbol{\gamma}_{12}^n$ and $\frac{d^2M_{12}e^{\xi_{12}y}\boldsymbol{\gamma}_{12}^n}{dy} = M_{12}\xi_{12}^2e^{\xi_{12}y}\boldsymbol{\gamma}_{12}^n$, we have

$$
\begin{aligned}
\hat{H}_{11} &= MM_{12}\begin{pmatrix} \xi_1^2 & 0 \\ 0 & \xi_2^2 \end{pmatrix} + M_{12}\begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix} + P_1M_{12} + \hat{S}_1M_{12},\\
\hat{H}_{12} &= MM_{34}\begin{pmatrix} \xi_3^2 & 0 \\ 0 & \xi_4^2 \end{pmatrix} + M_{34}\begin{pmatrix} \xi_3 & 0 \\ 0 & \xi_4 \end{pmatrix} + P_2M_{34} + \hat{S}_1M_{34},\\
\hat{H}_{22} &= -MM_{34}\begin{pmatrix} \xi_3^2 & 0 \\ 0 & \xi_4^2 \end{pmatrix} - M_{34}\begin{pmatrix} \xi_3 & 0 \\ 0 & \xi_4 \end{pmatrix} - P_2M_{34} + \hat{S}_2M_{34},\\
\hat{H}_{21} &= -MM_{12}\begin{pmatrix} \xi_1^2 & 0 \\ 0 & \xi_2^2 \end{pmatrix} - M_{12}\begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix} - P_1M_{12} + \hat{S}_2M_{12}.
\end{aligned}
$$

Assuming that $\hat{H}_{11}$ and $\hat{H}_{22}$ are invertible ,we thus get for the error coefficients $\boldsymbol{\gamma}_{12}^n$ the recurrence relation $\boldsymbol{\gamma}_{12}^n = \hat{H}_{11}^{-1}\hat{H}_{12}\hat{H}_{22}^{-1}\hat{H}_{21}\boldsymbol{\gamma}_{12}^{n-2}$. Hence, the convergence factor of the error in the Schwarz method (2) is determined by the spectral radius of the *iteration operator* given by

$$
\hat{H}(k, \hat{S}_1, \hat{S}_2) := \hat{H}_{11}^{-1}\hat{H}_{12}\hat{H}_{22}^{-1}\hat{H}_{21}. \tag{11}
$$

Convergence Factor: In order to ensure fast convergence of the Schwarz algorithm for all possible frequencies $k \in \mathbb{R}$, we have to choose operators $\hat{S}_j$ $(j = 1, 2)$ that make the convergence factor small [9, 5, 2]. The convergence factor is

$$
\rho_{\mathrm{OSM}}(k, \hat{S}_1, \hat{S}_2) := \rho(\hat{H}(k, \hat{S}_1, \hat{S}_2)) < 1, \tag{12}
$$

where $\rho(\hat{H})$ is the spectral radius of $\hat{H}$ for a fixed k and $\hat{S}_j$ $(j = 1, 2)$.

Optimal Operators: The symbols $\hat{S}_j$ (or equivalently operators S_j) are still free to be chosen at this point. It is possible to make the right hand of the transmission conditions (10) vanish, and to obtain an algorithm that converges in two iterations, if we choose

$$\hat{S}_1^* := -MM_{34}\begin{pmatrix}\xi_3^2 & 0\\ 0 & \xi_1^2\end{pmatrix}M_{34}^{-1} - M_{34}\begin{pmatrix}\xi_3 & 0\\ 0 & \xi_4\end{pmatrix}M_{34}^{-1} - P_2,$$
$$\hat{S}_2^* := MM_{12}\begin{pmatrix}\xi_1^2 & 0\\ 0 & \xi_1^2\end{pmatrix}M_{12}^{-1} + M_{12}\begin{pmatrix}\xi_1 & 0\\ 0 & \xi_2\end{pmatrix}M_{12}^{-1} + P_1,$$

and a lengthy calculation permits to simplify the preceding expressions, yielding

$$\hat{S}_1^* = \begin{pmatrix}|k| + \lambda_2(k) & \frac{ik}{|k|}\lambda_2(k)\\ \frac{-ik}{|k|}\lambda_2(k) & \frac{\lambda_2(k)}{|k|}(|k| + \lambda_2(k))\end{pmatrix},\ \hat{S}_2^* = \begin{pmatrix}|k| + \lambda_1(k) & \frac{-ik}{|k|}\lambda_1(k)\\ \frac{ik}{|k|}\lambda_1(k) & \frac{\lambda_1(k)}{|k|}(|k| + \lambda_1(k))\end{pmatrix}. \tag{13}$$

Some terms in these operators are not polynomials in ik, and thus the corresponding operators $S_j = \mathcal{F}_x^{-1}(\hat{S}_j)$ in real space are nonlocal in x, which is not convenient for implementations, since it requires convolution computations.

3 Optimized Schwarz methods and asymptotic performance

We would therefore like to approximate $\hat{S}_j^*$ by local operators that still give very fast convergence of the Schwarz iteration. The idea is to find local operators $\hat{S}_j$ that minimize the convergence factor (12) uniformly over a relevant range of frequencies, which leads to the min-max problem

$$\min_{\hat{S}_j}\quad (\max_{k\in[k_{\min},k_{\max}]}\rho_{\text{OSM}}(k,\hat{S}_1,\hat{S}_2)). \tag{14}$$

Although the problem we considered before is a continuous model on the infinite plane, the range of frequencies can be bounded by incorporating information about the actual discretized problem we intend to solve. In (14), $k_{\min}$ can in general be negative, but when the optimized $\hat{S}_j$ lead to an even convergence factor in k, as we will see later, we can equivalently assume $k_{\min} > 0$. Thus the minimal frequency component of the solution can be estimated by $k_{\min} = \frac{\pi}{L}$ for an interface of length L, and $k_{\max} = \frac{\pi}{h}$ with grid spacing h, see for example [9, 6].

Let $\hat{S}_j$ $(j = 1, 2)$ keep the sign, symmetry and parity of the optimal operator in (13), and let us denote $\hat{S}_j := \begin{pmatrix}\mathcal{S}_{11}^j(k) & \mathcal{S}_{12}^j(k)\\ \mathcal{S}_{21}^j(k) & \mathcal{S}_{22}^j(k)\end{pmatrix}$, with $\mathcal{S}_{21}^j(k) = -\mathcal{S}_{12}^j(k)$ $(j = 1, 2)$. We first study properties of $\hat{H}(k,\hat{S}_1,\hat{S}_2)$ for these $\hat{S}_j$, which can be obtained by a lengthy technical computation that will appear elsewhere [11].

Lemma 1 *Assuming that* $\mathcal{S}_{12}^j = ik\cdot f_{0j}(k)$ *and* $\mathcal{S}_{ll}^j$ *and* $f_{0j}(k)$ $(l, j = 1, 2)$ *are even functions of* k*, then* $\hat{H}(k,\hat{S}_1,\hat{S}_2)$ *is always of the form*

$$\hat{H}(k,\hat{S}_1,\hat{S}_2) = \begin{pmatrix}f_1(k) & ig_1(k)\\ ig_2(k) & f_2(k)\end{pmatrix},$$

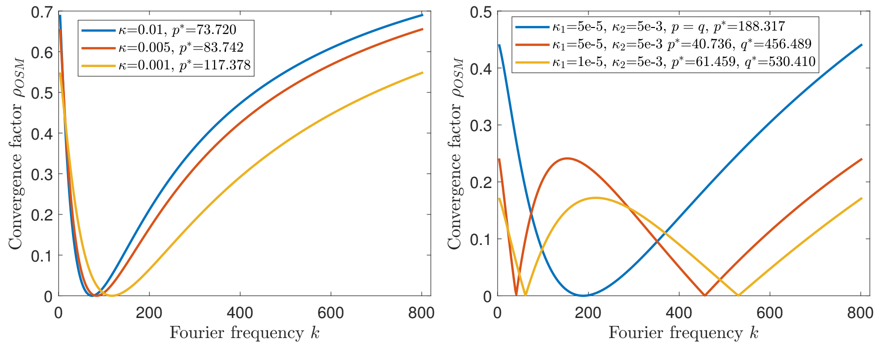

Fig. 1 Left: Optimized $\rho_{\text{OSM}}(k, p)$ for $\kappa_1 = \kappa_2$. Right: Optimized $\rho_{\text{OSM}}(k, p, q)$ for $\kappa_1 \neq \kappa_2$.

where $f_l(k)$ are even functions and $g_l(k)$ are odd functions of k for $k \in \mathbb{R}$ $(l = 1, 2)$. Furthermore, this implies that

- *the eigenvalues of $\hat{H}(k, \hat{S}_1, \hat{S}_2)$ are even functions of k for $k \in \mathbb{R}$,*
- *the optimized problem in (14) is equivalent to restricting $k \in \mathbb{R}^+$.*

Now, we derive optimized Robin-like transmission conditions for continuous and discontinuous coefficients κ.

The continuous case, $\kappa_1 = \kappa_2 := \kappa$: In this case, $\lambda_j(k) = \sqrt{k^2 + \frac{1}{\kappa}} = \lambda(k)$, and we introduce the structurally consistent approximations replacing k by the constant p for the non-local terms k and $\lambda_j(k)$, c.f. (13),

$$\hat{S}_1^c := \begin{pmatrix} p + \lambda(p) & \frac{ik}{p}\lambda(p) \\ \frac{-ik}{p}\lambda(p) & \frac{\lambda(p)}{p}(p + \lambda(p)) \end{pmatrix}, \quad \hat{S}_2^c := \begin{pmatrix} p + \lambda(p) & \frac{-ik}{p}\lambda(p) \\ \frac{ik}{p}\lambda(p) & \frac{\lambda(p)}{p}(p + \lambda(p)) \end{pmatrix},$$

with one free parameter p, where $p, k > 0$ (using superscript c for continuous to distinguish from the following discontinuous case). With $\hat{S}_j = \hat{S}_j^c$, the convergence factor $\rho_{\text{OSM}}(k, \hat{S}_1, \hat{S}_2)$ only depends on k and p, so we denote it by $\rho_{\text{OSM}}(k, p)$. A lengthy computation shows that $\hat{H}_{11}^{-1}\hat{H}_{12} = \hat{H}_{22}^{-1}\hat{H}_{21}$, and we obtain the following property to choose the maximum of the two eigenvalues of $\hat{H}$.

Lemma 2 *The eigenvalues $\mu^{\pm}(k, p)$ of $\hat{H}(k, \hat{S}_1^c, \hat{S}_2^c)$ are always positive, and*

$$sign(\mu^+(k, p) - \mu^-(k, p)) = sign(p - k).$$

From Fig.1 (left), we find that the optimized parameter p^* for continuous κ is characterized by an equioscillation property: $\rho_{\text{OSM}}(k_{\min}, p^*) = \rho_{\text{OSM}}(k_{\max}, p^*)$.

Theorem 1 *The optimized parameter p^* solved from $\mu^+(k_{\min}, p) = \mu^-(k_{\max}, p)$ is*

$$p^* \sim C_p h^{-\frac{1}{2}}, \quad C_p := \sqrt{\frac{L^2 - D^2}{4\kappa \frac{D^2}{L}}}, \quad D := \pi\sqrt{\kappa} - \sqrt{L^2 + \kappa\pi^2},$$

when $k_{\min} = \frac{\pi}{L}$ and $k_{\max} = \frac{\pi}{h}$. Furthermore, the asymptotic convergence factor of the resulting one-side optimized Schwarz method is

$$\min_{k\in[k_{\min},k_{\max}]} \rho_{\mathrm{OSM}}(k,p^*) \sim 1 - C\cdot h^{\frac{1}{2}}, \quad C := \frac{4C_p}{\pi}.$$

Proof We make the ansatz $p^* := C_p \cdot h^{-\frac{1}{2}}$. Expanding for small h, we obtain

$$\mu^+(k_{\min},p^*) = 1 - \frac{L(L^2-D^2)}{C_p\kappa\pi D^2}\sqrt{h} + O(h), \text{ where } D = \pi\sqrt{\kappa} - \sqrt{L^2+\kappa\pi^2},$$
$$\mu^-(k_{\max},p^*) = 1 - \frac{4C_p}{\pi}\sqrt{h} + O(h)$$

Solving $\mu^+(k_{\min},p) = \mu^-(k_{\max},p)$ asymptotically then determines p^*. □

The discontinuous case, $\kappa_1 \neq \kappa_2$: For p and q $(p,q>0)$ two free parameters, we introduce the structurally consistent approximations (superscript d for discontinuous)

$$\hat{S}_1^d = \begin{pmatrix} p+\lambda_2(p) & \frac{ik}{p}\lambda_2(p) \\ \frac{-ik}{p}\lambda_2(p) & \frac{\lambda_2(p)}{p}(p+\lambda_2(p)) \end{pmatrix}, \quad \hat{S}_2^d = \begin{pmatrix} q+\lambda_1(q) & \frac{-ik}{q}\lambda_1(q) \\ \frac{ik}{q}\lambda_1(q) & \frac{\lambda_1(q)}{q}(q+\lambda_1(q)) \end{pmatrix},$$

and study the associated convergence factor $\rho_{\mathrm{OSM}}(k,p,q)$ numerically. We show in Fig.1 (right) that

- the optimized parameters p^* and q^* are characterized by an equioscillation property: $\rho_{\mathrm{OSM}}(k_{\min},p^*,q^*) = \rho_{\mathrm{OSM}}(k_{\max},p^*,q^*) = \rho_{\mathrm{OSM}}(\bar{k},p^*,q^*)$,
- high contrast κ leads to fast convergence,
- the two parameters (two-sided Robin) give better convergence than the one parameter (one sided Robin) case earlier.

Numerically, we observe in Fig. 2 that the asymptotic performance is given by

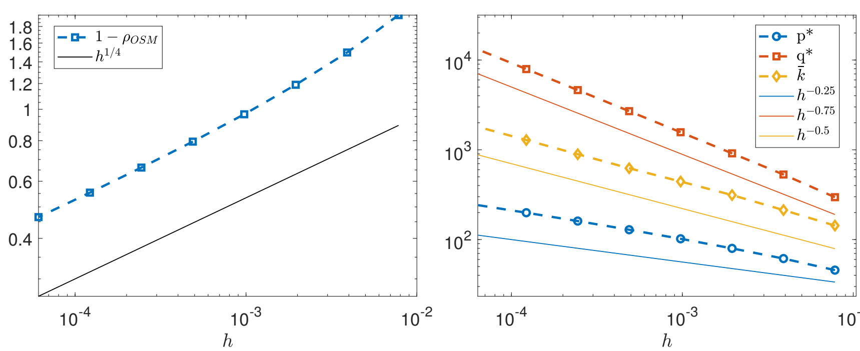

Fig. 2 Log–log plot of the convergence factor and the optimized parameters for $\kappa_1 = 10^{-5}$ and $\kappa_2 = 5\times10^{-3}$.

$$p^* = C_1 h^{-\frac{1}{4}},\ q^* = C_2 h^{-\frac{3}{4}}, \bar{k} = C_3 h^{-\frac{1}{2}}, \quad \rho_{\text{OSM}}(k, p^*, q^*) = 1 - C_0 h^{\frac{1}{4}} + O(h^{\frac{1}{2}}), \tag{15}$$

where C_0, C_1, C_2 and C_3 are constants.

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