Piece-wise Constant, Linear and Oscillatory: a Historical Introduction to Spectral Coarse Spaces with Focus on Schwarz Methods

Martin J. Gander and Laurence Halpern

1 Classical coarse spaces

In 1987, ROY NICOLAIDES introduced what we would now call a coarse space correction for the conjugate gradient method [30]:

> "In this paper, another way of improving the convergence of conjugate gradients is used. It can be used alone or in conjunction with preconditioners. Used alone, it is at least as efficient as the standard preconditioners on model problems. Used with preconditioning it appears from numerical experiments to give a method considerably better than either used separately–it seems that the approaches are in some sense complementary."

The idea of Nicolaides for an example Poisson problem is to deflate piece-wise constant functions on subdomains from the residual at each CG iteration ("we shall systematically interpret E's columns as being a basis for a subspace of certain slowly varying residual components"). From the quote above we see that he advocates to use this technique together with another preconditioner, realizing the two-level character this provides:

> "The method has something in common with a two-level multigrid scheme, although neither smoothing nor subgrids is explicitly used."

There was however no theoretical understanding yet at this point:

> "No theoretical predictions are available at present on the rate of convergence to be expected with preconditioned versions."

Deflation was also introduced independently by ZDENĚK DOSTÁL in [7] under the name of 'preconditioning by projector', and the special case of deflating eigenvectors was studied; see also [8] for a relation to Schur complement preconditioning.

Martin J. Gander
FSMP and Section de Mathématiques, Université de Genève, e-mail: martin.gander@unige.ch

Laurence Halpern
LAGA, Université Sorbonne Paris Nord e-mail: halpern@math.univ-paris13.fr

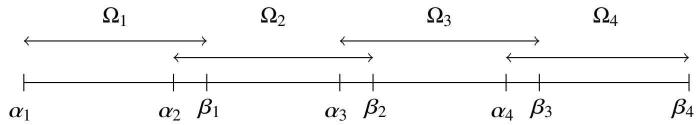


Fig. 1 One-dimensional overlapping domain decomposition.

In order to illustrate the performance of this piece-wise constant coarse space in the context of domain decomposition, we show a numerical experiment for the 1D Laplace problem in $\Omega = (0, 1)$,

$$\partial_{xx} u = 0, \quad u(0) = 0 \text{ and } u(1) = 1,$$

using the parallel Schwarz method introduced by Pierre-Louis Lions [26] at the first international conference on domain decomposition methods (DD1),

$$\begin{aligned} \partial_{xx} u_i^n = 0 \quad \text{in } \Omega_i, \qquad i = 1, .., I, \\ u_i^n(\alpha_i) = u_{i-1}^{n-1}(\alpha_i), \quad u_i^n(\beta_i) = u_{i+1}^{n-1}(\beta_i), \end{aligned} \tag{1}$$

for the decomposition shown in Figure 1 for $I = 4$. When this method is discretized, it is equivalent to Restricted Additive Schwarz (RAS) by Xiao-Chuan Cai and Markus Sarkis [4] for the linear system $A\mathbf{u} = \mathbf{f}$,

$$\mathbf{u}^n := \mathbf{u}^{n-1} + \sum_{i=1}^{I} \tilde{R}^T A_i^{-1} R(\mathbf{f} - A\mathbf{u}^{n-1}), \tag{2}$$

where R_i are restriction matrices of $\mathbf{u}^n$ to the subdomain Ω_i, $A_i := R_i A R_i^T$, and $\tilde{R}_i$ are restriction matrices for a non-overlapping partition; see [15] for more details and the proof of equivalence. In order to combine this with a piece-wise constant coarse correction, we use the $\mathbf{u}^n$ from RAS in (2) and then coarse correct them by computing $\mathbf{u}^n := \mathbf{u}^n + R^T A_c^{-1} R(\mathbf{f} - A\mathbf{u}^n)$, where R is a restriction to the piece-wise constant coarse space functions and $A_c := RAR^T$ is the coarse correction matrix on that space.

We show in Figure 2 the iterates without Krylov acceleration for RAS without and with piece-wise constant coarse correction in the top two rows. We see that the coarse correction indeed changes the iterates, but not by much. To see the true benefit from the coarse correction, we need to use more subdomains. We show the decay of the error for more and more subdomains in Figure 3 (left). We see that indeed with the piece-wise constant coarse space we obtain a scalable method[1], which would be termed "optimal" because of this, but are there better coarse spaces?

Max Dryja and Olof Widlund introduced in the same year as Nicolaides their seminal additive Schwarz method [34] which includes a different coarse space:

> "The first subspace V_0^h, which we also call V^H, is special. It is the space of continuous, piece-wise linear functions on the coarse mesh defined by the substructures Ω_i."

[1] Iteration numbers do not deteriorate when using more and more subdomains.

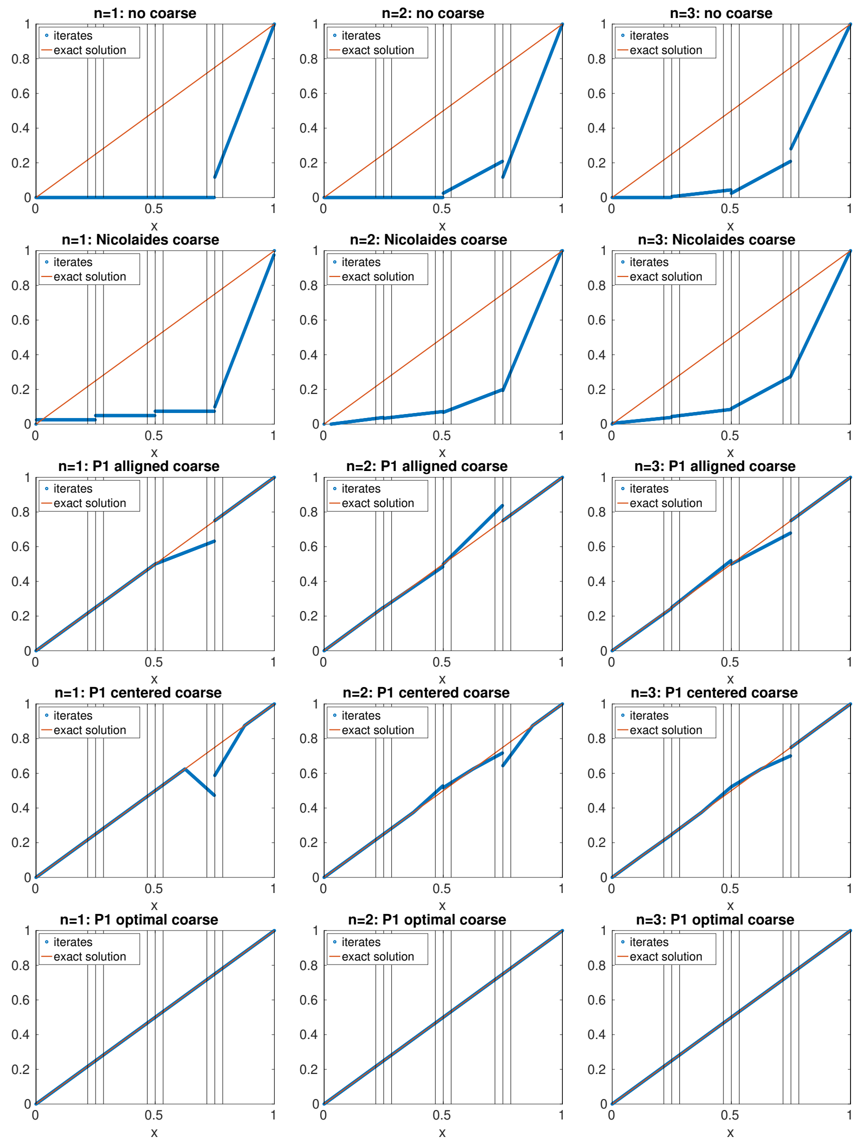


Fig. 2 First three parallel Schwarz iterates without coarse correction (top row), with piece-wise constant coarse correction (second row), with P1 coarse correction aligned with the subdomains (third row), with P1 coarse correction centered in the subdomains (fourth row) and optimal (best possible) coarse correction (last row).

Here the Ω_i correspond to triangles forming a non-overlapping decomposition of the domain, and in contrast to Nicolaides, the coarse functions are linear, not constant, on the subdomains. The results for this coarse space and our model problem are

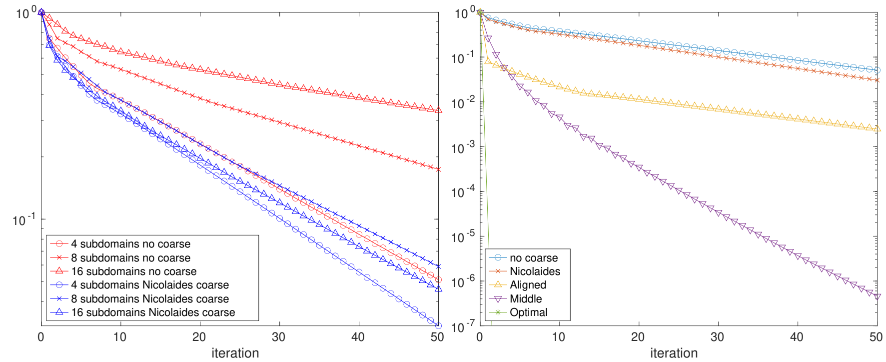


Fig. 3 Error of the parallel Schwarz method with and without piece-wise constant coarse space for increasing number of subdomains (left) and for various coarse spaces and four subdomains (right).

shown in Figure 2 (third row), and we see the coarse space works much better than the Nicolaides coarse space. At the first international conference on domain decomposition methods a year later, Olof Widlund presented an iterative substructuring variant for the piece-wise linear coarse correction on triangles [35], and Max Dryja an extension to three-dimensional problems [9], also in the context of substructuring.

Jan Mandel and Marian Brezina then studied the balancing domain decomposition method in [27]:

> "The Balancing Domain Decomposition (BDD) was introduced by Mandel [1993] by adding a coarse problem to an earlier method of De Roeck and Le Tallec [1991[2]], known as the Neumann-Neumann method . . . "
>
> ". . . a global coarse problem with one or few unknowns for each subdomain . . . "
>
> "The presence of the coarse problem now guarantees that the possibly singular local problems are consistent."

They transformed the bug of the classical Neumann-Neumann method to have floating subdomains with all Neumann conditions around that made the method not well posed into a feature: they determine the constant (in the Laplace case) by a coarse problem, which leads to a piece-wise constant coarse space aligned with the subdomains. The FETI method invented by Charbel Farhat and François-Xavier Roux in [12] also contains naturally the piece-wise constant modes in the projection step as a coarse space; for a theoretical analysis, see [11, 28]. Note that all these coarse spaces were developed independently of the work by Nicolaides.

Max Dryja, Barry Smith and Olof Widlund emphasize in [10] the great importance and challenge of good coarse space constructions:

> " The design, analysis, and implementation of the coarse space problem pose the most challenging technical problems in work of this kind."

They consider several richer coarse spaces than just a constant per substructure and compare them for primal Schur complement substructuring methods. A first

[2] Also at the first international conference on domain decomposition methods!

variant is using piece-wise linear coarse basis functions aligned with triangular substructures, and then additional piece-wise constant edge and face coarse functions are considered, harmonically extended into the subdomains, keeping the vertex functions. For all variants, detailed condition number estimates are provided, and compared to the earlier piece-wise constant coarse space.

We see that all these early coarse spaces were aligned with subdomain boundaries of the domain decomposition method. A generalization of the analysis that permits coarse spaces not aligned with the subdomains, also using ideas from non-overlapping methods, can be found in the book by Andrea Toselli and Olof Widlund [33]:

> " We introduce a shape-regular coarse mesh $\mathcal{T}_H$ on the domain Ω and the finite element space [...] of continuous, piece-wise linear functions on $\mathcal{T}_H$ [...] We stress that the fine mesh $\mathcal{T}$ need not be a refinement of $\mathcal{T}_H$."

Such general coarse spaces were studied at the continuous level in [18] with accurate estimates of the constants involved in the resulting condition number estimate. We show the performance of such a P1 non-aligned coarse space in Figure 2 (fourth row) with coarse points in the middle of the subdomains for our model problem. A comparison of the convergence as a function of the iterations is shown in Figure 3 on the right, where we see that the general position of the coarse points in the middle of the subdomains performs best so far. But is there an even better option?

2 Optimal coarse spaces and spectral approximations

It was first observed in [16] and then analyzed in more detail in [17, 19] that the position of the coarse nodes has indeed an important impact on the performance of the coarse space. For a large scale implementation of various coarse node positionings for Schwarz methods, see [23]. We show in Figure 2 in the last row the performance of a coarse space whose nodes are located to the left and right of the RAS non-overlapping interface. We see that this P1 coarse space transforms the two-level method into a direct solver, the solution is obtained within the subdomains after the coarse correction. This is also visible in the convergence curves in Figure 3 (right). Such coarse spaces are called optimal in the sense of better is not possible, not in the sense of scalable, and the idea is related to the algebraic multigrid construction in [3, 32].

New coarse spaces in domain decomposition methods are approximations of this optimal coarse space; see the Spectral Harmonically Enriched Multiscale (SHEM) coarse space [21, 20] for such a construction in a multiscale context. In higher spatial dimensions, this optimal coarse space simply needs to contain all discrete harmonic functions (functions that solve the homogeneous equation) in each subdomain, and is thus of the size of the number of interface variables of the subdomains. A first approximation for a decomposition into square subdomains is to add the historically successful Q1 functions aligned with each subdomain; see e.g. Figure 4 (left) for one of them. One can then enrich this coarse space by adding harmonically extended

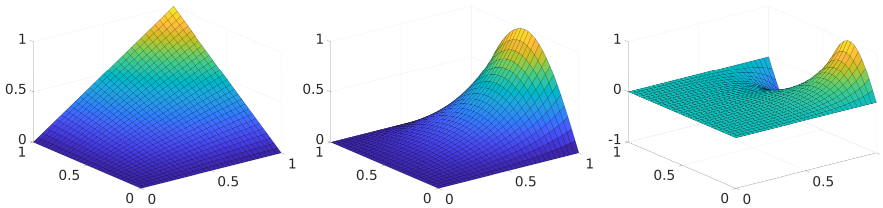
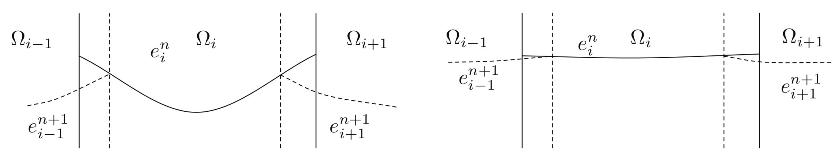


Fig. 4 First Q1 coarse space functions and two spectral enrichments.

sine functions; see Figure 4 (middle and right) to get a spectral coarse space. This construction is not restricted to square subdomains; see [21, 20, 5].

A seemingly different construction of a new coarse space was proposed by Frédéric Nataf, Hua Xiang, Victorita Dolean and Nicole Spillane in [29] for high contrast problems:

> "An effective two-level preconditioner is highly dependent on the choice of the coarse-grid subspace. We will now focus on the choice of the coarse space Z in the context of DDMs for problems of type (1.1) with heterogeneous coefficients."



> "Moreover, a fast decay for this value corresponds to a large eigenvalue of the DtN map, whereas a slow decay corresponds to small eigenvalues of this map because the DtN operator is related to the normal derivative at the interface and the overlap is thin."

From the drawing in their manuscript above, eigenmodes of the Dirichlet-to-Neumann (DtN) map with large eigenvalues will converge fast (left), while eigenmodes with small eigenvalues will converge slowly (right). Hence the idea is to use eigenmodes of the DtN map with small eigenvalues on each subdomain as coarse space. We show in Figure 5 the first four DtN modes for a square subdomain, and also mode 5 and 9. The first four modes look like they span the same space as the four Q1 coarse modes from before. Mode 5 contains a first sine component on the boundary like the enrichment mode in Figure 4 (middle); modes 6-8 (not shown) are similar. Mode 9 contains the second sine mode on the boundary, like the enrichment mode in Figure 4 (right), and modes 10-12 (not shown) are again similar. So the DtN coarse space seems to be related to the SHEM coarse space. This relation becomes even more evident if one uses eigenmodes of the DtN operator computed for each of the four boundaries of the square subdomain separately, since then they coincide with the modes shown in Figure 4 (middle, right)!

A highly successful coarse space, also for high contrast problems, was introduced by Nicole Spillane, Victorita Dolean, Patrice Hauret, Frédéric Nataf,

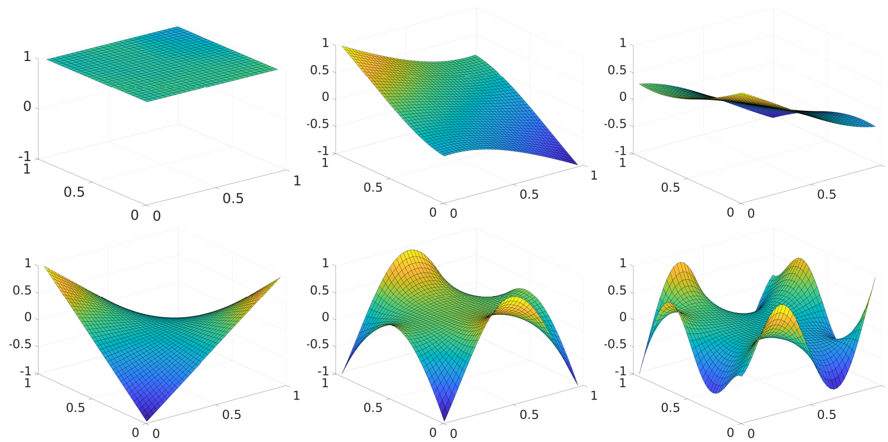


Fig. 5 DtN modes 1-4, 5, and 9.

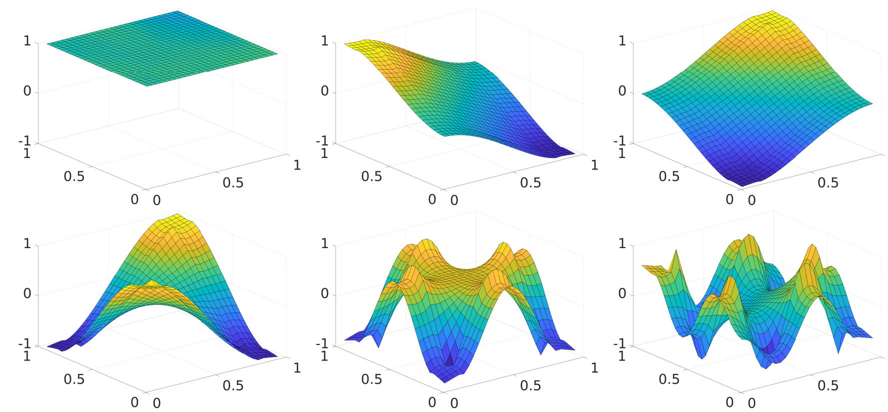


Fig. 6 GenEO modes 1-4, 5, and 9.

Clemens Pechstein and Robert Scheichl in [31], namely GenEO (Generalized Eigenvalue Problems in the Overlaps). The powerful idea of GenEO is to directly improve the Additive Schwarz convergence estimate by adding the corresponding slow modes from the estimate to the coarse space. The modes are also computed in each overlapping subdomain, following [6], by solving the eigenvalue problem

$$B_i \mathbf{u} = \lambda D A_i D \mathbf{u}, \tag{3}$$

where B_i is the Neumann subdomain matrix, A_i is the Dirichlet subdomain matrix, and D is a diagonal weighting matrix representing a partition of unity. We show the first 4 modes, and then also mode 5 and 9 in Figure 6. We see that they are

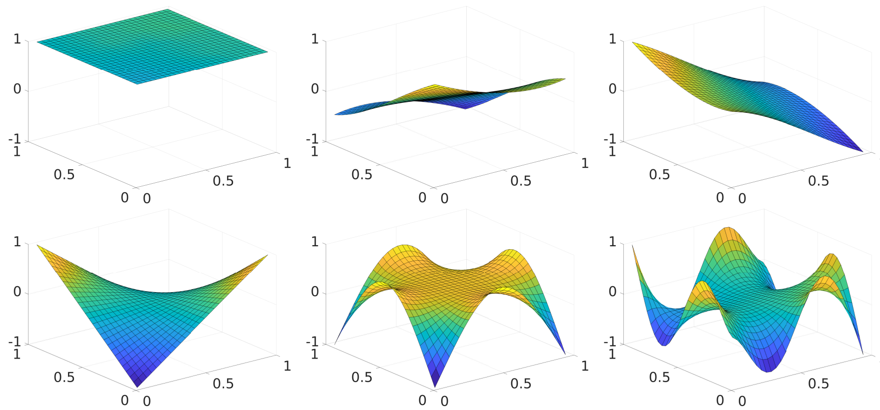


Fig. 7 GenEO modes 1-4, 5, and 9 without the partition of unity.

very similar to the DtN eigenmodes (mode 3, 4 and 9 just need to be multiplied by -1). If we remove the partition of unity in the eigenvalue problem (3), we get the modes shown in Figure 7. These are now very close to the DtN modes (up to multiplications by -1) in Figure 5, and we are working to prove that they in fact span the same coarse space. A comparison of the numerical performance of these coarse spaces can be found in [22]; this comparison was made before these relations were known. In [22], there is also a comparison with the coarse spaces introduced by Juan Galvis and Yalchin Efendiev in [13, 14], which are based on subdomain eigenfunctions in volume and thus not harmonic in the subdomains. Note that such volume eigenvalue coarse spaces have been already introduced for non-overlapping domain decomposition methods by Petter Bjørstad and Piotr Krzyżanowski almost a decade earlier [2], and this in an adaptive fashion (see also [1]):

> "It appears that this paper is the first to propose an adaptive algorithm that can construct an effective coarse space for problems of this kind".

Techniques from multiscale finite element methods were also used to construct coarse spaces for Schwarz methods: the ACMS (Approximate Component Mode Synthesis) coarse space by Alexander Heinlein, Axel Klawonn, Jascha Knepper and Oliver Rheinbach in [25] is using Schur complement eigenvalue problems on subdomain edges in order to construct coarse basis functions. This approach is in the simple Laplace case related to the SHEM enrichment functions shown in Figure 4 in the middle and on the right. The early coarse space from [10] for non-overlapping domain decomposition methods based on piece-wise constant edge (and face) functions became also the basis for a spectrally enriched coarse space under the name adaptive GDSW (Generalized Dryja Smith Widlund) coarse space, see [24], where the authors use for the enrichment Dirichlet to Neumann eigenfunctions at the interfaces, extended harmonically into the subdomains.

3 Conclusions

We gave a short historical and personal introduction to the fascinating research area of coarse space construction for domain decomposition methods. This is currently a very active field of research, and a complete understanding of best coarse spaces in terms of performance even for Laplace problems is only emerging. Corresponding intrinsic coarse space components for Schwarz methods can be found in [5], and their analysis is currently our focus.

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