# An Introduction to Heterogeneous Domain Decomposition Methods for Multi-Physics Problems

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## 1 Mono-Physics and Multi-Physics Problems

In order to understand research problems in multi-physics, it is instructive to first look at coupling conditions for mono-physics problems, which arise naturally in domain decomposition. To do so, we consider the model problem $\mathcal{L}u := (\eta - \Delta)u = f$ in a domain $\Omega$, with suitable boundary conditions. In this case, the solution $u$ we want to compute is well defined, and domain decomposition methods provide two techniques to couple solutions of such mono-physics problems: the first one comes from the alternating Schwarz method [20], see Figure 1 (left) for the historical domain and its decomposition. The alternating Schwarz method solves for $n = 1, 2, \ldots$

$$\mathcal{L}u_1^n = f \text{ in } \Omega_1, \quad u_1^n = u_2^{n-1} \text{ on } \Gamma_1, \qquad \mathcal{L}u_2^n = f \text{ in } \Omega_2, \quad u_2^n = u_1^n \text{ on } \Gamma_2, \tag{1}$$

starting with some $u_2^0$. At convergence, this method defines naturally coupling conditions that involve an overlap, namely the two subdomain solutions must satisfy $u_1 = u_2$ on $\Gamma_1$ and $u_2 = u_1$ on $\Gamma_2$. We thus found the classical *overlapping coupling conditions for second order mono-physics problems*. The second coupling technique comes from historical substructuring methods introduced by Przemieniecki in [19].

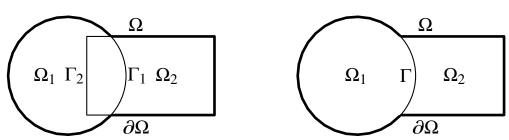


Fig. 1: Left: Schwarz coupling. Right: Przemieniecki or Schur coupling.

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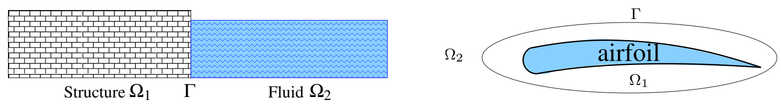


Fig. 2: Left: Fluid-structure interaction. Right: Flow around an airfoil.

For a Schwarz like example, see Figure 1 (right), but now with subdomains that do not overlap, Przemieniecki posed directly the coupled problem with Dirichlet and Neumann conditions,

$$\mathcal{L}u_1 = f \text{ in } \Omega_1, \quad u_1 = u_2 \text{ on } \Gamma, \qquad \mathcal{L}u_2 = f \text{ in } \Omega_2, \quad \partial_n u_2 = \partial_n u_1 \text{ on } \Gamma. \tag{2}$$

He solved it by first assuming the Dirichlet condition to be known, eliminating then the interior unknowns in the subdomains, and finally by imposing the Neumann coupling conditions obtained as equation for the Dirichlet traces, see [12] for more details. The equations on $\Gamma$ in (2) are the classical *non-overlapping coupling conditions for second order mono-physics problems*. These conditions can also be used to obtain an iterative algorithm, namely the Dirichlet-Neumann method, by solving

$$\mathcal{L}u_1^n = f \text{ in } \Omega_1, \quad u_1^n = u_2^{n-1} \text{ on } \Gamma, \qquad \mathcal{L}u_2^n = f \text{ in } \Omega_2, \quad \partial_n u_2^n = \partial_n u_1^n \text{ on } \Gamma, \tag{3}$$

which however needs a relaxation parameter for convergence, see [3, Section 4.7]. This naturally raises the question if these physical coupling conditions are good to obtain rapid convergence. One could consider for example Robin conditions in (3),

$$(\partial_{n_1} + p)u_1^n = (\partial_{n_1} + p)u_2^{n-1} \text{ on } \Gamma_1, \quad (\partial_{n_2} + p)u_2^n = (\partial_{n_2} + p)u_1^n \text{ on } \Gamma_2, \tag{4}$$

where $p$ can be a number, a function or even an operator as advocated by Lions [18]. One can now use both overlapping and non-overlapping ($\Gamma_1 = \Gamma_2 = \Gamma$) configurations, since the Robin conditions imply the coupling conditions on $\Gamma$ in (2) at convergence (just take the sum and difference of the Robin conditions). We call the Robin conditions *transmission conditions*, since they must transmit information as effectively as possible for fast convergence, a research field that led to optimized Schwarz methods, see [7] for an introduction.

For Multi-Physics Problems, we have to distinguish two situations. The first one is where the physics is truly different in different regions, as for example in fluid structure interaction, see Figure 2 (left). Here the solution $u$ we want to compute is also well defined, the coupling conditions are given by the physics of the problem along the interface $\Gamma$ between the fluid and the structure, and only non-overlapping techniques make sense. Once good physical coupling conditions are found, the question is what are good transmission conditions for fast convergence when one solves alternatingly the structure and fluid problems in $\Omega_1$ and $\Omega_2$, and which imply the coupling conditions on $\Gamma$ at convergence.

The second situation is when in principle we have a mono-physics problem in $\Omega$, but different physical models are used in different regions for computational savings, see Figure 2 (left) for a flow around an airfoil. Here one wants to use an expensive model only where it is necessary, in $\Omega_1$ close to the airfoil, and far away a cheaper model suffices in $\Omega_2$ to save computation time. The question is then what are good coupling conditions to get close to the expensive solution everywhere. For this one can consider non-overlapping techniques with one interface $\Gamma$, or also overlapping ones with two interfaces $\Gamma_1$ and $\Gamma_2$ at an overlapping distance, see Figure 1. Once good coupling conditions are found, again the question arises on what are good transmission conditions for fast iterative convergence when solving alternatingly on $\Omega_1$ the expensive and on $\Omega_2$ the cheap model, which also imply the good coupling conditions at convergence.

## 2 Truly Multi-Physics Problems

A typical example of a truly multi-physics problem can be found in [4]:

> "In a second circumstance, one may be obliged to consider truly different models to account for the presence of distinct physical problems within the same global domain. This case is usually indicated as multi-physics or multi-field problem."

The problem considered is the deformation of an artery. The fluid equations for the velocity field $\mathbf{u}$ and the pressure $p$ are

$$\begin{aligned} \rho_{\mathrm{f}}\left(\left.\frac{\partial \boldsymbol{u}}{\partial t}\right|_{x_0}+(\boldsymbol{u}-\boldsymbol{w}^{\mathrm{f}})\cdot\boldsymbol{\nabla}\boldsymbol{u}\right)-\operatorname{div}[\boldsymbol{\sigma}_{\mathrm{f}}(\boldsymbol{u},p)] &= \boldsymbol{f}_{\mathrm{f}} \quad \text{in } \Omega^{\mathrm{f}}(t),\\ \operatorname{div}\boldsymbol{u} &= 0 \quad \text{in } \Omega^{\mathrm{f}}(t),\\ \boldsymbol{u}=\boldsymbol{u}_{\mathrm{in}} \ \text{ on } \Gamma^{\mathrm{in}}(t), \quad \boldsymbol{\sigma}_{\mathrm{f}}(\boldsymbol{u},p)\cdot\boldsymbol{n}_{\mathrm{f}} &= \boldsymbol{g}_{\mathrm{f}} \quad \text{on } \Gamma^{\mathrm{out}}(t). \end{aligned}$$

The solid equations for the displacement $\mathbf{d}^s$ are

$$\begin{aligned} \rho_{\mathrm{s}}\frac{\partial^2\boldsymbol{d}^{\mathrm{s}}}{\partial t^2}-\operatorname{div}_{|x_0}(\boldsymbol{\sigma}_{\mathrm{s}}(\boldsymbol{d}^{\mathrm{s}})) &= \boldsymbol{f}_{\mathrm{s}} \quad \text{in } \Omega_0^{\mathrm{s}},\\ \boldsymbol{\sigma}_{\mathrm{s}}(\boldsymbol{d}^{\mathrm{s}})\cdot\boldsymbol{n}_{\mathrm{s}} &= \boldsymbol{g}_{\mathrm{s}} \quad \text{on } \partial\Omega_0^{\mathrm{s}}\setminus\Gamma_0, \end{aligned}$$

and the physical coupling conditions are ("Dirichlet" and "Neumann")

$$\begin{aligned} \boldsymbol{x}_t^{\mathrm{s}} = x_0+\lambda = \boldsymbol{x}_t^{\mathrm{f}}, &\qquad \boldsymbol{u}\circ\boldsymbol{x}_t^{\mathrm{f}} = \frac{\partial\lambda}{\partial t},\\ (\boldsymbol{\sigma}_{\mathrm{f}}(\boldsymbol{u},p)\cdot\boldsymbol{n}_{\mathrm{f}})\circ\boldsymbol{x}_t^{\mathrm{f}} &= -\boldsymbol{\sigma}_{\mathrm{s}}(\boldsymbol{d}^{\mathrm{s}})\cdot\boldsymbol{n}_{\mathrm{s}}, \end{aligned}$$

imposing the matching of the interface displacements from the fluid and solid subdomains, the continuity of the velocities and the normal stresses. The authors propose to use directly these coupling conditions also as transmission conditions and study the following methods:

*Dirichlet-Neumann (DN):* $P_k = P_{DN} = S_{\mathrm{s}}'(\lambda^k)$, for $\alpha_{\mathrm{f}}^k = 0$, $\alpha_{\mathrm{s}}^k = 1$,
*Neumann-Dirichlet (ND):* $P_k = P_{ND} = S_{\mathrm{f}}'(\lambda^k)$, for $\alpha_{\mathrm{f}}^k = 1$, $\alpha_{\mathrm{s}}^k = 0$,
*Neumann-Neumann (NN):* $P_k = P_{NN}$ with $\alpha_{\mathrm{f}}^k + \alpha_{\mathrm{s}}^k = 1$, $\alpha_{\mathrm{f}}^k, \alpha_{\mathrm{s}}^k \neq 0$.

Our interest for such multiphysics problems mainly focused on developing transmission conditions for fast convergence. In [8] we developed a non-overlapping optimized Schwarz method for jumping coefficient diffusion problems that converges independently of the mesh parameter, and faster and faster the bigger the jump becomes: the method truly benefits from the multi-physics nature of the problem. In [13] we designed and studied heterogeneous optimized Schwarz methods for coupling Helmholtz and Laplace equations; for more general elliptic problems, see [14, 15], for Stokes-Darcy see [16], and for a new technique to automatically obtain such transmission conditions through probing, see [11].

## 3 Multi-Physics Problems for Computational Savings

The classical mathematical approach for such problems is the technique of *matched asymptotic expansions*. For the model problem of advection diffusion with $a > 0$,

$$\nu\partial_{xx}u + a\partial_x u - \eta u = 0 \quad \text{in } (0,1), \quad u(0) = 0,\ u(1) = 1, \tag{5}$$

a regular expansion for $\nu$ small, $u = u_0 + \nu u_1 + \ldots$, gives

$$a\partial_x u_0 - \eta u_0 + \nu(\partial_{xx}u_0 + a\partial_x u_1 - \eta u_1) + \ldots = 0,$$

and therefore by matching terms $u_0(x) = e^{\frac{\eta}{a}(x-1)}$, $u_1(x) = -\frac{\eta^2}{a^3}e^{\frac{\eta}{a}(x-1)}(x-1)$ etc, which can not capture the zero boundary condition at $x = 0$. One thus introduces the stretched variable $x := \epsilon\xi$, $\tilde{u}(\xi) := u(\epsilon\xi)$, which yields $\partial_\xi\tilde{u}(\xi) = \partial_x u(\epsilon\xi)\epsilon$ and

$$\frac{\nu}{\epsilon^2}\partial_{\xi\xi}\tilde{u} + \frac{a}{\epsilon}\partial_x\tilde{u} - \eta\tilde{u} = 0 \quad \Longrightarrow \quad \partial_{\xi\xi}\tilde{u} + a\partial_x\tilde{u} - \nu\eta\tilde{u} = 0,$$

where we multiplied by $\epsilon$ and choose $\epsilon := \nu$. A regular expansion for $\nu$ small, $\tilde{u} = \tilde{u}_0 + \nu\tilde{u}_1 + \ldots$ now gives for the first term with $\tilde{u}_0(0) = 0$

$$\partial_{\xi\xi}\tilde{u}_0 + a\partial_\xi\tilde{u}_0 = 0 \quad \Longrightarrow \quad \tilde{u}_0(\xi) = C(e^{-a\xi} - 1).$$

The constant $C$ is then determined by asymptotic matching, $\lim_{\xi=\infty}\tilde{u}_0(\xi) = -C = e^{-\frac{\eta}{a}} = \lim_{x=0}u_0(x)$. We obtain the inner and outer solutions $\tilde{u}_0(x) = e^{-\frac{\eta}{a}}(1-e^{-\frac{ax}{\nu}})$, $u_0(x) = e^{\frac{\eta}{a}(x-1)}$, and the composite solution by summation, and subtraction of the common limit,

$$u_0^a(x) = -e^{-\frac{\eta}{a}}e^{-\frac{ax}{\nu}} + e^{\frac{\eta}{a}(x-1)}.$$

The exact solution of the problem is $u(x) = \frac{e^{\lambda_1 x} - e^{\lambda_2 x}}{e^{\lambda_1} - e^{\lambda_2}}$, with $\lambda_1 := \frac{-a+\sqrt{a^2+4\eta\nu}}{2\nu} = \frac{\eta}{a} - \frac{\eta^2}{a^3}\nu + O(\nu^2)$, $\lambda_2 := \frac{-a-\sqrt{a^2+4\eta\nu}}{2\nu} = -\frac{a}{\nu} - \frac{\eta}{a} + \frac{\eta^2}{a^3}\nu + O(\nu^2)$. We show in Figure 3 (left) the difference between the matched asymptotic solution and the exact one. Using asymptotic analysis, one can show

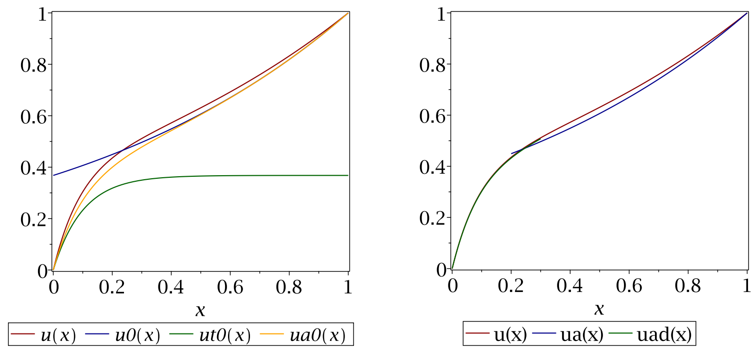


Fig. 3: Example for $a = 1$, $\eta = 1$, $\nu = 0.1$. Left: matched asymptotic expansions. Right: overlapping coupling from [5] with overlap $\gamma_{1,2} = (0.2, 0.3)$.

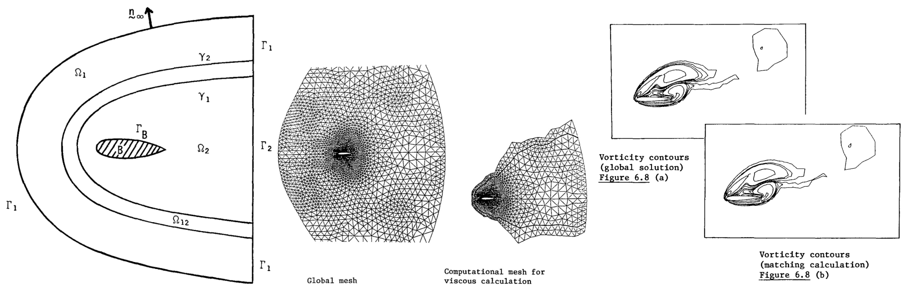


Fig. 4: Navier-Stokes and potential flow coupling from [5].

**Proposition 1 (Matched asymptotic expansion)** *For $\nu$ small, the matched asymptotic expansion approximation satisfies for $x = O(\nu^\alpha)$ the error estimate*

$$\|u - u_0^a\|_{L^\infty(0,x)} = \begin{cases} O(\nu) & 0 < \alpha < 1, \\ O(\nu^\alpha) & \alpha \geq 1. \end{cases} \tag{6}$$

An *optimal control method* for such a coupled solution was given in [5]:

> The main goal of this paper is to present a computational method for the coupling of two distinct mathematical models describing the same physical phenomenon, namely the flow of an *incompressible viscous fluid*. The basic idea is to replace the Navier-Stokes equations by the potential one in those regions where we can neglect the viscous effects and where the vorticity is small.

In order to achieve this coupling, the authors use an overlapping decomposition as in Figure 4 (left), and then impose on the Navier-Stokes equation the velocity $\mathbf{u} = \mathbf{v}$ on the interface $\gamma_1$, and on the potential flow the Dirichlet condition $\phi = \psi$ on the interface $\gamma_2$. They then determine $\mathbf{v}$ and $\psi$ that minimize the functional

$$J(\mathbf{v}, \psi) := \frac{1}{2} \int_{\Omega_{12}} |\mathbf{u} - \nabla\phi|^2,$$

where $\mathbf{u}$ solves the Navier-Stokes equation in $\Omega_2$ and $\phi$ the potential equation in $\Omega_1$. For the model problem of the matched asymptotic expansion we solve for $\gamma_2 < \gamma_1$

$$\nu\partial_{xx}u_{ad} + a\partial_x u_{ad} - \eta u_{ad} = 0 \text{ in } (0,\gamma_1), \quad a\partial_x u_a - \eta u_a = 0 \text{ in } (\gamma_2, 1),$$

$u_{ad}(0) = 0$, $u_a(1) = 1$, and $u_{ad}(\gamma_1) = \psi$ with $\psi$ which minimizes the norm $\|u_{ad} - u_a\|_{L^2(\gamma_1,\gamma_2)}$. We get with this approach the results shown in Figure 3 (right). Using asymptotic analysis, we obtain

**Proposition 2 (Advection error estimate, valid for all methods)** *At $x = O(\nu^\alpha)$, the advection approximation satisfies the error estimate*

$$u(x) - u_a(x) = \begin{cases} O(\nu) & 0 \le \alpha < 1, \\ O(1) & \alpha \ge 1. \end{cases} \tag{7}$$

**Proposition 3 (Optimal control method)** *For $\gamma_1 = O(\nu^\alpha)$, $\alpha \ge 0$, the optimal control method satisfies for the advection diffusion approximation the error estimate*

$$\|u - u_{ad}\|_{L^\infty(0,\gamma_1)} = \begin{cases} O(\nu) & 0 \le \alpha < 1, \\ O(\nu^{1-\beta}) & \alpha \ge 1, \gamma_2 = O(\nu^\beta), 0 \le \beta < 1, \\ O(1) & \textit{otherwise}. \end{cases} \tag{8}$$

The *$\chi$-method* from [1] is a different such coupling method. The idea is to add in the advection reaction diffusion equation a cut-function for the diffusion,

$$-\nu\chi_\delta(\Delta u) + \mathbf{a}\cdot\nabla u + bu = f \quad \text{in } \Omega, \quad \chi_\delta(s) := \begin{cases} 0, & |s| \le \delta, \\ s, & |s| > \delta. \end{cases}$$

The authors say: “We remark that the perturbed equation is at least as difficult to solve as the imperturbed equation”, but conceptually think it is better to solve the same equation on the entire domain. For our model problem, we obtain the results in Figure 5 (left). Using asymptotic analysis, one can show

**Proposition 4 ($\chi$-method)** *For $\delta = O(\nu^{-\alpha})$, $\alpha \ge 0$, the $\chi-$method for* (5) *satisfies in the advection diffusion region the error estimate*

$$\|u - u_{ad}\|_{L^\infty(0,\gamma_1)} = \begin{cases} O(\nu) & 0 \le \alpha < 1, \\ O(\nu^{2-\alpha}) & 1 \le \alpha \le 2, \\ O(1) & \alpha > 2. \end{cases} \tag{9}$$

A *non-overlapping DD coupling technique* ($\gamma := \gamma_1 = \gamma_2$) was proposed in [17]:

> “We deal with the coupling of hyperbolic and parabolic systems in a domain $\Omega$ divided into two disjoint subdomains $\Omega^+$ and $\Omega^-$ [. . . ] The justification of the interface conditions is based on a singular perturbation analysis, that is, the hyperbolic system is rendered parabolic by adding a small artificial “viscosity”. As this goes to zero, the coupled parabolic-parabolic problem degenerates into the original one, yielding some conditions at the interface. These we take as interface conditions for the hyperbolic-parabolic problem. Actually, we discuss two alternative sets of interface conditions according to whether the regularization procedure is variational or nonvariational.”

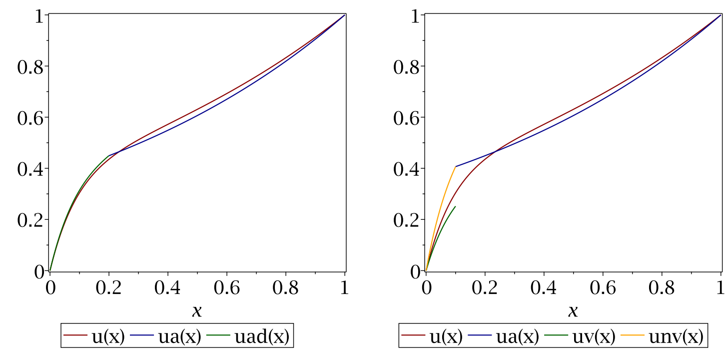


Fig. 5: Example for $a = \eta = 1$, $\nu = 0.1$. Left: $\mathcal{X}$-method with $\delta = 5$. Right: variational coupling with $L = 0.1$.

For our model problem, the variational condition is $(\nu\partial_x + a)u^v_{ad}(\gamma) = au_a(\gamma)$, and the non-variational condition is $u^{nv}_{ad}(\gamma) = u_a(\gamma)$. We show in Figure 5 (right) a computational result for the variational and non-variational conditions.

**Proposition 5 (DD coupling technique)** *For $\gamma = O(\nu^\alpha)$, $\alpha \geq 0$, we obtain in the advection diffusion region for the variational and non-variational approach*

$$\|u - u^v_{ad}\|_{L^\infty(0,\gamma)} = \begin{cases} O(\nu) & 0 \leq \alpha < 1, \\ O(\nu^\alpha) & \alpha \geq 1, \end{cases} \quad \|u - u^{nv}_{ad}\|_{L^\infty(0,\gamma)} = \begin{cases} O(\nu) & 0 \leq \alpha < 1, \\ O(1) & \alpha \geq 1. \end{cases} \tag{10}$$

In the PhD thesis [6], a fundamental new *optimization based method* was introduced:

> "L'objectif est alors d'essayer des conditions de transmission adéquates à la frontière de façon à minimiser l'erreur entre la solution du problème de transmission et celle de Navier Stokes complet dans tout le domaine."

The new idea is to find coupling conditions s.t. $||u - u_{\text{approx}}|| \longrightarrow$ min! Based on absorbing boundary condition techniques, this gives variational coupling conditions for our model problem, and non-variational ones for the inverse flow direction ($a < 0$). We introduced in [9, 10] a method based on the factorization $-\nu(\partial_x - \lambda_2)(\partial_x - \lambda_1)u = 0$, $\lambda_1 \geq 0$, $\lambda_2 \leq 0$. The idea consists in first solving the modified advection equation $u'_{\text{ma}} - \lambda_1 u_{\text{ma}} = 0$ on $(\gamma, 1)$ with the boundary condition $u_{\text{ma}}(1) = g$ where $g$ is an approximation at order $m$ of a function of $u(1)$ and $u'(1)$. We solve then the advection-diffusion equation with the boundary condition $(-\nu(u^{\text{fact}}_{\text{ad}})' + \nu\lambda_2 u^{\text{fact}}_{\text{ad}})(\gamma) = au_{\text{ma}}(\gamma)$ and we obtain the following error estimate:

**Proposition 6 (Factorization method)** *For $\gamma = O(\nu^\alpha)$, $\alpha \geq 0$,*

$$\|u - u^{\text{fact}}_{\text{ad}}\|_{L^\infty(0,\gamma)} = \begin{cases} O(\nu^m) & 0 \leq \alpha < 1, \\ O(\nu^{m+\alpha-1}) & \alpha \geq 1, \end{cases}$$

A further technique using partition of unity methods can be found in [2].

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