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Overlapping Schwarz for Linear and Nonlinear Parabolic Problems

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1.1 Introduction

The basic ideas underlying waveform relaxation were first suggested in the late 19th century by Picard and Lindelöf ([Lin94], [Lin93]). However much recent interest in waveform relaxation as a practical parallel method for the solution of stiff ordinary differential equations (ODE's) has been generated after the publication of a paper by Lelarasmee and coworkers [LRSV82] in the VLSI literature, and the paper by O'Leary and White [OW85] which introduced multi-splittings of matrices for the solution of linear systems of equations. Recent work in this field includes papers by Miekkala and Nevanlinna [MN87a], [MN87b], Nevanlinna [Nev89a], [Nev89b] Bellen and Zennaro [BZ93] and Jeltsch and Pohl [JP95].

The standard convergence result for a system of nonlinear ODE's needs the assumption that the splitting function is Lipschitz continuous in both arguments. It states superlinear convergence on any finite time interval $[0, T]$. This result can be found for example in Bjørhus [Bjø95a].

For a linear system of ODE's which is asymptotically stable Miekkala and Nevanlinna show in [MN87a] the existence of splittings such that the waveform relaxation algorithm converges linearly on the infinite time interval $[0, \infty)$. Jeltsch and Pohl [JP95] extend the work of [MN87a] to prove superlinear convergence of certain overlapping splittings on bounded time intervals. They also extend the results on unbounded time intervals to overlapping splittings for a certain class of problems.

However in all the results mentioned the constants in general depend badly on Δx if the linear ODE arises from a partial differential equation (PDE) which is discretized in space.

Motivated by the work of Bjørhus [Bjø95b], we show how one can use overlapping domain decomposition to obtain a waveform relaxation algorithm for the semi-discrete heat equation which converges at a rate independent of the mesh parameter Δx. The details of the analysis can be found in [GS96].

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1.2 Continuous Case

Consider the one dimensional inhomogeneous heat equation on the interval $[0, L]$,

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + f(x,t) && 0 < x < L,\ t > 0 \\ u(0,t) &= g_1(t) && t > 0 \\ u(L,t) &= g_2(t) && t > 0 \\ u(x,0) &= u_0(x) && 0 < x < L, \end{aligned} \tag{1.1}$$

where we assume enough smoothness on the data such that (1.1) has a unique bounded solution [Can84]. Given any function $f(t) : \mathbb{R}^+ \longrightarrow \mathbb{R}$ we define

$$||f(\cdot)||_\infty := \sup_{t>0} |f(t)|$$

We decompose the domain $\Omega = [0, L] \times [0, \infty)$ into two overlapping subdomains $\Omega_1 = [0, \beta L] \times [0, \infty)$ and $\Omega_2 = [\alpha L, L] \times [0, \infty)$ where $0 < \alpha < \beta < 1$. The solution $u(x, t)$ of (1.1) can now be obtained by composing the solutions $v(x, t)$ on Ω_1 and $w(x, t)$ on Ω_2, which satisfy the same inhomogeneous heat equation on the subdomains with the new interior boundary conditions $v(\beta L, t) = w(\beta L, t)$ and $w(\alpha L, t) = v(\alpha L, t)$ respectively. Note that $v(x, t) \equiv w(x, t)$ in the overlap. The system, which is coupled through the boundary, can be solved using an alternating Schwarz iteration, where the new function $v^{k+1}(x, t)$ on Ω_1 is obtained using the previous iterate $w^k(x, t)$ at the interior boundary and similarly on Ω_2. Let $d^k(x, t) := v^k(x, t) - v(x, t)$ and $e^k(x, t) := w^k(x, t) - w(x, t)$ and consider the error equations

$$\begin{aligned} \frac{\partial d^{k+1}}{\partial t} &= \frac{\partial^2 d^{k+1}}{\partial x^2} && 0 < x < \beta L,\ t > 0 \\ d^{k+1}(0,t) &= 0 && t > 0 \\ d^{k+1}(\beta L,t) &= e^k(\beta L,t) && t > 0 \\ d^{k+1}(x,0) &= 0 && 0 < x < \beta L \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} \frac{\partial e^{k+1}}{\partial t} &= \frac{\partial^2 e^{k+1}}{\partial x^2} && \alpha L < x < L,\ t > 0 \\ e^{k+1}(\alpha L,t) &= d^k(\alpha L,t) && t > 0 \\ e^{k+1}(L,t) &= 0 && t > 0 \\ e^{k+1}(x,0) &= 0 && \alpha L < x < L. \end{aligned} \tag{1.3}$$

Given any function $g(x, t) : [a, b] \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ we define

$$||g(\cdot,\cdot)||_{\infty,\infty} := \sup_{a<x<b,t>0} |g(x,t)|$$

Theorem 1.2.1 *The Schwarz iteration for the heat equation with two subdomains converges at a rate depending on the size of the overlap. The error on the two subdomains decays at the rate*

$$||d^{2k+1}(\cdot,\cdot)||_{\infty,\infty} \leq \left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^k ||e^0(\beta L,\cdot)||_\infty \tag{1.4}$$

$$||e^{2k+1}(\cdot,\cdot)||_{\infty,\infty} \leq \left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^k ||d^0(\alpha L,\cdot)||_\infty. \tag{1.5}$$

Proof The proof is obtained using the maximum principle of the heat equation and can be found in [GS96]. ■

1.3 Semi-Discrete Case

Consider the heat equation continuous in time, but discretized in space using a centered second order finite difference scheme on a grid with n grid points and $\Delta x = \frac{L}{n+1}$. This gives

$$\begin{array}{rcll} \frac{\partial \boldsymbol{u}}{\partial t} & = & A_{(n)}\boldsymbol{u} + \boldsymbol{f}(t) & \quad t > 0 \\ \boldsymbol{u}(0) & = & \boldsymbol{u}_0, & \end{array} \tag{1.6}$$

where the $n \times n$ matrix $A_{(n)}$ is given by

$$A_{(n)} = \frac{1}{(\Delta x)^2}\begin{bmatrix} -2 & 1 & & 0 \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -2 \end{bmatrix} \tag{1.7}$$

and $\boldsymbol{f}(t) = (f(\Delta x, t) + \frac{1}{(\Delta x)^2}g_1(t), f(2\Delta x, t), \ldots, f((n-1)\Delta x, t), f(n\Delta x, t) + \frac{1}{(\Delta x)^2}g_2(t))^T$, $\boldsymbol{u}_0 = (u_0(\Delta x), \ldots, u_0(n\Delta x))^T$.

We decompose the domain into two overlapping subdomains Ω_1 and Ω_2 as in figure 1.1. We assume for simplicity that αL falls on the grid point $i = a$ and βL on the grid

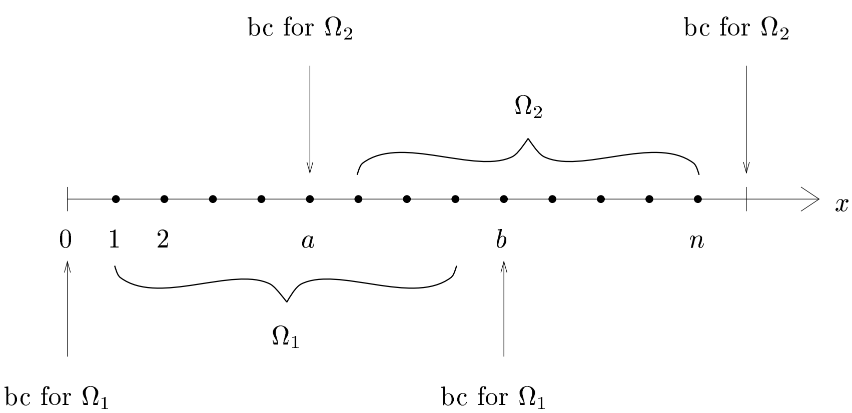

Figure 1.1 Decomposition in the semi-discrete case.

point $i = b$. We therefore have $a\Delta x = \alpha L$ and $b\Delta x = \beta L$. As in the continuous case, the solution $\boldsymbol{u}(t)$ of (1.6) can be obtained by composing the solutions $\boldsymbol{v}(t)$ on Ω_1 and $\boldsymbol{w}(t)$ on Ω_2, which satisfy the corresponding equations on the subdomains. Applying a Schwarz iteration as in the continuous case one obtains the error equations

$$\begin{array}{rcll} \frac{\partial \boldsymbol{d}^{k+1}}{\partial t} & = & A_{(b-1)}\boldsymbol{d}^{k+1} + \boldsymbol{f}^{(e^k)} & \quad t > 0 \\ \boldsymbol{d}^{k+1}(0) & = & \boldsymbol{0} & \end{array} \tag{1.8}$$

with $\boldsymbol{f}^{(e^k)} = (0,\ldots,0,\frac{1}{(\Delta x)^2}\boldsymbol{e}^k(b-a,t))^T$ and

$$\begin{aligned} \frac{\partial \boldsymbol{e}^{k+1}}{\partial t} &= A_{(n-a)}\boldsymbol{e}^{k+1} + \boldsymbol{f}^{(d^k)} \qquad t>0 \\ \boldsymbol{e}^{k+1}(0) &= \boldsymbol{0} \end{aligned} \tag{1.9}$$

with $\boldsymbol{f}^{(d^k)} = (\frac{1}{(\Delta x)^2}\boldsymbol{d}^k(a,t),0,\ldots,0)^T$.

Given any vector valued function $\boldsymbol{h}(t) : \mathbb{R}^+ \longrightarrow \mathbb{R}^n$ we define

$$||\boldsymbol{h}(\cdot,\cdot)||_{\infty,\infty} := \max_{1<j<n}\sup_{t>0}|\boldsymbol{h}(j,t)|$$

where $\boldsymbol{h}(j,t)$ denotes the j-th component of the vector $\boldsymbol{h}(t)$.

Theorem 1.3.1 *The Schwarz iteration for the semi-discrete heat equation with two subdomains converges at a rate depending on the size of the overlap. The error on the two subdomains decays at the rate*

$$\begin{aligned} ||\boldsymbol{d}^{2k+1}(\cdot,\cdot)||_{\infty,\infty} &\leq \left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^k ||\boldsymbol{e}^0(b-a,\cdot)||_\infty \\ ||\boldsymbol{e}^{2k+1}(\cdot,\cdot)||_{\infty,\infty} &\leq \left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^k ||\boldsymbol{d}^0(a,\cdot)||_\infty. \end{aligned}$$

Proof The proof uses the discrete maximum principle and can be found in [GS96]. ■

The results shown for two subdomains can be generalized to an arbitrary number of subdomains, although the analysis is more involved. The theorems corresponding to Theorem 1.2.1 and 1.3.1 can be found in [GS96].

1.4 The Algorithm in the Framework of Waveform Relaxation

For a linear initial value problem

$$\frac{\partial \boldsymbol{u}(t)}{\partial t} = A\boldsymbol{u}(t) + \boldsymbol{f}(t), \quad \boldsymbol{u}(0) = \boldsymbol{u}_0$$

the standard waveform relaxation algorithm is based on a splitting of the matrix A into $A = M + N$ which yields

$$\frac{\partial \boldsymbol{u}(t)}{\partial t} = M\boldsymbol{u}(t) + N\boldsymbol{u}(t) + \boldsymbol{f}(t), \quad \boldsymbol{u}(0) = \boldsymbol{u}_0.$$

This system of ODE's is solved using an iteration of the form

$$\frac{\partial \boldsymbol{v}^{k+1}}{\partial t} = M\boldsymbol{v}^{k+1} + N\boldsymbol{v}^k + \boldsymbol{f}, \quad \boldsymbol{v}^{k+1}(0) = \boldsymbol{u}_0. \tag{1.10}$$

where the starting function $\boldsymbol{v}^0(t)$ is chosen as a constant function $\boldsymbol{v}^0(t) = \boldsymbol{u}_0$. In the case of Block-Jacobi the matrix M is chosen to be block diagonal, for example for two subblocks

$$M = \begin{bmatrix} D_1 & \\ & D_2 \end{bmatrix}, \tag{1.11}$$

and N contains the remaining off diagonal blocks. This allows for solution of the subsystems D_i, $i = 1, 2$ in equation (1.10) in parallel. In the case where A equals $A_{(n)}$ from the semi-discrete heat equation (1.6), the waveform relaxation algorithm with Block-Jacobi splitting computes the same iterates as the Schwarz domain decomposition algorithm presented in subsection 1.3 with overlap Δx (i.e. one grid point only). This result can be generalized to an arbitrary number of subdomains, as shown in [GS96].

To extend this analogy to arbitrary overlaps, the concept of *multi-splittings* is needed, which was first introduced by O'Leary and White in [OW85] for solving large systems of linear equations on a parallel computer. The idea was generalized to nonlinear problems by White in [Whi86]. Jeltsch and Pohl generalized multi-splittings to linear systems of ODE's and waveform relaxation in [JP95].

Let A, M_i, N_i and E_i, $i = 1, 2$ be real $n \times n$ matrices. The set of ordered triples (M_i, N_i, E_i) for $i = 1, 2$ is called a *multi-splitting* of A if

1. $A = M_i - N_i$ for $i = 1, 2$
2. The matrices E_l are nonnegative diagonal matrices and satisfy

$$E_1 + E_2 = I. \tag{1.12}$$

Using the waveform relaxation algorithm, we get two new approximations $\boldsymbol{v}_1^{k+1}$ and $\boldsymbol{v}_2^{k+1}$ at each step according to

$$\frac{\partial \boldsymbol{v}_i^{k+1}}{\partial t} = M_i \boldsymbol{v}_i^{k+1}(t) + N_i \boldsymbol{v}_i^k + \boldsymbol{f}_i, \quad \boldsymbol{v}_i^{k+1}(0) = \boldsymbol{u}_0,\ i = 1, 2 \tag{1.13}$$

which are combined using the matrices E_i to form a new approximation $\boldsymbol{v}^{k+1}$ by $\boldsymbol{v}^{k+1} = E_1 \boldsymbol{v}_1^{k+1} + E_2 \boldsymbol{v}_2^{k+1}$. Note that the two equations in (1.13) can be solved in parallel and in addition, components of $\boldsymbol{v}_i^{k+1}$ where E_i has a zero on the diagonal do not have to be computed at all provided they do not couple to other components of $\boldsymbol{v}_i^{k+1}$ where E_i has a non zero diagonal entry. Jeltsch and Pohl prove in [JP95] that the multi-splitting algorithm converges superlinearly on a finite time interval $[0, T]$ for all splittings and matrices A, and on an infinite time interval linearly if A is an M-matrix and the splitting is an M-splitting. However in the case of the semi-discrete heat equation, the rate of convergence in their analysis may depend badly on Δx since their level of generality includes the Schwarz method with one grid point overlap and spectral radius $1 - O(\Delta x^2)$ - the block Jacobi algorithm (1.11). Jeltsch and Pohl also observe, on the basis of numerical experiments, that some overlap appears to be beneficial. Using the analysis described here we are able to substantiate and quantify this observation in the specific case of the heat equation. Consider the E_i chosen in such a way that the domain decomposition algorithm described in the previous section is recovered. This can be obtained by choosing the two splittings of A according to the two subdomains of the domain decomposition and letting E_i have the value one on the diagonal in the interior of the corresponding subdomain Ω_i, including the first point of the overlap, some arbitrary distribution in the overlap satisfying (1.12) and zero in the interior of the other subdomain. Then the intermediate solutions $\boldsymbol{v}_i^{k+1}$ computed by the multi-splitting algorithm for the heat equation are identical to the solutions computed by the domain decomposition algorithm described in the previous section. Thus, in this case, multi-splitting gives a Δx independent rate of convergence.

Note that one could save half of the computation time by computing only even iterates on Ω_1 and odd iterates on Ω_2 or vice versa, since these two solution sequences are independent of one another. In the terminology of Domain Decomposition this would correspond to the multiplicative Schwarz algorithm with red black ordering whereas the multi-splitting algorithm corresponds to the additive Schwarz algorithm.

The important point here is that our algorithm converges linearly independent of the mesh size on unbounded time intervals. Hence the multi-splitting algorithm for the semi-discrete heat equation, which computes identical iterates, must converge at the same rate. Thus for certain PDE's the analysis of Jeltsch and Pohl can be refined to give Δx independent rates of convergence if sufficient overlap is used.

Although the results are proved for linear problems, an experiment in the next section shows that similar results may be proved for some nonlinear problems.

1.5 Numerical Experiments

We perform numerical experiments to measure the actual convergence rate of the algorithm. We consider first the linear example problem

$$\begin{array}{rcll} \frac{\partial u}{\partial t} & = & \frac{\partial^2 u}{\partial x^2} + 5e^{-(t-2)^2-(x-\frac{1}{4})^2} & 0 < x < 1,\ 0 < t < 3 \\ u(0,t) & = & 0 & 0 < t < 3 \\ u(1,t) & = & e^{-t} & 0 < t < 3 \\ u(x,0) & = & x^2 & 0 < x < 1. \end{array} \tag{1.14}$$

To solve the semi-discrete heat equation, we use the backward euler method in time. The experiment is done splitting the domain $\Omega = [0,1] \times [0,3]$ into the two subdomains $\Omega_1 = [0,\alpha] \times [0,3]$ and $\Omega_2 = [\beta,1] \times [0,3]$ for three pairs of values $(\alpha,\beta) \in \{(0.4,0.6),(0.45,0.55),(0.48,0.52)\}$. As initial guess for the iteration we use the constant value one. Figure 1.2 shows the convergence of the algorithm on the grid point b for $\Delta x = 0.01$ and $\Delta t = 0.01$. The solid line is the predicted bound on the convergence rate according to Theorem 1.3.1 and the dashed line is the measured one. The measured error displayed is the difference between the numerical solution on the whole domain and the solution obtained from the domain decomposition algorithm. We also checked the robustness of the method by refining the time step and obtained similar results.

Now consider the nonlinear example problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 15(u - u^3) \qquad 0 < x < 1,\ 0 < t < 3 \tag{1.15}$$

with the same initial and boundary conditions as in the linear case. We discretize in space as before and use the backward Euler method in time for the Laplacian, keeping the nonlinear part explicit. Figure 1.3 shows the convergence of the algorithm on the grid point b for $\Delta x = 0.01$ and $\Delta t = 0.01$ using the same overlaps as in the linear case.

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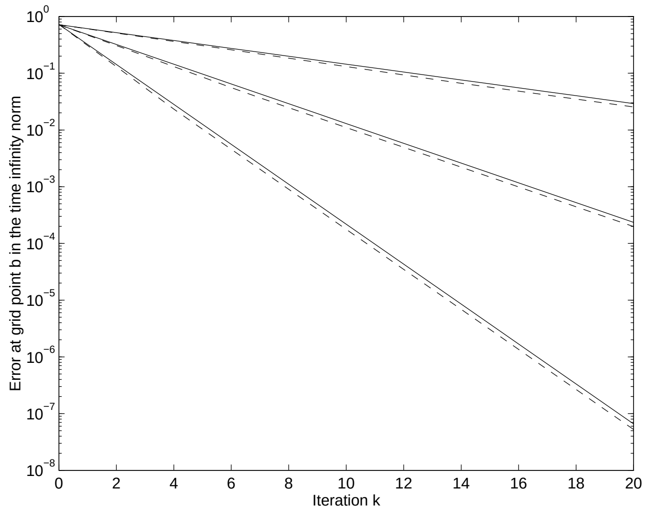

Figure 1.2 Theoretical and measured decay rate of the error for two subdomains and three different sizes of the overlap for the linear example problem

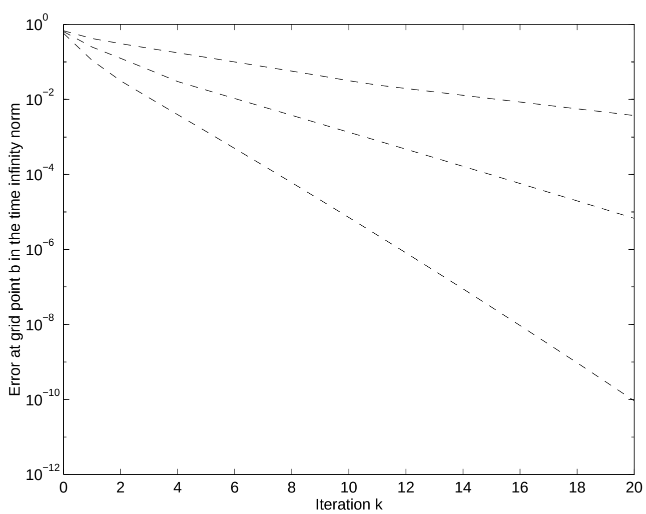

Figure 1.3 Measured decay rate of the error for two subdomains and three different sizes of the overlap for the nonlinear example problem

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