Overlapping Schwarz Waveform Relaxation for Advection Reaction Diffusion Equations

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February 16, 2009

Abstract

We analyze overlapping Schwarz waveform relaxation for linear advection reaction diffusion equations. We prove linear convergence of the algorithm on unbounded time intervals and superlinear convergence on bounded time intervals. In both cases the convergence rates are shown to depend on the size of the overlap. The linear convergence result also depends on the number of subdomains, because it is limited by the classical steady state result of overlapping Schwarz for elliptic problems. The superlinear convergence result however is independent of the number of subdomains. Thus overlapping Schwarz waveform relaxation does not need a coarse space for robust convergence independent of the number of subdomains, if the algorithm is in the superlinear convergence regime. Numerical experiments confirm our analysis.

1 Introduction

Overlapping Schwarz waveform relaxation is a long name for an algorithm which simply solves evolution problems in parallel. It got its name as follows: the distribution of the computation is achieved by partitioning the spatial domain into overlapping subdomains, like in the classical Schwarz method [Sch70]. However on subdomains, time dependent problems are solved in the iteration and thus the

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algorithm is also of waveform relaxation type [LRSV82]. Hence the name overlapping Schwarz waveform relaxation. These algorithms have been introduced in [GZ97, GS98a, GZ02] and independently in [GK02] for the solution of evolution problems in a parallel environment with slow communication links, since they permit to solve over several time steps before communicating information to the neighboring subdomains. They are ideal if one wants to use large existing networks of PC's as a super-computer. An earlier analysis of the same type of algorithm for hyperbolic problems can be found in [Bjø95b, Bjø95a]. For non-linear reaction diffusion equations in one dimension, see [Gan98].

These algorithms stand in contrast to the classical approach in domain decomposition for evolution problems, where time is first discretized uniformly using an implicit discretization and then at each time step a problem in space only is solved using domain decomposition, see for example [Meu91] and [Cai91, Cai94]. The main disadvantage of the classical approach is that one is forced to use the same time step in all subdomains and thus looses one of the main features of domain decomposition, namely to treat subdomains numerically differently. A second disadvantage is that one needs to exchange information at each time step. Overlapping Schwarz waveform relaxation is a remedy for both problems. An interesting variant using moving meshes can be found in [HR07], and the performance of the algorithm can be enhanced using more effective transmission conditions, see [GHN99, Mar05, GH07, BGH09].

In this paper we study classical overlapping Schwarz waveform relaxation for space decompositions in all generality for the linear advection reaction diffusion equation in d dimensions. We prove linear convergence of the algorithm on unbounded time intervals and superlinear convergence on bounded time intervals. Both results are obtained at the continuous level, which leads to algorithms that converge independently of the mesh size if the overlap is held constant.

2 The Schwarz Waveform Relaxation Algorithm

We consider as our guiding example the advection reaction diffusion equation for a scalar density $u(t, \boldsymbol{x})$ on a bounded domain $\Omega \subset \mathbb{R}^d$ with boundary $\partial\Omega$,

$$\begin{aligned} \mathcal{L}(u) := -\frac{\partial u}{\partial t} + \nu\Delta u + \boldsymbol{a}\cdot\nabla u + cu &= f \quad \text{in } \Omega \times (0,T), \\ u &= g \quad \text{on } \partial\Omega \times (0,T), \\ u(0,\cdot) &= u_0 \quad \text{in } \Omega, \end{aligned} \tag{1}$$

where $\nu > 0$ is the viscosity parameter, $\boldsymbol{a}$ is the advection field and c represents a source term. Without loss of generality, we assume that $c \leq 0$; if not, a change of

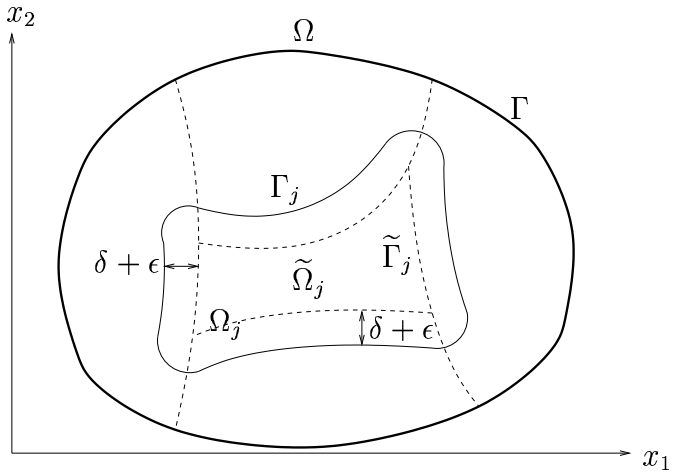

Figure 1: The construction of an overlapping decomposition in two dimensions.

variables $v = ue^{-\sigma t}$ leads to an equation with negative reaction term for a suitable choice of σ. For initial data $u_0 \in C^0(\Omega)$, boundary data $g \in C^0(0,T;C^0(\Omega))$ and $f \in C^0(0,T;C^0(\Omega))$ this problem has a unique solution $u \in C^1(0,T;C^2(\Omega))$. This solution satisfies a maximum principle: if $f = 0$, then the solution $u(t,\boldsymbol{x})$ attains its maximum either on the initial line, at $t = 0$, or on the boundary, $\boldsymbol{x} \in \partial\Omega$, see [RR96].

To compute an approximate solution of (1) on a parallel computer, we partition the domain Ω into overlapping subdomains, and then use an algorithm which requires only subdomain solutions to construct an approximation of the solution on the entire domain. A partition into overlapping subdomains can be obtained as follows: we first partition Ω into N non-overlapping subdomains $\widetilde{\Omega}_j$ with boundaries $\partial\widetilde{\Omega}_j$, $j = 1, 2, \ldots, N$, as shown in Figure 1 for an example in two dimensions. We denote the boundaries of the subdomain $\widetilde{\Omega}_j$ interior to the domain Ω by $\widetilde{\Gamma}_j$ and the part of the boundary shared with $\partial\Omega$ by $\widetilde{\Gamma}_j^0$. Then we construct an overlapping decomposition Ω_j with boundary $\partial\Omega_j$ by enlarging each $\widetilde{\Omega}_j$ so that the boundaries Γ_j of the new subdomains interior to Ω are at least a distance $\delta + \epsilon$ away from $\widetilde{\Gamma}_j$. We denote again the part of the boundary $\partial\Omega_j$ shared with $\partial\Omega$ by Γ_j^0.

We further introduce for technical reasons a partition of unity $\chi_j \in C^\infty$ associated with the non-overlapping subdomains $\widetilde{\Omega}_j$ such that $\chi_j = 1$ in $\widetilde{\Omega}_j$ and $\chi_j = 0$ outside, except for small transition layer centered around the interior boundary $\widetilde{\Gamma}_j$ of width $2\epsilon > 0$ where χ_j decays smoothly to zero.

To solve the parabolic problem (1), the overlapping Schwarz waveform relax-

ation algorithm starts with an initial guess $u^0(t,\boldsymbol{x}) \in C^1(0,T;C^2(\Omega))$ and constructs successive approximations $u^n(t,\boldsymbol{x}) \in C^1(0,T;C^2(\Omega))$ by solving subdomain problems in the subdomains Ω_j only, using as boundary conditions the traces of u^{n-1}. Once all the subdomain solutions u_j^n are obtained, the new global iterate u^n is defined using the partition of unity, $u^n := \sum_{j=1}^N \chi_j u_j^n$, and hence the new iterate satisfies again $u^n \in C^1(0,T;C^2(\Omega)$. The Schwarz waveform relaxation algorithm is thus given by

$$\begin{array}{rcll} \mathcal{L}(u_j^n) & = & f & \text{in } \Omega_j \times (0,T), \\ u_j^n & = & u^{n-1} & \text{on } \Gamma_j \times (0,T), \\ u_j^n & = & g & \text{on } \Gamma_j^0 \times (0,T), \\ u_j^n(0,\cdot) & = & u_0 & \text{in } \Omega_j, \end{array} \qquad u^n = \sum_{j=1}^N \chi_j u_j^n. \tag{2}$$

Algorithm (2) corresponds to a Jacobi Schwarz iteration which can be done in parallel: each subdomain solution u_j^n can be computed independantly of the others, the only data needed is u^{n-1}. One can also consider a Gauss Seidel Schwarz iteration, which would need a special coloring of subdomains to remain a parallel algorithm: if subdomains with the same color do not touch each other, then subdomains of the same color can be solved in parallel using the boundary values coming from subdomains with different colors. We focus in the analysis on the Jacobi version only, analogous results for the Gauss-Seidel versions can be obtained similarly.

3 Linear Convergence on Long Time Intervals

To obtain a result valid on long time intervals, we consider the case where $T = \infty$. We define the integer distance quantity m_j for each subdomain Ω_j to be the least number of subdomains one has to pass through to touch the boundary $\partial\Omega$, and also the maximum $m := \max_j m_j$. We further define the index sets $I_l := \{j : m_j = l\}$ so that the index set I_l contains the indices of all the subdomains which are within distance l of the boundary. We further define for bounded functions $g \in C^1(0,T;C^2(\Omega))$ the norm

$$||g||_\Omega := \sup_{t>0, \boldsymbol{x}\in\Omega} |g(t,\boldsymbol{x})|.$$

Theorem 3.1 (Linear Convergence) *The iterates u^n defined by the overlapping Schwarz waveform relaxation algorithm (2) converge on long time intervals $t \in [0,T)$, $T \leq \infty$, at least linearly to the solution u of (1),*

$$||u^{n(m+2)} - u||_\Omega \leq (\gamma(m,\delta))^n \, ||u^0 - u||_\Omega, \tag{3}$$

where $\gamma(m,\delta)$ is a number strictly less than one and independent of n.

Proof It suffices by linearity to consider the homogeneous problem, $f(\boldsymbol{x},t) = g(\boldsymbol{x},t) = u_0(\boldsymbol{x}) = 0$ in (1) and to prove convergence of the iterates u^n to zero. The idea of the proof is to construct a sequence of elliptic upper bounds on the iterates u_j^n on each subdomain, and then to apply the maximum principle argument from [Lio89] for the elliptic upper bounds. For n fixed, we define $U^n := ||u^n||_\Omega$ and using the maximum principle for parabolic problems, the steady state solution $\tilde{u}_j^{n+1}$ of the elliptic problem

$$\begin{aligned} \nu\Delta\tilde{u}_j^{n+1} + \boldsymbol{a}\cdot\nabla\tilde{u}_j^{n+1} + c\tilde{u}_j^{n+1} &= 0 \quad \text{in } \Omega_j, \\ \tilde{u}_j^{n+1}(\cdot) &= U^n \quad \text{on } \Gamma_j, \\ \tilde{u}_j^{n+1}(\cdot) &= 0 \quad \text{on } \Gamma_j^0, \end{aligned} \tag{4}$$

is an upper bound on the modulus of u_j^{n+1}. Now $\tilde{u}_j^{n+1}$ also satisfies a maximum principle and for $j \in I_0$ we have $\tilde{u}_j^{n+1} < U^n$ in the interior of $\widetilde{\Omega}_j$, since $\tilde{u}_j^{n+1}$ satisfies on the part Γ_j^0 of its boundary a homogeneous boundary condition. We now define

$$U^{n+1} := \sup_{\boldsymbol{x}\in\widetilde{\Omega}_l, l\in I_0} \tilde{u}_l^{n+1}(\boldsymbol{x}),$$

and by the argument above there exists a constant $\gamma_1 < 1$, such that $U^{n+1} \le \gamma_1 U^n$. Note that γ_1 depends on the size of the overlap, $\gamma_1 = \gamma_1(\delta)$, but not on n, because $\tilde{u}_j^{n+1}$ is a linear function of the boundary condition. Note also that for $j \notin I_0$ we have $\tilde{u}_j^{n+1}$ not necessarily less than U^n, since $\tilde{u}_j^{n+1}$ has the value U^n on all its boundaries and thus we could have $\tilde{u}_j^{n+1} \equiv U^n$.

Now for the next iteration by definition part of the boundary of the subdomains Ω_j with $j \in I_1$ lie strictly within $\widetilde{\Omega}_l$ with $l \in I_0$ and therefore for $j \in I_1$ the solution $\tilde{u}_j^{n+2}$ of the elliptic problem

$$\begin{aligned} \nu\Delta\tilde{u}_j^{n+2} + \boldsymbol{a}\cdot\nabla\tilde{u}_j^{n+2} + c\tilde{u}_j^{n+2} &= 0 \quad && \text{in } \Omega_j, \\ \tilde{u}_j^{n+2}(\cdot) &= U^n \quad && \text{on } \Gamma_j \cap \widetilde{\Omega}_l,\ l \notin I_0, \\ \tilde{u}_j^{n+2}(\cdot) &= U^{n+1} \quad && \text{on } \Gamma_j \cap \widetilde{\Omega}_l,\ l \in I_0, \end{aligned} \tag{5}$$

is an upper bound on the modulus of u_j^{n+2}. Since $U^{n+1} \le \gamma_1(\delta)U^n$ we have by the maximum principle $\tilde{u}_j^{n+2} < U^n$ in $\widetilde{\Omega}_j$ for $j \in I_1$ and defining U^{n+2} similarly to U^{n+1} before, we find $U^{n+2} \le \gamma_2(\delta)U^n$ for some constant $\gamma_1(\delta) \le \gamma_2(\delta) < 1$ independent of n. By induction, we find at step $n+m+1$ for the iterates in the subdomains Ω_j with $j \in I_m$ the elliptic upper bound

$$\begin{aligned} \nu\Delta\tilde{u}_j^{n+m+1} + \boldsymbol{a}\cdot\nabla\tilde{u}_j^{n+m+1} + c\tilde{u}_j^{n+m+1} &= 0 \quad && \text{in } \Omega_j, \\ \tilde{u}_j^{n+m+1}(\cdot) &= U^n \quad && \text{on } \Gamma_j \cap \widetilde{\Omega}_l,\ l \notin I_{m-1}, \\ \tilde{u}_j^{n+m+1}(\cdot) &= U^{n+m} \quad && \text{on } \Gamma_j \cap \widetilde{\Omega}_l,\ l \in I_{m-1}, \end{aligned} \tag{6}$$

and $\tilde{u}_j^{n+m+1} < U^n$ in $\widetilde{\Omega}_j$. Defining U^{n+m+1} as before we find $U^{n+m+1} \leq \gamma_{m+1}(\delta)U^n$ for some constant $\gamma_1(\delta) \leq \gamma_2(\delta) \leq \ldots \leq \gamma_{m+1}(\delta) < 1$, independent of n. Now for the next iteration step $n+m+2$ all the u_j^{n+m+2} have boundary values less than or equal to $U^{n+m+1} \leq \gamma_{m+1}(\delta)U^n$, since they come from iteration step $k+m+1$ in the interior of neighboring subdomains. Defining $\gamma(m,\delta) := \gamma_{m+1}(\delta)$ the result follows. ■

Note that the above estimate for $\gamma(m,\delta)$ is conservative, since $\gamma(m,\delta)$ was derived assuming worst case behavior. $\gamma(m,\delta)$ will also depend on the shape of the subdomains, which does not appear in the above argument because of the generality of the domain decomposition we used.

The convergence result we derived on unbounded time domains depends on the number of subdomains, as one can see explicitly from the dependence of γ on m. The more subdomains one uses, the longer it takes for information to propagate from the outer boundary of Ω to the inner subdomains. This is because the steady state solution is limiting the convergence rate, and the steady state solution does not see the zero initial condition. This is different in the superlinear convergence analysis of the next section.

4 Superlinear Convergence for Bounded Time Domains

We now consider a bounded time interval, $T < \infty$. Like in the unbounded time domain case we are solving at each step n of the waveform relaxation iteration (2) subproblems using the boundary information from the neighboring subdomains at step $n-1$. We are interested in estimating the decay of u_j^n in (2) as one moves away from the boundary of the subdomain Ω_j for short time t. We define the infinity norm of a function $g(\boldsymbol{x},t)$ on the boundary Γ_j of subdomain Ω_j by

$$||g(\cdot,\cdot)||_{\Gamma_j} := \sup_{\boldsymbol{x}\in\Gamma_j, 0<t<T} |g(\boldsymbol{x},t)|.$$

We first estimate the decay of the iterates on the boundary of the subdomains over one step of the iteration. To do this, we need an estimate of the solution in a ball with radius δ, $B(\delta) \subset \mathbb{R}^d$, of the problem

$$\begin{array}{rcll} \mathcal{L}(v) & = & 0 & \boldsymbol{x} \in B(\delta), \quad 0 < t < T, \\ v(\boldsymbol{x},t) & = & g(t) & \boldsymbol{x} \in \partial B(\delta), \quad 0 < t < T, \\ v(\boldsymbol{x},0) & = & 0 & \boldsymbol{x} \in B(\delta). \end{array} \tag{7}$$

Lemma 4.1 *For $g(t) \geq 0$, non-decreasing, and $c \leq 0$ the solution v of (7) for $B(\delta) \subset \mathbb{R}^d$ is bounded from above at the center of the ball $\boldsymbol{x} = 0$ by*

$$v(\mathbf{0}, t) \leq \int_0^t K(\frac{\delta}{\sqrt{d}}, t - \tau) g(\tau) d\tau,$$

where the kernel $K(x,t)$ is given by

$$K(x,t) = \sum_{j=1}^{d} \cosh(x a_j/(2\nu)) \frac{x}{\sqrt{\pi\nu} t^{3/2}} e^{-\frac{a_j^2 - 4\nu c}{4\nu} t} e^{-\frac{x^2}{4\nu t}}.$$

Proof We inscribe a hypercube $Q(\delta)$ in $\mathbb{R}^d$ with side $\frac{2\delta}{\sqrt{d}}$ into the ball $B(\delta)$ and consider the decay of $\tilde{v}$ in the hypercube $Q(\delta)$,

$$\begin{array}{rcll} \mathcal{L}(\tilde{v}) & = & 0 & \boldsymbol{x} \in Q(\delta), \quad 0 < t < T, \\ \tilde{v}(\boldsymbol{x}, t) & = & g(t) & \boldsymbol{x} \in \partial Q(\delta), \quad 0 < t < T, \\ \tilde{v}(\boldsymbol{x}, 0) & = & 0 & \boldsymbol{x} \in Q(\delta). \end{array} \tag{8}$$

Evaluating $\tilde{v}$ at the center of the hypercube $Q(\delta)$ we obtain an upper bound on v by the maximum principle and the monotonicity of $g(t)$. Now an upper bound on $\tilde{v}$ at the center of the hypercube $Q(\delta)$ can be obtained by summing the half space solutions $\hat{v}_j$ of the advection reaction diffusion equation for each of the $2d$ faces of the hypercube,

$$\begin{array}{rcll} \mathcal{L}(\hat{v}_j) & = & 0 & \boldsymbol{x} \in \mathbb{R}^d, \ x_j > 0, \quad 0 < t < T, \\ \hat{v}_j(\boldsymbol{x}, t) & = & g(t) & \boldsymbol{x} \in \mathbb{R}^d, \ x_j = 0, \quad 0 < t < T, \\ \hat{v}_j(\boldsymbol{x}, 0) & = & 0 & \boldsymbol{x} \in \mathbb{R}^d, \ x_j \geq 0, \end{array} \tag{9}$$

where x_j denotes the j-th component of $\boldsymbol{x} \in \mathbb{R}^d$. The solution of (9) does only depend on x_j in space and is independent of x_i, $i \neq j$, because $g(t)$ is independent of $\boldsymbol{x}$. It is therefore easily obtained via Laplace transform,

$$\hat{v}_j(\boldsymbol{x}, t) = \int_0^t \hat{K}(x_j, t - \tau) g(t - \tau) d\tau$$

with the kernel $\hat{K}$ given by

$$\hat{K}_j(x,t) = \frac{x}{2\sqrt{\pi\nu} t^{3/2}} e^{-\frac{x^2 + (a_j^2 - 4\nu c) t^2 + 2 x a_j t}{4\nu t}}.$$

Summing this solution for each side of the hypercube means summing the kernels and noting that opposite sides of the hypercube differ in the kernel only in the advection direction, we find the kernel given in the Lemma. ■

Lemma 4.2 *In d dimensions, for $c \leq 0$, the iterates u_j^{n+1} of the overlapping Schwarz waveform relaxation algorithm (2) satisfy on the boundary of all subdomains Ω_j the estimate*

$$\max_j ||u_j^{n+1}||_{\Gamma_j} \leq 2d \cosh(\delta \bar{a}/(2\nu\sqrt{d})) erfc(\frac{\delta}{2\sqrt{d\nu T}}) \max_j ||u_j^n||_{\Gamma_j},$$

where $\bar{a} := \max_j |a_j|$.

Proof We define $U_n(T) := \max_j(||u_j^n||_{\Gamma_j})$. Because of the overlapping property we are interested in the magnitude of $u_j^n(\boldsymbol{x}, t)$ at distance δ from the boundary Γ_j for short time. An upper bound is given by Lemma 4.1,

$$U_{n+1}(T) = \max_j(||u_j^{n+1}||_{\Gamma_j}) \leq \int_0^T K(\frac{\delta}{\sqrt{d}}, T-\tau) U_n(\tau) d\tau. \tag{10}$$

Now taking the supremum of $U_n(\tau)$ (which is attained at T) out of the integral, and estimating one part of the exponential in the kernel K by

$$e^{-\frac{a_j^2 - 4\nu c}{4\nu} t} \leq 1, \quad 0 \leq t \leq T, \tag{11}$$

where we used $c \leq 0$, we obtain

$$U_{n+1}(T) \leq 2d \cosh(\delta \bar{a}/(2\nu\sqrt{d})) \int_0^T K_h(\frac{\delta}{\sqrt{d}}, T-\tau) d\tau U_n(T)$$

where the heat kernel K_h is given by

$$K_h(x,t) = \frac{x}{2\sqrt{\pi\nu} t^{3/2}} e^{-\frac{x^2}{4\nu t}}.$$

Applying the variable transform

$$y := \frac{\delta}{2\sqrt{d\nu(T-\tau)}}$$

to the integral leads to

$$U_{n+1}(T) \leq 2d \cosh(\delta \bar{a}/(2\nu\sqrt{d})) \mathrm{erfc}(\frac{\delta}{2\sqrt{d\nu T}}) U_n(T).$$

■

Note that a slightly sharper convergence estimate could be obtained by using the additional exponential decay in (11).

Lemma 4.2 can be used to derive an arbitrary fast linear upper bound on the convergence rate by shortening the time interval $(0,T)$, since

$$\lim_{x\to\infty} \mathrm{erfc}(x) = 0.$$

This can be acheived in the algorithm by introducing time windows: the global time interval $(0,T)$ is split into equal time windows and the computation is performed in each time window separately, where it converges much faster. This technique is often used for waveform relaxation algorithms, see for example [Nev89].

But this indicates that the fundamental convergence rate of the algorithm is faster than linear, and we derive now an upper bound on the decay of the error over n steps of the iteration which leads to a superlinear convergence result.

Lemma 4.3 *In d dimensions, the maximum of the iterates u_j^n of the overlapping Schwarz waveform relaxation algorithm (2) decays on the boundary of all subdomains Ω_j in the infinity norm at least at the rate*

$$\max_j ||u_j^n||_{\Gamma_j} \leq \left(2d\cosh(\delta\bar{a}/(2\nu\sqrt{d}))\right)^n erfc(\frac{n\delta}{2\sqrt{d\nu T}}) \max_j ||u_j^0||_{\Gamma_j}.$$

Proof By iteration of inequality (10) in Lemma 4.2 we get a bound in form of a convolution, namely

$$\begin{aligned} U_n(T) \quad \leq \quad & \left(2d\cosh(\delta\bar{a}/(2\nu\sqrt{d}))\right)^n \int_0^T K_h(\frac{\delta}{\sqrt{d}}, T-s_1)\cdots \\ & \int_0^{s_{n-1}} K_h(\frac{\delta}{\sqrt{d}}, s_{n-1}-s_n) ds_n \cdots ds_1 U_0(T). \end{aligned}$$

To unfold the convolutions, note that the Laplace transform of a convolution is the product of the Laplace transformed kernels. In our case the Laplace transform of the kernel, see [AS64], is

$$\int_0^\infty e^{-st} K_h(\frac{\delta}{\sqrt{d}}, t) dt = e^{-\frac{\delta}{2\sqrt{d}}\sqrt{s}},$$

and thus the n-fold convolution is the product of identical exponentials in the Laplace transformed domain,

$$e^{-\frac{n\delta}{2\sqrt{d}}\sqrt{s}}.$$

Back-transforming this expression, we find

$$U_n(T) \leq \left(2d\cosh(\delta\bar{a}/(2\nu\sqrt{d}))\right)^n \int_0^T K_h(\frac{n\delta}{\sqrt{d}}, T-\tau) d\tau U_0(T).$$

Using a similar variable transform as in Lemma 4.2 the result follows. ■

Defining for bounded functions $g(\boldsymbol{x},t) : \Omega \times [0,T) \to \mathbb{R}$ the norm

$$||g(\cdot,\cdot)||_T := \sup_{\boldsymbol{x}\in\Omega, 0<t<T} |g(\boldsymbol{x},t)|,$$

we have the following convergence result.

Theorem 4.4 (Superlinear Convergence) *For $c \leq 0$ the overlapping Schwarz waveform relaxation algorithm (2) in $\mathbb{R}^d$ converges superlinearly on bounded time intervals $t \in [0, T < \infty)$ in the infinity norm,*

$$\max_j ||u_j^n(\cdot,\cdot)||_T \leq \left(2d\cosh(\delta\bar{a}/(2\nu\sqrt{d}))\right)^n erfc(\frac{n\delta}{2\sqrt{dT}}) \max_j ||u_j^0(\cdot,\cdot)||_T. \qquad (12)$$

Proof The proof follows from Lemma 4.3 and the maximum principle. ■
There are two interesting facts to note about Theorem 4.4: first the convergence rate is independent of the number of subdomains, there is no dependence on a parameter m related to the number of subdomains as in Theorem 3.1. Second the superlinear convergence rate is faster than the superlinear convergence rate found for classical waveform relaxation algorithms. The classical result gives a contraction governed by a factorial, see [MN87], with asymptotic expansion

$$\frac{(CT)^n}{n!} = \left(\frac{1}{\sqrt{2\pi}} + O(n^{-1})\right) e^{-n\ln n + (1+\ln(CT))n - \frac{1}{2}\ln n} \sim e^{-n\ln n},$$

whereas the new result in Theorem 4.4 gives a contraction with asymptotic expansion

$$C_1^n \mathrm{erfc}(\frac{C_2 n}{\sqrt{T}}) = \left(\frac{\sqrt{T}}{C_2\sqrt{\pi}} + O(n^{-2})\right) e^{-\frac{C_2^2}{T}n^2 + \ln(C_1)n - \ln n} \sim e^{-n^2},$$

an improvement due to the diffusion.

5 Numerical Experiments

We present now numerical experiments for the advection reaction diffusion equation (1) on the unit square in two dimensions,

$$\frac{\partial u}{\partial t} = \nu\Delta u + \mathbf{a}\nabla u + cu, \quad \text{in } \Omega = (0,1)\times(0,1),\ t \in (0,T]. \qquad (13)$$

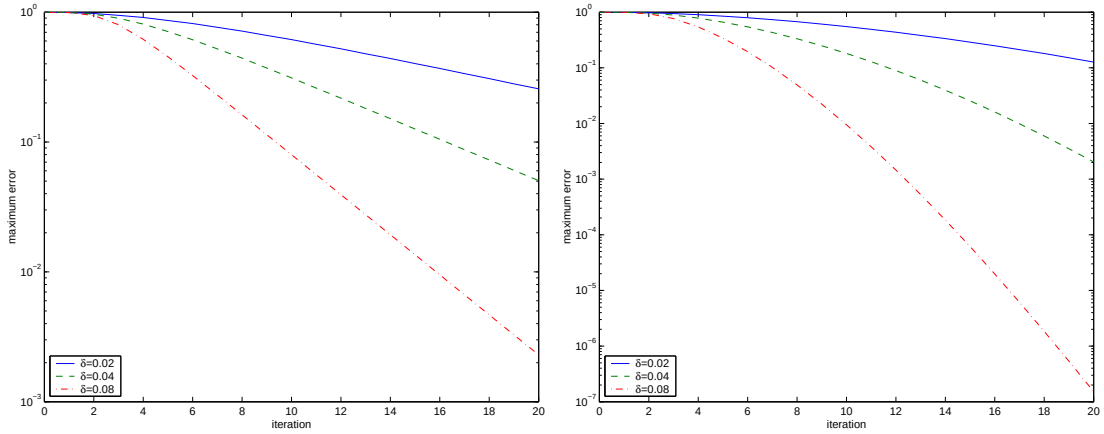

Figure 2: Error in the maximum norm in space and time as a function of the iteration index n for different overlap sizes, on the left when integrating over a relatively long time interval, where the algorithm converges linearly, and on the right over a shorter time interval, where the algorithm converges superlinearly.

We simulate directly the error equations and thus employ homogeneous initial and boundary conditions and look for the zero solution. The advection is chosen to be diagonal, $\boldsymbol{a} := (1, 1)$ and the other parameters are $c = -1$ and $\nu = \frac{1}{2}$. We decompose the domain into several overlapping subdomains (small squares) with equal size and overlap in both spatial directions. We discretize in space using centered finite differences, and the problem is solved using backward Euler in time. To see linear convergence, the problem is integrated over a relatively long time interval $t \in (0, 1]$, and to see superlinear convergence the problem integrated over a shorter time interval $t \in (0, 0.1]$. We chose for the spatial discretization step $h = \frac{1}{50}$, and for the time step $\Delta t = \frac{1}{500}$. Figure 2 shows the algorithm in the linear and superlinear convergence regime when the problem is solved using four subdomains, for different values of the overlap parameter $\delta \in \{0.02, 0.04, 0.08\}$. This illustrates the two different convergence behaviors of the overlapping Schwarz waveform relaxation algorithm shown in Theorem 3.1 and 4.4, and also that increasing the overlap leads to faster convergence, as expected.

We next investigate the dependence of the convergence on the number of subdomains. Theorem 3.1 for the linear convergence regime shows that the decay of the error depends on the number of subdomains, the parameter m appears in equation (3), which is similar to the results found for the heat equation in [GS98b]. Thus for a long time interval, the overlapping Schwarz waveform relaxation algorithm does not scale with respect to the number of subdomains. This is illustrated in Figure

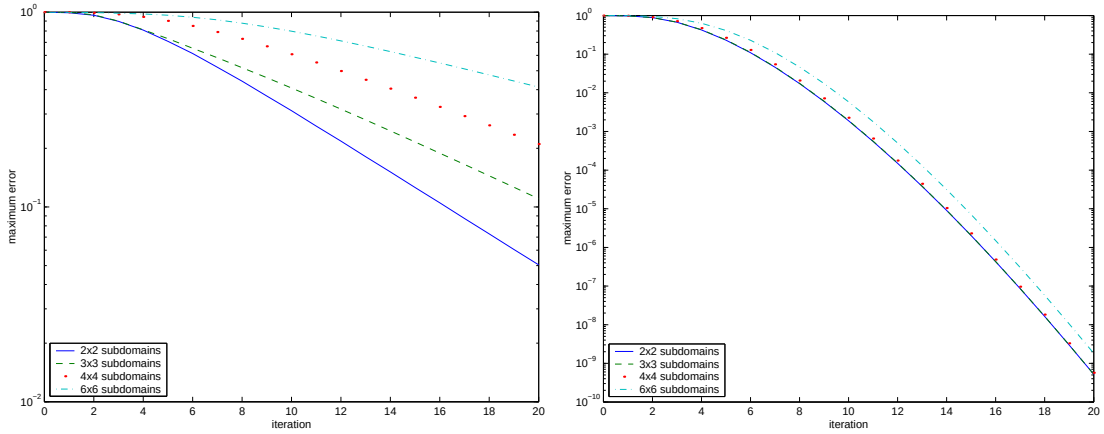

Figure 3: Influence of the number of subdomains on the algorithm: on the left for a long time interval where the linear convergence rate depends on the number of subdomains, and on the right for a short time interval where the superlinear convergence rate is independent of the number of subdomains.

3 on the left, for the case of the decomposition of the unit square into 2×2, 3×3, 4×4 and 6×6 subdomains, using each time the same overlap parameter $\delta = 0.04$, and we integrate the equation on the time interval $t \in (0, 2]$. In the superlinear convergence regime however the convergence rate is independent of the number of subdomains, as we showed in Theorem 4.4. This is confirmed in the numerical experiment shown in Figure 3 on the right, and corresponds to the result found for the heat equation in [GZ97].

Both results can be understood intuitively, by looking at the sequence of errors at the end of the time window given in Figure 4 for the case of 6×6 subdomains, when the algorithm is used over the long time window. In this long time window case, the initial condition, where the error is zero, has very little influence on the error at the end of the time window, and thus only the zero error on the boundary of the domain contributes to reduce the error at the end of the time interval. This leads therefore to a convergence behavior comparable to the classical Schwarz algorithm for steady, elliptic problems, whose convergence depends on the number of subdomains.

The situation is very different if the same algorithm is used over a short time window. The corresponding errors at the end of the short time window are shown in Figure 5. Here, the error decay is dominated by the initial condition, which is correct (i.e. the error is zero initially) for all subdomains. Since one integrates only over a short time, the error is small in all subdomains, independent of their distance

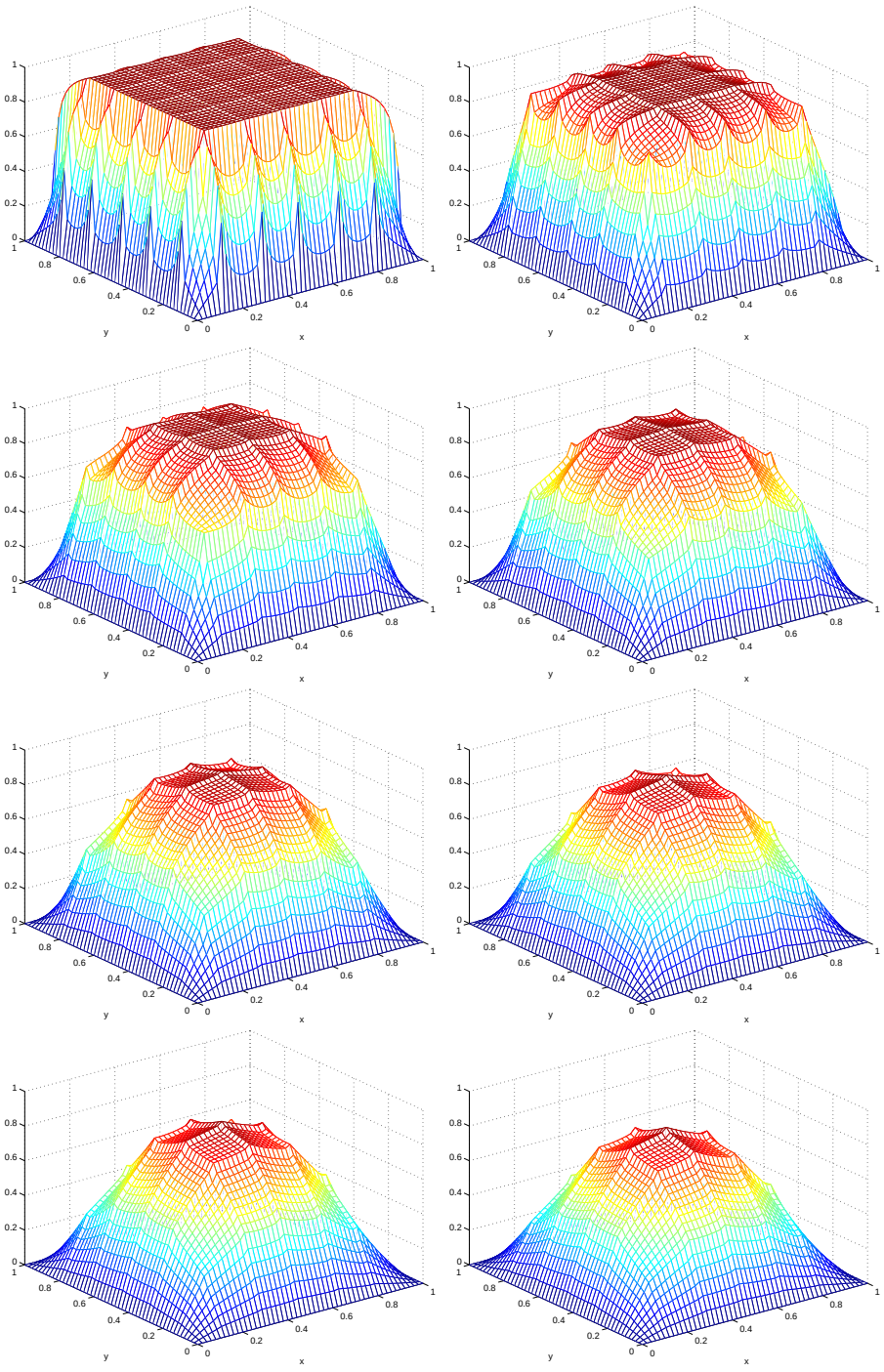

Figure 4: When used over long time windows, the error decay must be propagated by the algorithm from the boundary inward across subdomains, and thus the convergence depends on the number of subdomains.

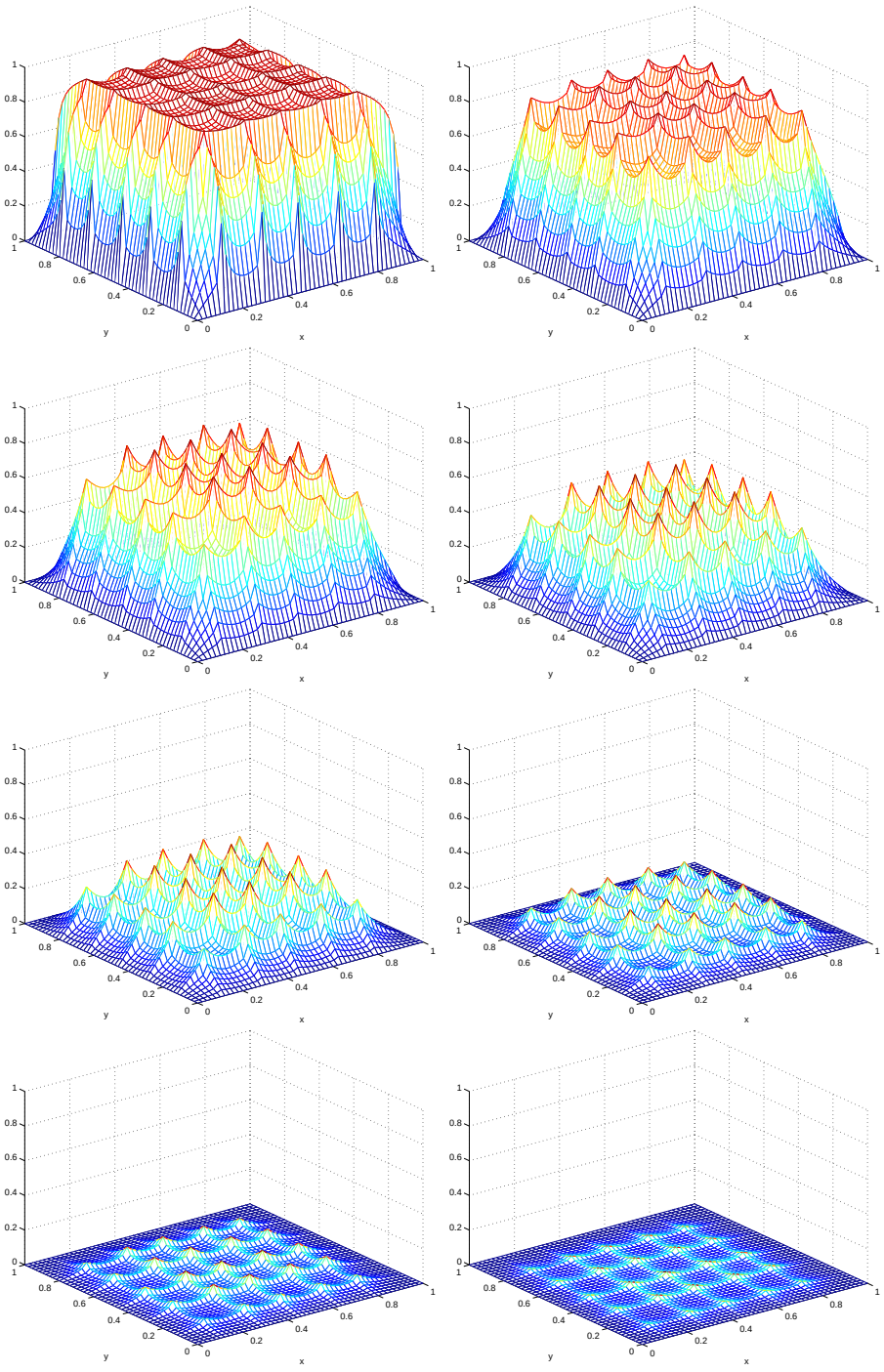

Figure 5: When used over short time windows, the error decay is mostly influenced by the initial condition, where the error equals zero, and hence the error is reduced uniformly over all subdomains, and thus the convergence does not depend on the number of subdomains.

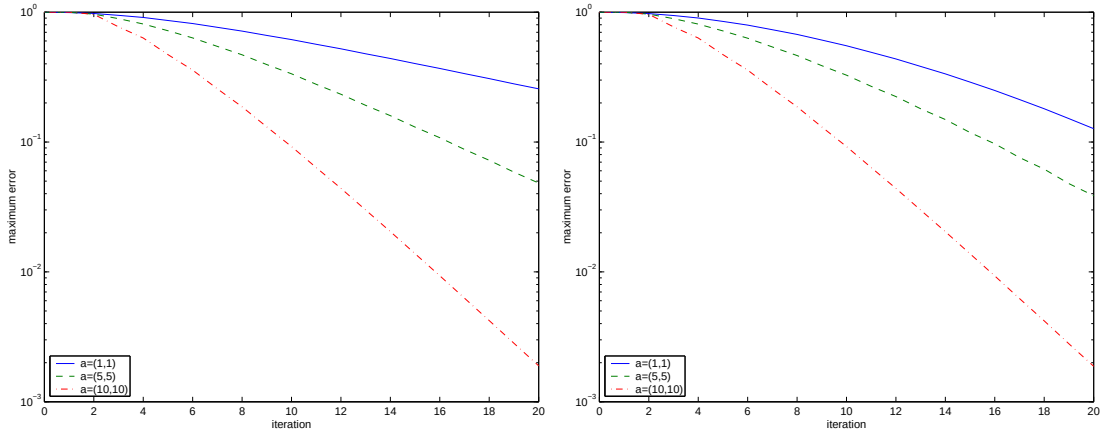

Figure 6: Influence of advection strength on the convergence behavior of the Schwarz waveform relaxation algorithm, on the left over a long time interval, and on the right over a short time interval.

to the boundary of the original domain, and thus convergence is independent of the number of subdomains.

We finally show the influence of the advection on the convergence of the algorithm in Figure 6. We chose the same parameters as in Figure 3, except that we fixed now the overlap parameter to $\delta = 0.02$, and varied the advection strength, $\boldsymbol{a} = (1, 1)$, $\boldsymbol{a} = (5, 5)$, and $\boldsymbol{a} = (10, 10)$. One can see that the algorithm converges faster when the advection increases.

6 Conclusions

We have shown that the overlapping Schwarz waveform relaxation algorithm for general linear advection reaction diffusion equations in arbitrary dimension with very general domain decomposition exhibits two different types of convergence regimes: on unbounded time intervals the algorithm converges linearly at a rate depending on the overlap, the problem parameters and the number of subdomains. On bounded time intervals, the convergence is superlinear, also depending on the overlap and the problem parameters, but it is independent of the number of subdomains. The main property used in the convergence analysis is a maximum principle, and hence similar results can be derived for other equations satisfying a maximum principle. For the superlinear convergence rate, in addition the decay of the Green's function needs to be estimated. We believe that the algorithm will converge superlinearly for many other time dependent partial differential equations

when applied over short time intervals, because the initial condition is then most important for the solution. Waveform relaxation algorithms in general for systems of ordinary differential equations have a zero spectral radius on bounded time intervals, see [Nev89]. It would be interesting for example to analyze the Stokes problem.

The algorithm presented in this paper can be improved by using other transmission conditions. In particular the use of Robin transmission conditions leads to substantially faster algorithms, see [GH07], which also converge without overlap between the subdomains, see also [BGH09] for higher order transmission conditions. Such algorithms, called optimized Schwarz waveform relaxation algorithms, have also been studied for the wave equation, see [GHN03] for the one dimensional case, which includes also a detailed analysis of the discretized case, and [GH04] for the case of higher dimensional wave equations. To prove convergence with overlap is however much more difficult for optimized Schwarz waveform relaxation algorithms in the case of general decompositions like the ones used in this paper, this is the subject of futur research.

Acknowledgment: The second author acknowledges the support of the Swiss National Science Foundation grants 200020-117577/1 and 200021-107988.

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