Title: Waveform Relaxation

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Waveform Relaxation

Synonyms

Dynamic Iteration

Mathematics Subject Classification

65F08, 65F10, 65L05, 65M55

Short Description

Waveform relaxation methods are iterative methods to solve time dependent problems. They start with an initial guess of the solution over the entire time interval of interest, and produce iteratively better and better approximations to the solution over the entire time interval at once.

Description

Classical Waveform Relaxation Methods

Waveform relaxation algorithms were invented for circuit simulation [9]. The idea is to partition large scale circuits into subcircuits, as shown for the historical MOS-Ring oscillator from [9] in Figure 1. Using Kirchhoff's and Ohm's laws, one obtains a system

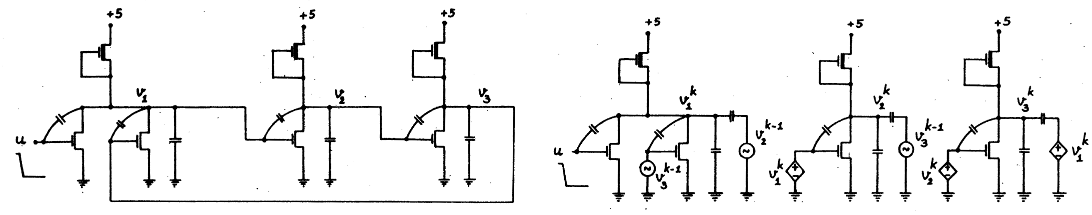

Fig. 1. Historical example of a waveform relaxation decomposition

of ordinary differential equations (ODEs) of the form

$$\frac{dv_1}{dt} = f_1(v_1, v_2, v_3), \quad \frac{dv_2}{dt} = f_2(v_1, v_2, v_3), \quad \frac{dv_3}{dt} = f_3(v_1, v_2, v_3). \tag{1}$$

for the unknown voltages v_1, v_2, v_3. When the circuit is partitioned into subcircuits, coupling terms are replaced by artificial sources, providing signals from the previous iteration, as shown in Figure 1 on the right. This relaxation of signals, called waveforms in the community, led to the name waveform relaxation. Mathematically, this relaxation corresponds for given initial waveforms $v_1^0(t)$, $v_2^0(t)$, $v_3^0(t)$ to the iteration

$$\frac{dv_1^k}{dt} = f_1(v_1^k, v_2^{k-1}, v_3^{k-1}), \quad \frac{dv_2^k}{dt} = f_2(v_1^k, v_2^k, v_3^{k-1}), \quad \frac{dv_3^k}{dt} = f_3(v_1^k, v_2^k, v_3^k), \quad k = 1, 2, \ldots, \tag{2}$$

which is like a Gauss-Seidel method for linear systems, and is thus called Gauss-Seidel waveform relaxation. Naturally also a more parallel Jacobi waveform relaxation can be used.

Waveform relaxation methods are very much related to the classical method of successive approximations by Picard in 1890 [16], where all arguments on the right in (2) would be taken at iteration index $k-1$, and they have similar convergence properties: convergence is superlinear, i.e.

$$||\mathbf{v}^k - \mathbf{v}|| \leq \frac{(CT)^k}{k!}||\mathbf{v}^0 - \mathbf{v}||, \quad \mathbf{v} := (v_1, v_2, v_3),\ \mathbf{v}^k := (v_1^k, v_2^k, v_3^k),\ k = 0, 1, \ldots, \quad (3)$$

where $(0,T)$ is the time interval of simulation and C is a constant related to the Lipschitz constant of $\mathbf{f} := (f_1, f_2, f_3)$. This result was shown for the Picard iteration by Lindelöf in 1894 [10], and for waveform relaxation by Miekkala and Nevanlinna [12], see also [14; 15] and the review paper [13]. From (3) we see that convergence is very fast for T small, and hence it is good to partition long time intervals into shorter so called time windows to apply the algorithm on each time window separately.

Schwarz Waveform Relaxation

Waveform relaxation can be applied to evolution partial differential equations (PDEs) after discretization in space. It is however more interesting to decompose directly the domain, like the circuit, by domain decomposition, as proposed in the PhD thesis of Morten Bjørhus for hyperbolic systems, and by Gander and Stuart for parabolic problems [4]. Classical Schwarz waveform relaxation for the heat equation,

$$\frac{\partial u}{\partial t} = \nu \Delta u \quad \in \Omega \subset \mathbb{R}^2, \qquad (4)$$

is based on an overlapping decomposition of Ω into subdomains Ω_i as shown in Figure 2, and given by the iteration

$$\begin{aligned} \frac{\partial u_i^k}{\partial t} &= \nu \Delta u_i^k + f && \text{in } \Omega_i \times (0,T), \\ u_i^k(\cdot,\cdot,0) &= u_0 && \text{in } \Omega_i, \\ u_i^k &= u_j^{k-1} && \text{on } \Gamma_{ij} \times (0,T). \end{aligned} \qquad (5)$$

The global iterate can then for example be defined by $u^k := u_i^k$ in $\widetilde{\Omega}_i \times [0,T]$, or using a partition of unity for more smoothness. Schwarz waveform relaxation algorithms also

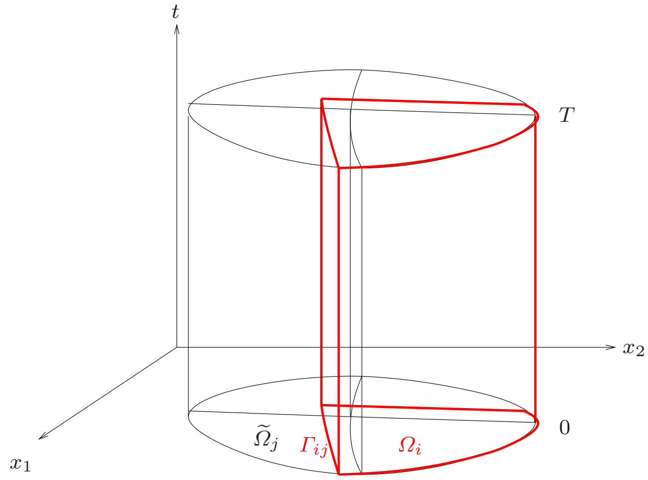

Fig. 2. Space-Time domain decomposition for Schwarz Waveform Relaxation, where $\widetilde{\Omega}_i$ are the non-overlapping subdomains from which the overlapping decomposition Ω_i is constructed by enlarging each $\widetilde{\Omega}_i$ by a layer of width $\frac{\delta}{2}$, leading to the overlapping space-time subdomains $\Omega_i \times (0, T)$

converge superlinearly for diffusive problems [6; 5], like the heat equation, with an error estimate of the form

$$||u^k - u|| \leq C^k \mathrm{erfc}(\frac{k\delta}{2\sqrt{\nu T}})||u^0 - u||,$$

where δ represents the overlap. However they converge asymptotically faster than classical waveform relaxation algorithms, since $C^k \mathrm{erfc}(\frac{k\delta}{\sqrt{\nu T}}) \sim e^{-k^2}$, whereas for classical waveform relaxation we have $\frac{(CT)^k}{k!} \sim e^{-k \ln k}$. One can furthermore show that Schwarz waveform relaxation algorithms applied to diffusive problems still converge linearly over long time intervals, see [4], a result that also holds for classical waveform relaxation applied to dissipative systems of ODEs. For the wave equation, and more generally for hyperbolic systems, where the speed of propagation is finite, one can show that Schwarz waveform relaxation algorithms converge in a finite number of steps, see for example [2].

One can obtain much faster Schwarz waveform relaxation algorithms, if one replaces the transmission conditions in (5) by

$$\mathcal{B}_{ij}(u_i^k) = \mathcal{B}_{ij}(u_j^{k-1}) \quad \text{on } \Gamma_{ij} \times (0, T), \tag{6}$$

where the transmission operators $\mathcal{B}_{ij}$ are chosen to improve information transfer between subdomains. For Robin transmission conditions, $\mathcal{B}_{ij} := \partial_{n_{ij}} + p$ with $\partial_{n_{ij}}$ denoting the normal derivative, the parameter p was optimized for advection reaction diffusion equation in [3], and higher order transmission operators were optimized in [1], for the wave equation see [2]. For fixed overlap, these optimized Schwarz waveform relaxation algorithms converge very rapidly, independently of the mesh parameters, and over short time intervals also independently of the number of subdomains, there is no need for a coarse grid. Optimized waveform relaxation algorithms have also been developed for circuits, where better information transfer was obtained by exchanging combinations of voltage and current values.

Since optimized Schwarz waveform relaxation methods converge even without overlap, they are also an excellent modeling tool to couple different physics or different mathematical models directly in space time, like in fluid structure interaction or in ocean atmosphere coupling.

Multigrid Waveform Relaxation

In the case of linear systems of equations, one can accelerate the basic Jacobi or Gauss-Seidel iterations by using them only as a smoother on coarser and coarser grids to obtain a multigrid method. Lubich and Ostermann [11] proposed in the same spirit to use the Jacobi or Gauss-Seidel waveform relaxation algorithm as a smoother on coarser and coarser spatial grids in the space time waveform relaxation iteration. Note that there is no coarsening in time in this multigrid waveform relaxation algorithm, time is kept continuous. The algorithm has convergence properties like multigrid applied to stationary problems, and is also more robust than the parabolic multigrid method proposed earlier by Hackbusch in [7], where one applies the smoother for the stationary problem on several time levels in parallel. A complete space-time multigrid method was proposed by Horton and Vandewalle in [8]: this method considers the entire space-time grid and the problem posed thereon, and performs a multigrid iteration by both coarsening in space and time. The authors show that care must be taken in choosing the coarsening strategy, as well as the prolongation and restriction operations, in order to obtain a good space-time multigrid method.

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