Analysis of Overlap in Waveform Relaxation Methods for RC Circuits

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1 Introduction

Classical Waveform Relaxation (WR) was introduced in 1981 for circuit solver applications [5]. In WR, large systems of differential equations modeling electric circuits are partitioned into small subcircuits, which are then solved separately, and an iteration is used to get better and better approximations to the overall solution of the underlying large circuit. For classical WR, smart partitioning is very important to enhance the convergence rate, while optimized WR uses more effective transmission conditions to enhance the convergence rate, and thus permits also partitioning at less suitable locations in the circuit without negatively affecting the convergence rate. We study here for the first time the influence of overlapping subcircuits in classical and optimized WR methods applied to RC circuits.

2 The RC Circuit Equations

Circuit equations are obtained from a given circuit using Modified Nodal Analysis (MNA), a major invention that led for circuits to a similar assembly procedure like the finite element method [4]. The MNA circuit equations for the RC circuit of length N shown in Figure 1 are

$$\dot{\mathbf{v}} = \begin{bmatrix} b_1 & c_1 & & & \\ a_1 & b_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{N-2} & b_{N-1} & c_{N-1} \\ & & & a_{N-1} & b_N \end{bmatrix} \mathbf{v} + \mathbf{f}, \tag{1}$$

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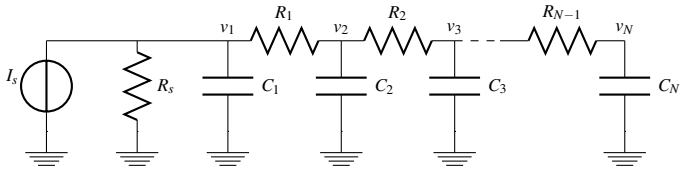

Fig. 1: Finite RC circuit of length N.

where the entries in the tridiagonal matrix are given by

$$\begin{cases} a_i = \frac{1}{R_i C_{i+1}}, \\ c_i = \frac{1}{R_i C_i}, \end{cases} \quad i = 1,2,..,N-1, \quad b_i = \begin{cases} -(\frac{1}{R_s} + \frac{1}{R_1})\frac{1}{C_1}, & i = 1, \\ -(\frac{1}{R_{i-1}} + \frac{1}{R_i})\frac{1}{C_i}, & i = 2,3,...,N-1, \\ -\frac{1}{R_{N-1}C_N}, & i = N. \end{cases}$$

The resistances R_i and capacitances C_i are strictly positive constants. The source term on the right-hand side is given by $\mathbf{f}(t)=(I_s(t)/C_1, 0, ...0)^T$ for some current function $I_s(t)$, and we need to specify initial voltage values $\mathbf{v}(0) = (v_1^0, v_2^0, .., v_N^0)^T$ at time $t = 0$ to solve this system.

3 The Classical WR Algorithm

To define the classical WR algorithm, we partition the circuit in Figure 1 with the voltages $\mathbf{v}$ to be determined into two sub-circuits with unknown voltages $\mathbf{u}$ and $\mathbf{w}$. For convenience in the analysis that will follow, we assume N to be even, and we renumber the nodes: instead of using the numbering from 1 to N, we use the numbering from $-\frac{N}{2}+1$ to $\frac{N}{2}$, see Figure 2. We thus have $\mathbf{v} := (v_{-\frac{N}{2}+1}, ..., v_{-1}, v_0, v_1, ...v_{N/2})^T$, which is still of length N, and

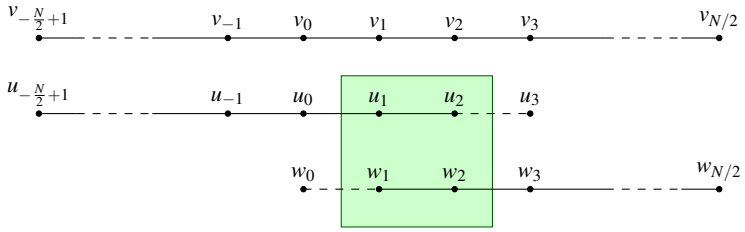

Fig. 2: Decomposition into two sub-circuits with two nodes overlap.

$$\mathbf{u} := (u_{-\frac{N}{2}+1}, ..., u_{n-2}, u_{n-1}, u_n)^T, \quad u_j = v_j \text{ for } j = -\tfrac{N}{2}+1, \ldots, n,$$
$$\mathbf{w} := (w_1, w_2, ..., w_{\frac{N}{2}})^T, \quad w_j = v_j \text{ for } j = 1, \ldots, \tfrac{N}{2},$$

which are of length $\frac{N}{2}+n$ and $\frac{N}{2}$, since we added n nodes to subcircuit $\mathbf{u}$ to have an overlap of n nodes. The classical WR algorithm applied to the two sub-systems is

$$\begin{aligned}
\dot{\mathbf{u}}^{k+1} &= \begin{bmatrix} b_{-\frac{N}{2}+1} & c_{-\frac{N}{2}+1} & & \\ \ddots & \ddots & \ddots & \\ & a_{n-2} & b_{n-1} & c_{n-1} \\ & & a_{n-1} & b_n \end{bmatrix} \begin{bmatrix} u_{-\frac{N}{2}+1} \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix}^{k+1} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ c_n u_{n+1}^{k+1} \end{bmatrix} + \begin{bmatrix} f_{-\frac{N}{2}+1} \\ \vdots \\ f_{n-1} \\ f_n \end{bmatrix}, \\
\dot{\mathbf{w}}^{k+1} &= \begin{bmatrix} b_1 & c_1 & & \\ a_1 & b_2 & c_2 & \\ & \ddots & \ddots & \ddots \\ & & a_{\frac{N}{2}-1} & b_{\frac{N}{2}} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{\frac{N}{2}} \end{bmatrix}^{k+1} + \begin{bmatrix} a_0 w_0^{k+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{\frac{N}{2}} \end{bmatrix},
\end{aligned} \tag{2}$$

where u_{n+1}^{k+1} and w_0^{k+1} are determined in classical WR by the transmission conditions

$$u_{n+1}^{k+1} = w_{n+1}^k \quad \text{and} \quad w_0^{k+1} = u_0^k. \tag{3}$$

Note that in these transmission conditions, we exchange voltages at the interfaces. The two subsystems are given the initial voltages $\mathbf{u}(0) = (v^0_{-\frac{N}{2}+1}, ..., v^0_{n-1}, v^0_n)^T$ and $\mathbf{w}(0) = (v_1^0, v_2^0, .., v^0_{\frac{N}{2}})^T$, and the initial waveforms u_0^0, w_{n+1}^0 are needed to start the WR algorithm.

To simplify our analysis of the convergence factor, we assume that all resistors and capacitors are the same, $R := R_i$ and $C := C_i$ for all $i \in \mathbb{Z}$, which implies

$$b := b_i \quad \text{and} \quad a := a_i = c_i \quad \text{for all } i \in \mathbb{Z}, \tag{4}$$

and for our RC circuit $b = -2a$. To further simplify the analysis, we also assume that the circuit is of infinite length, $N \to \infty$, and by linearity it suffices to analyze the homogeneous problem corresponding to the error equations, and to study convergence to the zero solution. Taking a Laplace transform in time with Laplace parameter $s \in \mathbb{C}$ of the WR algorithm (2), we get in the homogeneous case when $N \to \infty$

$$\begin{aligned}
s\hat{\mathbf{u}}^{k+1} &= \begin{bmatrix} \ddots & \ddots & \ddots & \\ & a & b & a \\ & & a & b \end{bmatrix} \begin{bmatrix} \vdots \\ \hat{u}_{n-1} \\ \hat{u}_n \end{bmatrix}^{k+1} + \begin{bmatrix} \vdots \\ 0 \\ a\hat{w}_{n+1}^k \end{bmatrix}, \\
s\hat{\mathbf{w}}^{k+1} &= \begin{bmatrix} b & a & & \\ a & b & a & \\ & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \vdots \end{bmatrix}^{k+1} + \begin{bmatrix} a\hat{u}_0^k \\ 0 \\ \vdots \end{bmatrix}.
\end{aligned} \tag{5}$$

Lemma 1. *Let $a > 0$, $b < 0$, $i = \sqrt{-1}$, and $s := \sigma + i\omega$, with $\sigma > 0$. If $-b \geq 2a$, then the roots $\lambda_{1,2} := \frac{s-b \pm \sqrt{(b-s)^2 - 4a^2}}{2a}$ of the characteristic equation $a\hat{u}_{j-1}^{k+1} + (b-s)\hat{u}_j^{k+1} + a\hat{u}_{j+1}^{k+1} = 0$ of the subsystems in (5) satisfy $|\lambda_2| < 1 < |\lambda_1|$.*

Proof. Since $a > 0$, $b < 0$ and $-b \geq 2a$, we can write $b = -(2+\varepsilon)a$ for some $\varepsilon \geq 0$. Let $p + iq := \sqrt{(b-s)^2 - 4a^2}$, for $p, q \in \mathbb{R}$, with $p > 0$. We then obtain with $\sigma > 0$ that

$$\begin{aligned} |\lambda_1| &= |\frac{s-b+\sqrt{(b-s)^2-4a^2}}{2a}| = |\frac{\sigma + i\cdot\omega + (2+\varepsilon)a}{2a} + \frac{1}{2a}(p + i\cdot q)| \\ &= |(1 + \frac{\varepsilon a + \sigma + p}{2a} + \frac{i}{2a}(\omega + q)| > 1. \end{aligned}$$

Now by Vieta's formulas, $\lambda_1\lambda_2 = 1$, which implies $|\lambda_2| < 1$ and thus completes the proof.

Theorem 1 (Convergence factor for Classical WR with Overlap). *The convergence factor of the classical WR algorithm* (5) *with n nodes overlap is*

$$\rho_{cla}(s,a,b) = \Big(\frac{1}{\lambda_1^2}\Big)^{n+1}. \tag{6}$$

Proof. The iterate $\mathbf{u}^{k+1}$ for the first subsystem satisfies the recurrence relation

$$a\,\hat{u}_{j-1}^{k+1} + (b-s)\,\hat{u}_j^{k+1} + a\,\hat{u}_{j+1}^{k+1} = 0 \qquad \text{for } j = \ldots, n-2, n-1, n, \tag{7}$$

whose solution is $\hat{u}_j^{k+1} = A^{k+1}\lambda_1^j + B^{k+1}\lambda_2^j$ for $j = \ldots, n-2, n-1, n$. Since the solution u_j^{k+1} must remain bounded for all j, we must have $B^{k+1} = 0$. Substituting $j = n$ into (7), we can determine A^{k+1} and obtain the general solution

$$\hat{u}_j^{k+1} = \Big(-\frac{a}{a\lambda_1^{-1} + (b-s)}\Big)\cdot\Big(\frac{1}{\lambda_1^n}\Big)\cdot\lambda_1^j\cdot\hat{w}_{n+1}^k \qquad \text{for } j = \ldots, n-2, n-1, n. \tag{8}$$

Similarly, we obtain for the second subsystem

$$\hat{w}_j^{k+1} = \Big(\frac{-a}{(b-s) + a\lambda_2}\Big)\cdot\lambda_2^{j-1}\cdot\hat{u}_0^k \qquad \text{for } j = 1, 2, \ldots. \tag{9}$$

Combining (8) and (9) and using Vieta's formulas $\lambda_1 + \lambda_2 = \frac{s-b}{a}$ and $\lambda_1\lambda_2 = 1$ then gives

$$\begin{aligned} \hat{u}_j^{k+1} &= \Big(\tfrac{-a}{a\lambda_1^{-1}+(b-s)}\Big)\cdot\Big(\tfrac{-a}{(b-s)+a\lambda_2}\Big)\cdot\Big(\tfrac{\lambda_2^n}{\lambda_1^n}\Big)\cdot\lambda_1^j\hat{u}_0^{k-1} \\ &= \Big(\tfrac{1}{\lambda_1^2}\Big)^{n+1}\hat{u}_j^{k-1} =: \rho_{cla}(s,a,b)\hat{u}_j^{k-1}, \end{aligned}$$

and similarly we find also for the second subsystem $\hat{w}_j^{k+1} = \rho_{cla}(s,a,b)\hat{w}_j^{k-1}$, which concludes the proof.

We see that the convergence factor $\rho_{cla}(s,a,b)$ is the same for all nodes in both subsystems, and since $|\lambda_1| > 1$, classical WR always converges, and convergence becomes faster when increasing the number of nodes the subsystems overlap. In the case $|b| = 2a$ however, $|\rho_{cla}(s,a,b)| \to 1$ when $s \to 0$, which indicates slow convergence for this case.

Remark 1. Theorem 1 implies $\hat{u}_j^{2k} = (\rho_{cla}(s,a,b))^k \hat{u}_j^0$ and $\hat{w}_j^{2k} = (\rho_{cla}(s,a,b))^k \hat{w}_j^0$. Using the Parseval-Plancherel identity, one can then obtain in the time domain

$$\|u_j^{2k}(t)\|_\sigma \leq \Big(\sup_{\omega\in\mathbb{R}} \rho_{cla}(s,a,b)\Big)^k \|u_j^0(t)\|_\sigma,\ \|w_j^{2k}(t)\|_\sigma \leq \Big(\sup_{\omega\in\mathbb{R}} \rho_{cla}(s,a,b)\Big)^k \|w_j^0(t)\|_\sigma,$$

where $\|x(t)\|_\sigma := \|e^{-\sigma t}x(t)\|_{L^2}$. For $\sigma = 0$, we thus obtain convergence in L^2.

4 The Optimized WR Algorithm

New transmission conditions were proposed in [1] for WR, namely

$$\begin{aligned} (u_{n+1}^{k+1} - u_n^{k+1}) + \alpha u_{n+1}^{k+1} &= (w_{n+1}^k - w_n^k) + \alpha w_{n+1}^k, \\ (w_1^{k+1} - w_0^{k+1}) + \beta w_0^{k+1} &= (u_1^k - u_0^k) + \beta u_0^k, \end{aligned} \tag{10}$$

where α and β are weighting factors that can be optimized to obtain more rapid convergence, leading to optimized waveform relaxation algorithms (OWR). If we divide the first equation in (10) by α and the second by β, we see that α and β represent resistances, and the new transmission conditions thus exchange both voltages and currents at the interfaces. Note also that the classical transmission conditions (3) become a special case when taking very large values of α and β.

Theorem 2 (Convergence factor for OWR with Overlap). *The convergence factor of the OWR algorithm with n nodes overlap is*

$$\rho_{opt}(s,a,b,\alpha,\beta) = \Big(\frac{1}{\lambda_1^2}\Big)^n \cdot \Big(\frac{\alpha+1-\lambda_1}{\lambda_1(1+\alpha)-1}\Big).\Big(\frac{\lambda_1+\beta-1}{1+(\beta-1)\lambda_1}\Big). \tag{11}$$

Proof. The transmission conditions (10) can we rewritten as

$$u_{n+1}^{k+1} = \frac{u_n^{k+1}}{1+\alpha} + w_{n+1}^k - \frac{w_n^k}{1+\alpha}, \quad w_0^{k+1} = -\frac{w_1^{k+1}}{\beta-1} + u_0^k + \frac{u_1^k}{\beta-1}.$$

Proceeding with these values as in the proof of Theorem 1 then leads to (11).

We see that OWR contains an extra term in its convergence factor, compared to classical WR, and with a good choice of α and β this term can be made smaller than one and thus leads to better convergence. To obtain the best possible convergence, we need to solve the min-max problem

$$\min_{\alpha,\beta}\Big(\max_{s}|\rho_{opt}(s,a,b,\alpha,\beta)|\Big). \tag{12}$$

To simplify this min-max problem in the complex plane, the following Lemma is useful:

Lemma 2. *Let $b<0$, $a>0$, $-b\geq 2a$, $\alpha>0$ and $\beta<0$. Then the convergence factor $\rho_{opt}(s,a,b,\alpha,\beta)$ is an analytic function in the right half of the complex plane.*

Proof. We need to show that the denominator of $\rho_{opt}(s,a,b,\alpha,\beta)$ does not have any zeros in the right half of the complex plane. We show this by contradiction. Assume there is a zero. Then $\lambda_1=0$ or $(1+\alpha)\lambda_1-1=0$ or $1+(\beta-1)\lambda_1=0$. The first case is not possible since under the given assumptions $|\lambda_1|>1$. Considering the second case we have $\lambda_1=\frac{1}{1+\alpha}$. Since $\alpha>0$, $|\lambda_1|=|\frac{1}{1+\alpha}|<1$ which is a contradiction. Similarly, the third case can not hold since $\beta<0$, which concludes the proof.

Since $\rho_{opt}(s,a,b,\alpha,\beta)$ is analytic in the right half of the complex plane, i.e for $s=\sigma+i\omega$, $\sigma\geq 0$, by the maximum principle for analytic functions, its maximum in modulus is attained on the boundary. Let $s=r\cdot e^{i\theta}$, where $r\in[0,\infty)$and $\theta\in[-\pi/2,\pi/2]$. From the definition of λ_1 given in Lemma 1, we observe that $\lim_{r\to\infty}\lambda_1=\infty$ and hence $\lim_{r\to\infty}\rho_{opt}(s,a,b,\alpha,\beta)=\lim_{r\to\infty}\left(\frac{-1}{(\alpha+1)(\beta-1)}\right)\cdot\left(\frac{1}{\lambda_1^2}\right)^n=0$. Thus the maximum lies on the boundary when $\theta=\pm\pi/2$ and $r<\infty$, i.e. when $\sigma=0$. For $\sigma=0$, one can show that $|\rho_{opt}(\omega,a,b,\alpha,\beta)|$ is symmetric in ω, and hence it is sufficient to optimize the convergence factor for $\omega\geq 0$. To simplify the min-max problem further, we use the fact that in our RC circuit, both sub-systems have very similar electrical properties. Since we assumed furthermore that all circuit elements have the same value, it makes sense to choose $\beta=-\alpha$, which can be interpreted as having the same current flow between the subsystems, just into opposite directions. Therefore, the min-max problem (12) simplifies to

$$\min_{\alpha}\Big(\max_{\omega\geq 0}|\rho_{opt}(\omega,a,b,\alpha)|\Big),\quad \rho_{opt}(\omega,a,b,\alpha)=\Big(\frac{\alpha+1-\lambda_1}{\lambda_1(1+\alpha)-1}\Big)^2\cdot\Big(\frac{1}{\lambda_1^2}\Big)^n. \tag{13}$$

Theorem 3 (Asymptotically optimized α). *For an RC circuit of infinite length with $b=-(2+\varepsilon)a$, where $\varepsilon\to 0$, the optimized parameter α^* for n nodes overlap is*

$$\alpha^*=\Big(\frac{\varepsilon}{n}\Big)^{1/3}. \tag{14}$$

Proof. This result can be proved using asymptotic analysis: one can show that the solution to the min-max problem (13) is given by equioscillation when $\varepsilon\to 0$, i.e α^* satisfies $|\rho_{opt}(\bar{\omega},a,b,\alpha^*)|=|\rho_{opt}(0,a,b,\alpha^*)|$ and $\frac{\partial}{\partial\omega}\rho_{opt}(\bar{\omega},a,b,\alpha^*)=0$ for some interior maximum point $\bar{\omega}>0$. The details are however too long and technical for this short paper, and will appear in [2].

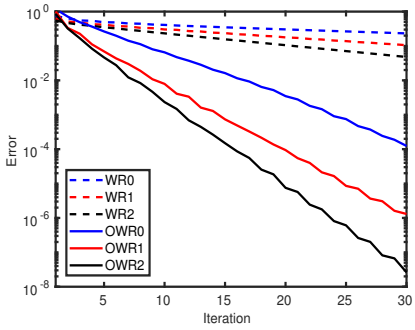

Fig. 3: Convergence for long time $T = 1000$.

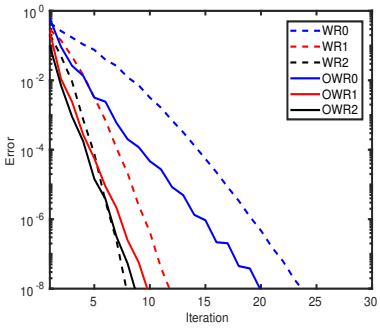

Fig. 4: Convergence for short time $T = 2$.

5 Numerical Results

We simulate an RC circuit of length $N = 80$ with $R = 0.5\,k\Omega$, $C = 0.63\mu F$, $a = \frac{1}{RC}$ and $b = -(2+\varepsilon)a$. We apply Backward Euler with $\Delta t = 0.1$, and simulate directly the error equations, starting with a random initial guess. In Figure 3, we show for $\varepsilon = 10^{-4}$ the influence of overlap on the convergence of classical and optimized WR (e.g. WR2 means WR with overlap 2) for a long time interval $(0,T)$, $T = 1000$. We see that OWR converges much faster than classical WR, see also Figure 5 for a theoretical comparison of the convergence factors. For a short time interval, $T = 2$, classical WR is already very fast, see Figure 4. We determined the optimal choice of α for these experiments solving the min-max problem (13) numerically. Next, we compare this min-max approach with the asymptotic optimization for $b = -(2+\varepsilon)a$ from Theorem 3, and also with running the algorithm for many choices of α numerically. Figure 6 shows that all three give similar results. Finally, we show in Figure 7 and 8 a comparison of the convergence factors for the differently optimized α for two choices of ε.

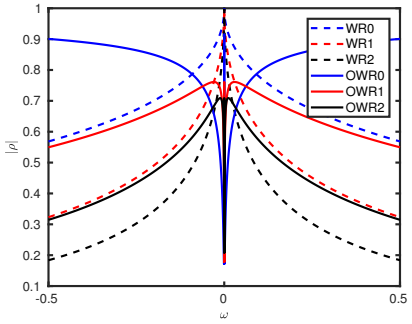

Fig. 5: Effect of overlap

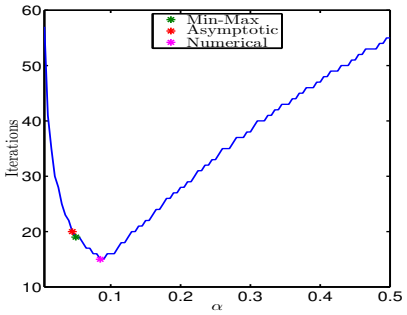

Fig. 6: Comparison of the optimized α

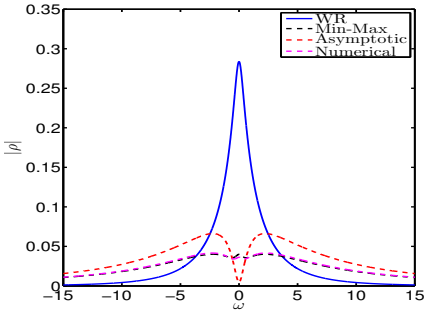

Fig. 7: Convergence factor for optimized α by different methods for $\varepsilon = 10^{-1}$.

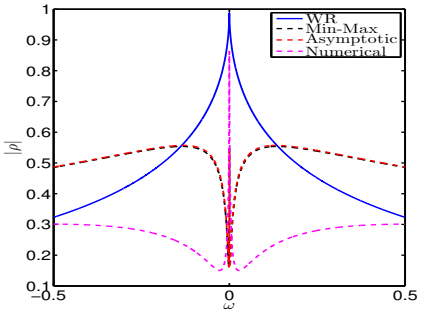

Fig. 8: Convergence factor for optimized α by different methods for $\varepsilon = 10^{-5}$.

6 Conclusion

We studied here for the first time the influence of overlap on the convergence of classical and optimized waveform relaxation algorithms for RC circuits. We defined an optimization problem which permits to obtain a theoretically optimized parameter leading to the fastest possible convergence of the optimized variant. Our analysis shows that overlap enhances the performance of both algorithm variants, which we also illustrated by numerical experiments. While the optimized variant converges much faster when used on long time intervals compared to the classical one, for short time intervals the optimization is less important. We finally compared numerically three different approaches to obtain the optimized parameter in the transmission conditions, and observed that the three methods give similar parameters.

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