On Optimal Coarse Spaces for Domain Decomposition and Their Approximation

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1 Definition of the Optimal Coarse Space

We consider a general second order elliptic model problem

$$\mathscr{L}u = f \quad \text{in } \Omega \tag{1}$$

with some given boundary conditions that make the problem well posed. We decompose the domain Ω first into non-overlapping subdomains $\tilde{\Omega}_j$, $j = 1,2,\ldots,J$, and to consider also overlapping domain decomposition methods, we construct overlapping subdomains Ω_j from $\tilde{\Omega}_j$ by simply enlarging them a bit. All domain decomposition methods provide at iteration n solutions u_j^n on the subdomains $\tilde{\Omega}_j$, $j = 1,2,\ldots,J$ (or on Ω_j in the case of overlapping methods, but then we just restrict those to the non-overlapping decomposition $\tilde{\Omega}_j$ to obtain an overall approximate solution on which we base our coarse space construction). We want to study here properties of the correction that needs to be added to these subdomain solutions in order to obtain the solution u of (1). This would be the best possible correction a coarse space can provide, independently of the domain decomposition method used, and it allows us to define an optimal coarse space, which we then approximate.

Since the u_j^n are subdomain solutions, they satisfy equation (1) on their corresponding subdomain,

$$\mathscr{L}u_j^n = f, \quad \text{in } \tilde{\Omega}_j. \tag{2}$$

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Defining the error

$$e_j^n(x) := u(x) - u_j^n(x), \quad x \in \tilde{\Omega}_j,$$

we see that the error satisfies the homogeneous problem in each subdomain,

$$\mathscr{L} e_j^n = 0 \quad \text{in } \tilde{\Omega}_j. \tag{3}$$

At the interface between the non-overlapping subdomains $\tilde{\Omega}_j$ the error is in general not continuous, and also the normal derivative of the error is not continuous, since the subdomain solutions u_j^n in general do not have this property[1]. The best coarse space, which we call optimal coarse space, must thus contain piecewise harmonic functions on $\tilde{\Omega}_j$ to be able to represent the error.

2 Computing the Optimal Coarse Correction

Having identified the optimal coarse space, we need to explain a general method to determine the optimal coarse correction in it. While two different approaches for specific cases can be found in [8, 7], we present now a completely general approach: let us denote the interface between subdomain $\tilde{\Omega}_j$ and $\tilde{\Omega}_i$ by Γ_{ji}, and let the jumps in the Dirichlet and Neumann traces between subdomain solutions be denoted by

$$g_{ji}^n(x) := u_j^n(x) - u_i^n(x), \quad h_{ji}^n(x) := \partial_{n_j} u_j^n(x) + \partial_{n_i} u_i^n(x), \quad x \in \Gamma_{ji}, \tag{4}$$

where ∂_{n_j} denotes the outer normal derivative of subdomain $\tilde{\Omega}_j$. Then the error satisfies the transmission problem

$$\begin{aligned} \mathscr{L} e_j^n &= 0 && \text{in } \tilde{\Omega}_j, \\ e_j^n(x) - e_i^n(x) &= g_{ji}^n(x) && \text{on } \Gamma_{ji}, \\ \partial_{n_j} e_j^n(x) + \partial_{n_i} e_i^n(x) &= h_{ji}^n(x) && \text{on } \Gamma_{ji}. \end{aligned} \tag{5}$$

Its solution lies in the optimal coarse space, and when added to the iterates u_j^n, we obtain the solution: the domain decomposition method has become a direct solver, it is nilpotent, independently of the domain decomposition method and the problem we solve: no better coarse correction is possible!

We now give a weak formulation of the transmission problem (5). To simplify the exposition, we use the case of the Laplacian, $\mathscr{L} := -\Delta$. We multiply the partial differential equation from (5) in each subdomain $\tilde{\Omega}_j$ by a test function v_j and integrate by parts to obtain

$$\int_{\tilde{\Omega}_j} \nabla e_j^n \cdot \nabla v_j \,\mathrm{d}x - \sum_i \int_{\Gamma_{ji}} \frac{\partial e_j^n}{\partial n_j} v_j \,\mathrm{d}s = 0. \tag{6}$$

[1] For certain methods, continuity of the normal derivative is however assured, like in the FETI methods, or continuity of the Dirichlet traces, like in the Neumann-Neumann method or the alternating Schwarz method. This can be used to reduce the size of the optimal coarse space.

If we denote by $\tilde{e}^n$ and $\tilde{v}$ the functions defined on all of Ω by the piecewise definition $\tilde{e}^n|_{\tilde{\Omega}_j} := e_j^n$ and $\tilde{v}|_{\tilde{\Omega}_j} := v_j$, then we can combine (6) over all subdomains $\tilde{\Omega}_j$ to obtain

$$\int_\Omega \nabla \tilde{e}^n \cdot \nabla \tilde{v}\, \mathrm{d}x - \sum_{j>i} \int_{\Gamma_{ji}} \left(\frac{\partial e_j^n}{\partial n_j} v_j + \frac{\partial e_i^n}{\partial n_i} v_i \right) \mathrm{d}s = 0. \tag{7}$$

If we impose now continuity on the test functions v_j, i.e. $\tilde{v}$ to be continuous, then (7) becomes

$$\int_\Omega \nabla \tilde{e}^n \cdot \nabla \tilde{v}\, \mathrm{d}x - \sum_{j>i} \int_{\Gamma_{ji}} \left(\frac{\partial e_j^n}{\partial n_j} + \frac{\partial e_i^n}{\partial n_i} \right) \tilde{v}\, \mathrm{d}s = 0, \tag{8}$$

and we can use the data of the problem to remove the normal derivatives,

$$\int_\Omega \nabla \tilde{e}^n \cdot \nabla \tilde{v}\, \mathrm{d}x - \sum_{j>i} \int_{\Gamma_{ji}} h_{ji}^n \tilde{v}\, \mathrm{d}s = 0. \tag{9}$$

It is therefore natural to choose a continuous test function $\tilde{v}$ to obtain a variational formulation of the transmission problem (5), a function in the space

$$V := \{ v : v|_{\tilde{\Omega}_j} =: v_j \in H_1(\tilde{\Omega}_j),\ v_j = v_i \text{ on } \Gamma_{ji} \}. \tag{10}$$

Now the jump in the Dirichlet traces of the errors would in general be imposed on the trial function space,

$$U^n := \{ u : u|_{\tilde{\Omega}_j} =: u_j \in H_1(\tilde{\Omega}_j),\ u_j - u_i = g_{ji}^n \text{ on } \Gamma_{ji} \}, \tag{11}$$

so the complete variational formulation for (5) is:

$$\text{find } \tilde{e}^n \in U^n, \text{ such that } \int_\Omega \nabla \tilde{e}^n \cdot \nabla \tilde{v}\, \mathrm{d}x - \sum_{j>i} \int_{\Gamma_{ji}} h_{ji}^n \tilde{v}\, \mathrm{d}s = 0 \quad \forall \tilde{v} \in V. \tag{12}$$

To discretize the variational formulation (12), we have to choose approximations of the spaces V and U^n, and both spaces contain interior Dirichlet conditions. In a finite element setting, it is natural to enforce the homogeneous Dirichlet conditions in V_h strongly if the mesh is matching at the interfaces, i.e. we just impose the nodal values to be the same for V_h.

While at the continuous level, the optimal coarse correction lies in an infinite dimensional space except for 1d problems, see [5, 7], at the discrete level this space becomes finite dimensional. It is in principle then possible to use the optimal coarse space at the discrete level and to obtain a nilpotent method, i.e. a method which converges after the coarse correction, see for example [9, 8, 11, 10], and also [1] for conditions under which classical subdomain iterations can become nilpotent. It is however not very practical to use these high dimensional optimal coarse spaces, and we are thus interested in approximations.

3 Approximations of the Optimal Coarse Space

We have seen that the optimal coarse space contains functions which satisfy the homogeneous equation in each non-overlapping subdomain $\tilde{\Omega}_j$, i.e. they are harmonic in $\tilde{\Omega}_j$. To obtain an approximation of the optimal coarse space, it is therefore sufficient to define an approximation for the functions on the interfaces Γ_{ji}, which are then extended harmonically inside $\tilde{\Omega}_j$. A natural way to approximate the functions on the interfaces is to use a Sturm-Liuville eigenvalue problem, and then to select eigenfunctions which correspond to modes on which the subdomain iteration of the domain decomposition methods used is not effective. This can be done either for the entire subdomain, for example choosing eigenfunctions of the Dirichlet to Neumann operator of the subdomain, see [2], or any other eigenvalue problem along the entire boundary of the subdomain $\tilde{\Omega}_j$, or piecewise on each interface Γ_{ji}, in which case also basis functions relating cross points need be added [11, 10], see also the ACMS coarse space [12] and references therein. This can be done solving for example lower dimensional counterparts of the original problem along the interface Γ_{ji} with boundary conditions one at one end, and zero at the other, creating something like hat functions around the crosspoint. Doing this for example for a rectangular domain decomposed into rectangular subdomains for Laplace's equation, this would just generate Q1 functions on each subdomain. It is important however to not force these function to be continuous across subdomains, since they have to solve approximately the transmission problem (5) whose solution is not continuous, except for specific methods[2]. So the resulting coarse basis function is not a hat function with one degree of freedom, but it is a discontinuous hat function with e.g. four degrees of freedom if four subdomains meet at that cross point.

Different approaches not based on approximating an optimal coarse space, but also using eigenfunctions in the coarse space to improve specific inequalities in the convergence analysis of domain decomposition methods are GenEO [14], whose functions are also harmonic in the interior of subdomains, and [3, 4], where volume eigenfunctions are used which are thus not harmonic within subdomains. For a good overview, see [13].

4 Concrete Example: the Parallel Schwarz Method

We consider the high contrast diffusion problem $\nabla \cdot (a(x,y)\nabla u) = f$ in $\Omega = (0,1)^2$ with two subdomains $\Omega_1 = (0, \frac{1+\delta}{2}) \times (0,1)$ and $\Omega_2 = (\frac{1-\delta}{2}, 1) \times (0,1)$. The classical parallel Schwarz method is converging most slowly for low frequencies along the interface $x = \frac{1}{2}$, i.e. error components represented in the Laplacian case by $\sin(k\pi y)$, $k = 1, 2, \ldots, K$ for some small integer K, see for example [6]. These are precisely the eigenfunctions of the eigenvalue problem one obtains when using separation of variables, which in our high contrast case is

[2] see footnote 1

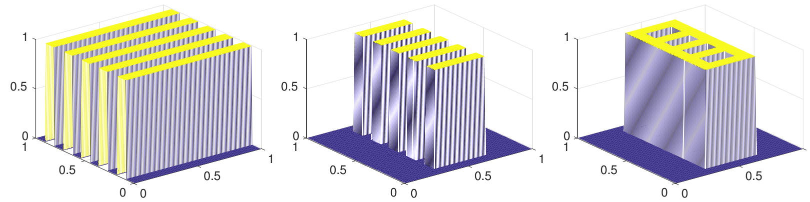

Fig. 1 An example with long channels, shortened channels, and closed shortened channels

$$\partial_y(a_\Gamma \partial_y \phi_k) = \lambda_k a_\Gamma \phi_k, \tag{13}$$

where a_Γ denotes the trace of the high contrast parameter along the interface, in our simple example $a_\Gamma(y) := a(\frac{1}{2}, y)$, and $\lambda_k \in \mathbb{R}$ denotes the eigenvalues and $\phi_k : (0,1) \mapsto \mathbb{R}$ the associated eigenfunctions, $k = 1, 2, \ldots$. So already in the case of Laplace's equation, it would be good to enrich a classical Q_1 coarse space aligned with the decomposition with harmonically extended eigenfunctions $\phi_k := \sin(k\pi y)$, $k = 1, 2, \ldots, K$ into the subdomains. We now illustrate why this is even more important in the case of high contrast channels, the $a(x,y)$ of which are shown in Figure 1. We show in Figure 2 the performance of a classical parallel Schwarz method with two subdomains for increasing overlap sizes. We see that for the case of the long channels, increasing the overlap improves the performance of the classical Schwarz methods as for the Laplacian[3], and nothing special happens between overlap $41h$ and overlap $43h$. This is however completely different for the shortened channel case, independently if they are closed or not, were increasing the overlap does not help at all, until suddenly changing from overlap $41h$ and overlap $43h$, the method becomes fast. This can be easily understood by the maximum principle, and is illustrated in Figure 3 which shows the errors in the subdomains. We clearly see that due to the fast diffusion the error propagates rapidly from the interface into the subdomains, and the maximum principle indicates slow convergence, as long as the overlap does not contain the shortened channels. As soon as the overlap contains the

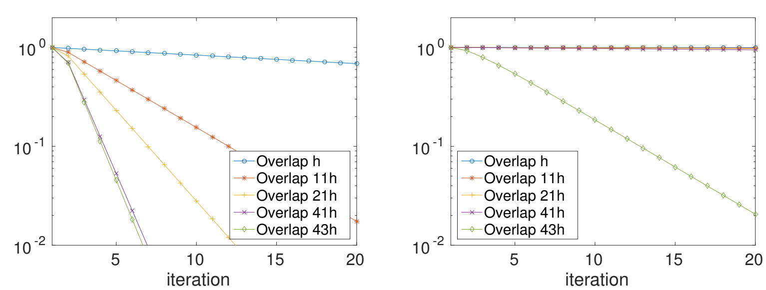

Fig. 2 Convergence behavior of a classical parallel Schwarz method for long high contrast channels (left), and shortened high contrast channels (right).

[3] the same happens if inclusions are only contained within the subdomains, outside the overlap

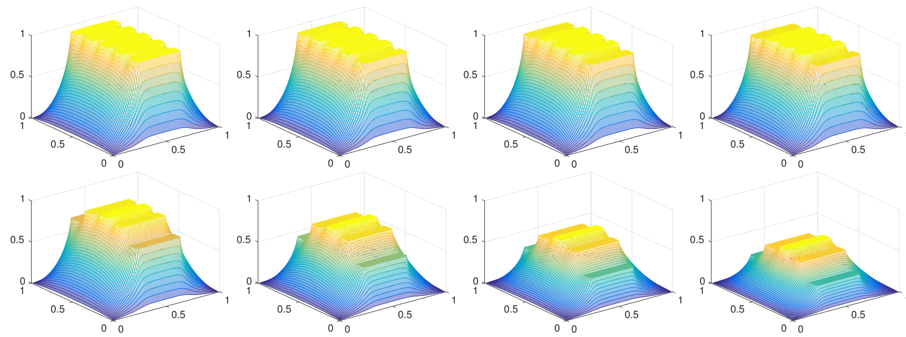

Fig. 3 Error for the first four iterations in the shortened channel case: Top overlap $41h$, and bottom overlap $43h$. We see that slightly more overlap suddenly leads to much more rapid convergence.

shortened channels, convergence becomes rapid. This is very different for the long channels, as illustrated in Figure 4. Here the channels touch the outer boundary of the domain, and the maximum principle indicates rapid convergence.

The case of shortened channels is precisely the situation where the convergence mechanism of the underlying domain decomposition method has problems, and if one can not afford a large enough overlap, a well chosen coarse space can help. It suffices to add harmonically extended low frequency modes of the cheap, lower dimensional interface eigenvalues problem to the coarse space, leading to the so called Spectrally Harmonically Enriched Multiscale coarse space (SHEM), see [11, 10]. Figure 5 shows that the eigenfunctions of the cheap interface eigenvalue problem are almost identical to the eigenfunctions obtained from the expensive DtN eigenvalue problem on the shortened channels from [2, 12], and still very similar to the ones of the DtN eigenvalue problem on the shortened closed channels, except for the first one, where the DtN eigenvalue problem for the shortened channel case sees the connection leading to a lowest mode like for one wider channel. We show in Figure 6 on the left the eigenvalues of the cheap interface eigenvalue problem, com-

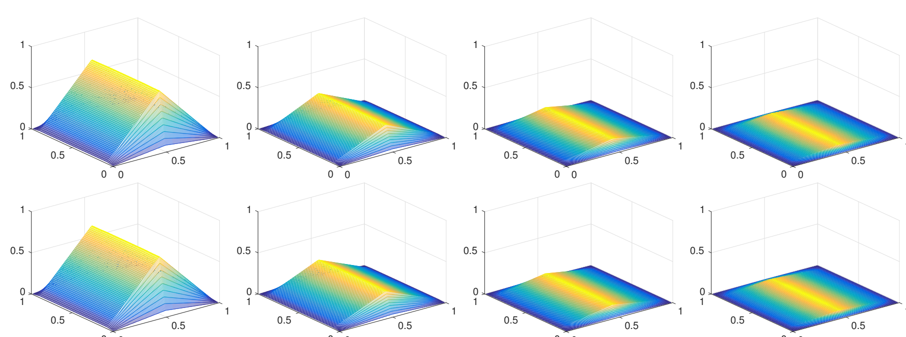

Fig. 4 Error for the first four iterations in the long channel case: Top overlap $41h$, and bottom overlap $43h$. We see that slightly more overlap leads only to slightly more rapid converge.

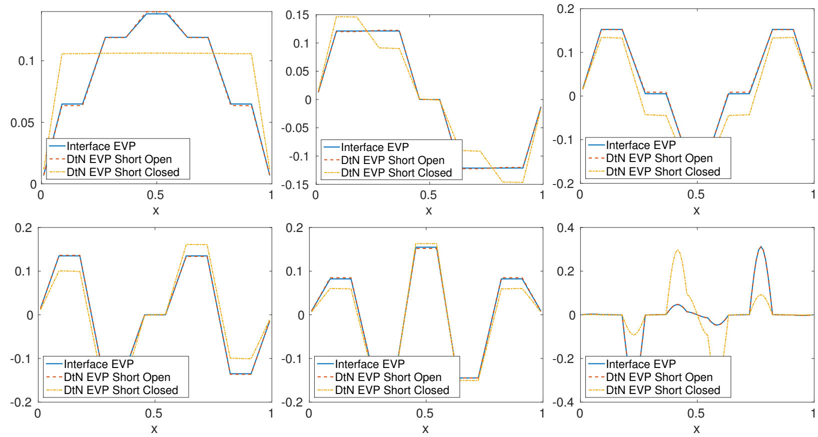

Fig. 5 Eigenfunctions of the different eigenvalue problems compared

pared to the eigenvalues of the expensive DtN-operator on the shortened channels and the shortened closed channels. They all indicate via the smallest eigenvalues that there are five channels, and five coarse functions are needed for good convergence, see Figure 6 on the right. The DtN-eigenvalue problem for the shortened closed channels also indicates that there is only one eigenvalue going to zero when the contrast becomes large. To obtain good convergence, it is however also in the closed shortened channel case necessary to include five enrichment functions in the coarse space, see Figure 7. It thus suffices as in SHEM to use the inexpensive interface eigenvalue problem to construct an effective approximation of the optimal coarse space, see [10] for simulations in the more general case of many subdomains and contrast functions.

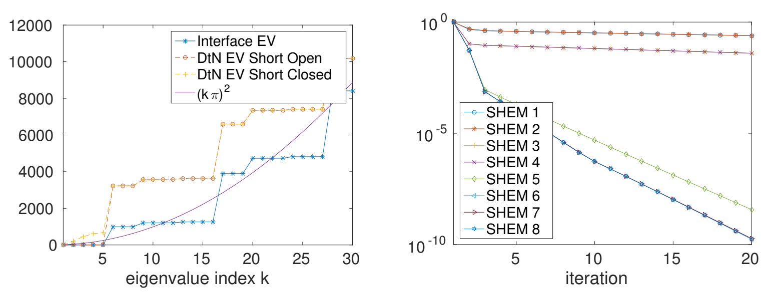

Fig. 6 Left: staircase behavior of the eigenvalues. Right: convergence with optimized coarse space on shortened channels

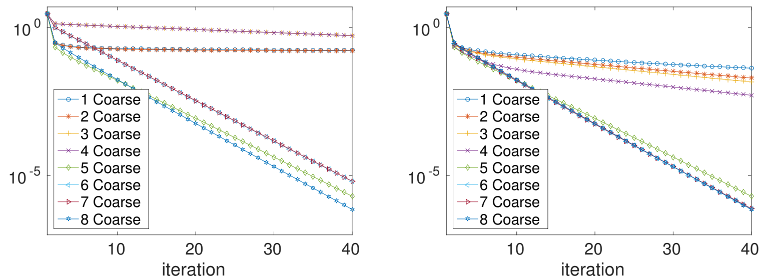

Fig. 7 Shortened closed channels problem. Left: coarse space based on the cheap interface eigenvalue problem. Right: coarse space based on the expensive DtN eigenvalue problem which sees the exact structure inside the subdomain, i.e. the shortened closed channels. Note that the performance is essentially the same: in both cases one needs 5 or more enrichment functions to get good convergence.

5 Conclusions

We introduced the concept of optimal coarse spaces for general elliptic problems and domain decomposition methods with and without overlap, i.e. coarse spaces which lead to convergence in one iteration. We then gave a variational formulation for the associated coarse problem that needs to be solved in each iteration. We also explained how to approximate this coarse problem, by employing coarse space components for which the underlying domain decomposition method exhibits slow convergence, and illustrated this approach with a high contrast problem containing channels. The main advantage of our construction is that it is not based on a convergence analysis, but on the domain decomposition iteration itself, and can thus also be applied to methods where general convergence analyses are not yet available, like for example RAS [8, 10] and optimized Schwarz methods [7].

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