Iterative Methods for Helmholtz and Maxwell Equations

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There is a certain confusion about which equation is really called the Helmholtz equation, since in the literature on meteorology and climate simulation, also the similar looking equation with the opposite sign on the zeroth order term is called Helmholtz equation, due to early publications using this terminology [2, 3]. Even standard textbooks have adopted this terminology, see for example [4] (sometimes also the eigenvalue problem is called the Helmholtz equation [5]). This summer during a conference, I was asked if maybe Helmholtz himself had already considered both equations, and so looked in the collected works of Helmholtz [6]. The contributions of Helmholtz to advancing science are vast, he worked on hydrodynamics, acoustics (physical and physiological), electrodynamics, galvanism (the contraction of muscles stimulated by electric current), optics (physical and physiological), and even psychology. In a beautiful paper about the understanding of organ pipes [7], see Figure 1, I then found *the Helmholtz equation*. Helmholtz describes in his paper the problem of the open end of the pipe, and using a domain decomposition approach, he connects the outer spherical solution to the inner one in the tube, in order to determine an appropriate boundary condition, a problem that had not been satisfactorily addressed before, as described at the beginning of the paper, see Figure 1.

XVI.

Theorie der Luftschwingungen in Röhren mit offenen Enden.

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Die mathematische Theorie der Orgelpfeifen ist von den bedeutendsten mathematischen Physikern vielfältig behandelt worden, aber seit den ersten Schritten, welche D. Bernoulli und Euler gethan haben, und durch welche die Hauptzüge der Erscheinung eine annähernde Erklärung fanden, um keinen wesentlichen Schritt vorgerückt. Der Grund davon hat hauptsächlich darin gelegen, dass die Mathematiker es nicht wagten die Annahme aufzugeben, dass die Bewegung der Lufttheilchen im Innern der Röhre überall ihrer Axe parallel gerichtet, und sowohl die Geschwindigkeit wie der Druck in allen Punkten desselben Querschnittes der Röhre gleich gross sei. Diese von den ersten Bearbeitern der Einfachheit wegen gemachte Annahme ist ganz unbedenklich für die von offenen Enden entfernteren Theile einer cylindrischen oder prismatischen Röhre, aber in der Nähe offener Enden, wo die ebenen Wellen der Röhre in den freien Raum überzugehen anfangen, um sich dort in Form kugeliger Wellen auszubreiten, ist jene Annahme nicht mehr zulässig, da es klar ist, dass ein solcher Uebergang nicht sprungweise geschehen kann. Bernoulli, Euler und Lagrange

und Schallschwingungen erregen. In allen anderen Theilen der Luftmasse ist $q = 0$, und dort sind daher die Functionen Ψ der Bedingung unterworfen:

$$0 = k^2\Psi + \frac{d^2\Psi}{dx^2} + \frac{d^2\Psi}{dy^2} + \frac{d^2\Psi}{dz^2}. \qquad (3\text{b})$$

Ich werde im Folgenden den immer wiederkehrenden Ausdruck:

$$\frac{d^2\Phi}{dx^2} + \frac{d^2\Phi}{dy^2} + \frac{d^2\Phi}{dz^2}$$

nach dem Vorgang von Green mit $\nabla_x \Phi$, oder wo es unzweideutig ist, mit $\nabla \Phi$ bezeichnen.

So- mit lässt sich jede stetige und eindeutige Function Ψ, welche in allen Theilen des Raumes S der Gleichung genügt:

$$\triangle\Psi + k^2\Psi = 0,$$

als Geschwindigkeitspotential von Erregungspunkten ausdrücken, die blos längs der Oberfläche von S ausgebreitet sind.

Figure 1. Copy of the beginning of the seminal publication of Helmholtz from [6], and an unusual shorthand notation for the Laplacian (the gradient symbol ∇), according to Helmholtz due to Green; only in one place later, Helmholtz uses the now common symbol Δ for the Laplacian (shown on the right).

The numerical approximation of solutions of the Helmholtz equation poses two main difficulties: the first one is related to the approximation problem. Usually, in order to represent a signal accurately, about 10 points per wavelength are sufficient. This is also true for solutions of the Helmholtz equation, but unfortunately, when one discretizes the equation itself on a grid with about 10 points per wavelength, the solution one obtains can be very inaccurate, not because there are not enough points to accurately represent it, but because the discretized operator gives a solution with a substantial phase error. This is the so called pollution effect, see for example [8] and references therein.

The second fundamental difficulty is that iterative methods for the solution of linear systems have historically been derived for discretizations of diffusive problems, especially Laplace's equation, and all the intuition and analysis that went into the development of these methods used fundamental properties of the discretized Laplace equation. Unfortunately, all these intuitions are incorrect for the Helmholtz equation: there is no maximum principle, classical iterative methods are not smoothers for the Helmholtz equation, there is no minimization principle. In [1] one can find detailed explanations why Krylov methods, ILU preconditioners, multigrid and classical domain decomposition methods fail when used for the Helmholtz equation. Only specialized methods for the Helmholtz equation should be used, and in particular a new class of domain decomposition methods, called optimized Schwarz methods, is quite effective [9, 10]. The time harmonic Maxwell's equations present the same two difficulties as the Helmholtz equation, and optimized Schwarz methods have been developed for them, see e.g. [11].

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