Overlapping Schwarz Waveform Relaxation for Parabolic Problems in Higher Dimensions

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Abstract

We quantify the effect of overlap on the convergence rate of the waveform relaxation algorithm for parabolic problems. We prove linear convergence of the algorithm on unbounded time intervals and superlinear convergence on bounded time intervals for the heat equation in n dimensions as a model problem. We describe how these results can be extended to more general parabolic problems.

1 Introduction

There are two classical convergence results for waveform relaxation algorithms for ordinary differential equations (ODEs): (i) for linear systems of ODEs on unbounded time intervals one can show linear convergence of the algorithm under some dissipation assumptions on the splitting ([14], [13], [4] and [12]); (ii) for nonlinear systems of ODEs (including linear ones) on bounded time intervals one can show superlinear convergence assuming a Lipschitz condition on the splitting function ([14], [2] and [3]).

If waveform relaxation is applied to partial differential equations (PDEs) discretized in space, the convergence results (i) and (ii) may depend on the discretization parameter for classical matrix splittings. The convergence rates then deteriorate as one refines the mesh.

Jeltsch and Pohl propose in [12] a multi-splitting algorithm with overlap. They prove results (i) and (ii) for their algorithm, but the convergence rates are again mesh dependent. They show however

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numerically for a discretized one dimensional heat equation that increasing the overlap accelerates the convergence of the waveform relaxation algorithm. Similar observations were made by Burrage et al [5] in higher dimensions and Cai in [6].

Gander and Stuart [9] relate the overlap in the multi-splitting algorithm to physical overlap of subdomains for the one dimensional heat equation. They quantify how the overlap affects the convergence rate and prove result (i) independent of the mesh parameter, provided the physical overlap is hold constant. Independently Giladi and Keller [11] proved result (ii) for an overlapping domain decomposition approach independent of the mesh parameter.

The dependence of overlap on the convergence rate in higher dimensions is the topic of this note. We derive results (i) and (ii) for a heat equation in $\mathbb{R}^n$. We show in the last section how these results can be extended to more general parabolic problems.

2 Model Problem

We are interested to solve a parabolic partial differential equation in n dimensions using waveform relaxation. We consider the heat equation on a bounded domain $\Omega \subset \mathbb{R}^n$ with a smooth boundary $\partial\Omega$ as our guiding example,

$$\begin{array}{rcll} \frac{\partial u}{\partial t} & = & \Delta u + f(\boldsymbol{x},t) & \boldsymbol{x} \in \Omega, 0 < t < T \\ u(\boldsymbol{x},t) & = & g(\boldsymbol{x},t) & \boldsymbol{x} \in \partial\Omega, 0 < t < T \\ u(\boldsymbol{x},0) & = & u_0(\boldsymbol{x}) & \boldsymbol{x} \in \Omega. \end{array} \tag{2.1}$$

We assume that the initial condition $u_0(\boldsymbol{x})$ and the boundary condition $g(\boldsymbol{x},t)$ are bounded piecewise continuous and $f(\boldsymbol{x},t)$ is continuous. This gives existence and uniqueness of a solution ([8], pp 40). Central in our analysis is the maximum principle satisfied by the solution $u(\boldsymbol{x},t)$ of (2.1) for $f(\boldsymbol{x},t) \equiv 0$:

Theorem 2.1 (Maximum Principle) *For $f(\boldsymbol{x},t) \equiv 0$ the solution $u(\boldsymbol{x},t)$ attains its maximum and minimum value either at $t = 0$ or for $\boldsymbol{x} \in \partial\Omega$.*

Proof The proof can be found for example in Friedman ([8], pp 34). ■

We decompose the domain Ω into N subdomains Ω_j with smooth boundaries $\partial\Omega_j$, $j = 1, 2, \ldots, N$ as shown for a two dimensional example in Figure 1. It is important to notice that we require the decomposition to satisfy the following overlap property: there exists a δ such that for each $\boldsymbol{x} \in \partial\Omega_j - \partial\Omega$, $j = 1, 2, \ldots, N$ the ball $B(\boldsymbol{x}, \delta)$

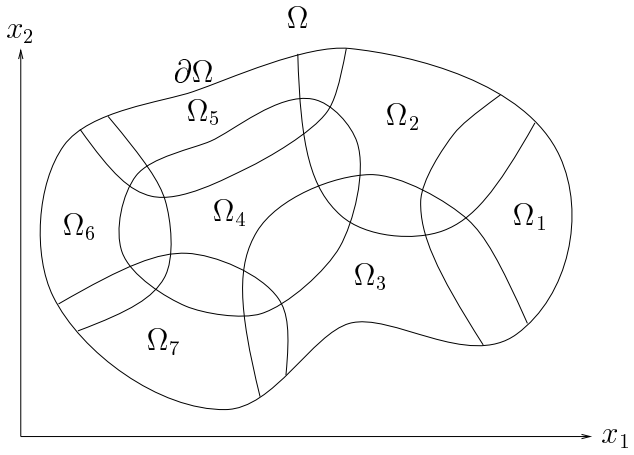

Figure 1: Decomposition into two overlapping subdomains.

with radius δ centered at $\boldsymbol{x}$ lies completely inside $\Omega_k \cup \Omega'$ for some $k \neq j$ where $\Omega' := \mathbb{R}^n - \Omega$. Note that this does note require that internal boundaries cross the boundary of Ω transversally. To solve the parabolic problem (2.1) we propose a waveform relaxation iteration which solves for the solution variable u_j on each subdomain Ω_j using as the boundary condition the values from the neighboring subdomains.

Naturally the question arises which subdomains can be solved simultaneously in parallel independently of one another. This corresponds to coloring the subdomains so that subdomains with the same color do not touch each other. Then subdomains of the same color can be solved in parallel using the boundary values coming from subdomains with different colors.

3 Linear Convergence for Unbounded Time Domains

We first consider the case where $T = \infty$. On each subdomain Ω_j we solve at each step $k+1$ of the waveform relaxation iteration the subproblem

$$\begin{array}{rcll} \dfrac{\partial u_j^{k+1}}{\partial t} & = & \Delta u_j^{k+1} + f(\boldsymbol{x},t) & \boldsymbol{x} \in \Omega_j, 0 < t < T \\ u_j^{k+1}(\boldsymbol{x},t) & = & u_l^k(\boldsymbol{x},t) & \boldsymbol{x} \in \Gamma_{jl}, 0 < t < T \\ u_j^{k+1}(\boldsymbol{x},t) & = & g(\boldsymbol{x},t) & \boldsymbol{x} \in \Gamma_{j0}, 0 < t < T \\ u(\boldsymbol{x},0) & = & u_0(\boldsymbol{x}) & \boldsymbol{x} \in \Omega_j, \end{array} \tag{3.1}$$

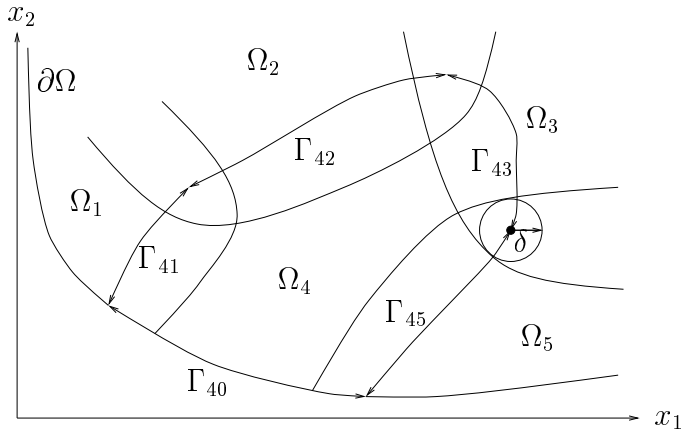

Figure 2: Passing of boundary information

for $j = 1, 2, \ldots, N$, using the boundary information from the neighboring subdomains at step k. To pass the boundary information, the boundary of Ω_j is decomposed into disjoint subsets Γ_{jl}, $l = 1, \ldots, N$ such that the Euclidean distance of $\boldsymbol{x} \in \Gamma_{jl}$ from the boundary of Ω_l is at least δ. This is possible because of the overlap property. Furthermore denote by Γ_{j0} the part of the boundary that Ω_j shares with Ω. An example for one subdomain with its neighbors in two dimensions is shown in Figure 2. The error $e_j^{k+1}(\boldsymbol{x}, t)$ between the real solution $u(\boldsymbol{x}, t)$ of (2.1) and the iterates $u_j^{k+1}(\boldsymbol{x}, t)$ of (3.1) satisfies the homogeneous equation

$$\begin{array}{rcll} \dfrac{\partial e_j^{k+1}}{\partial t} & = & \Delta e_j^{k+1} & \boldsymbol{x} \in \Omega_j, 0 < t < T \\ e_j^{k+1}(\boldsymbol{x}, t) & = & e_l^k(\boldsymbol{x}, t) & \boldsymbol{x} \in \Gamma_{jl}, 0 < t < T \\ e_j^{k+1}(\boldsymbol{x}, t) & = & 0 & \boldsymbol{x} \in \Gamma_{j0}, 0 < t < T \\ e_j^{k+1}(\boldsymbol{x}, 0) & = & 0 & \boldsymbol{x} \in \Omega_j. \end{array} \tag{3.2}$$

Define the integer distance quantity m_j for each subdomain Ω_j to be the least number of subdomains one has to pass through to touch the boundary $\partial\Omega$, and define $m := \max_j m_j$. Furthermore define the index sets $I_l := \{j : m_j = l\}$ so that the index set I_l contains the indices of all the subdomains which are within distance l of the boundary. Defining for bounded functions $g(\boldsymbol{x}, t) : \Omega \times [0, \infty) \to \mathbb{R}$ the norm

$$||g(\cdot, \cdot)||_\infty := \sup_{\boldsymbol{x} \in \Omega, t > 0} |g(\boldsymbol{x}, t)|$$

we have the following

Lemma 3.1 *The error of the waveform relaxation algorithm decays at the rate*

$$\max_j ||e_j^{k+m+1}(\cdot,\cdot)||_\infty \leq \gamma \max_j ||e_j^k(\cdot,\cdot)||_\infty \tag{3.3}$$

where γ is a number strictly less than one, independent of k.

Proof Fix k and define $E^k := \max_j ||e_j^k(\cdot,\cdot)||_\infty$ and note that on each subdomain the solution $\tilde{e}_j^{k+1}$ of the elliptic problem

$$\begin{aligned} 0 &= \Delta \tilde{e}_j^{k+1} && \boldsymbol{x} \in \Omega_j \\ \tilde{e}_j^{k+1}(\boldsymbol{x}) &= E^k && \boldsymbol{x} \in \Gamma_{jl} \\ \tilde{e}_j^{k+1}(\boldsymbol{x}) &= 0 && \boldsymbol{x} \in \Gamma_{j0} \end{aligned} \tag{3.4}$$

is an upper bound on the modulus of e_j^{k+1}. Now $\tilde{e}_j^{k+1}$ satisfies a maximum principle and for $j \in I_0$ $\tilde{e}_j^{k+1} < E^k$ in the interior of Ω_j, since $\tilde{e}_j^{k+1}$ satisfies on part of the boundary of Ω_j a homogeneous boundary condition. Note that for $j \notin I_0$ we have $\tilde{e}_j^{k+1}$ not necessarily strictly less than E^k since $\tilde{e}_j^{k+1}$ might have the value E^k on all its boundaries and thus by the maximum principle $\tilde{e}_j^{k+1} \equiv E^k$. Define

$$E^{k+1} := \sup_{\substack{\boldsymbol{x} \in \Gamma_{jl} \\ 1 \leq j \leq N \\ l \in I_0}} \tilde{e}_l^{k+1} \leq \gamma_1 E^k$$

for some constant $\gamma_1 < 1$. Note that γ_1 does not depend on k since the error $\tilde{e}_j^{k+1}$ is a linear function of the boundary condition. Now for the next iteration by definition part of the boundary of subdomains Ω_j with $j \in I_1$ lie strictly within Ω_l with $l \in I_0$ and therefore for $j \in I_1$ the solution $\tilde{e}_j^{k+2}$ of the elliptic problem

$$\begin{aligned} 0 &= \Delta \tilde{e}_j^{k+2} && \boldsymbol{x} \in \Omega_j \\ \tilde{e}_j^{k+2}(\boldsymbol{x}) &= E^k && \boldsymbol{x} \in \Gamma_{jl},\ l \notin I_0 \\ \tilde{e}_j^{k+2}(\boldsymbol{x}) &= E^{k+1} && \boldsymbol{x} \in \Gamma_{jl},\ l \in I_0 \end{aligned} \tag{3.5}$$

is an upper bound on the modulus of e_j^{k+2}. Since $E^{k+1} \leq \gamma_1 E^k$ we have by the maximum principle $\tilde{e}_j^{k+2} < E^k$ in the interior and defining E^{k+2} similarly to E^{k+1} before, we find $E^{k+2} \leq \gamma_2 E^k$ for some constant $\gamma_2 < 1$ independent of k. By induction we find at step $k+m$ for the error in the subdomains Ω_j with $j \in I_m$ the elliptic upper bound

$$\begin{aligned} 0 &= \Delta \tilde{e}_j^{k+m} && \boldsymbol{x} \in \Omega_j \\ \tilde{e}_j^{k+m}(\boldsymbol{x}) &= E^k && \boldsymbol{x} \in \Gamma_{jl},\ l \in I_m \\ \tilde{e}_j^{k+m}(\boldsymbol{x}) &= E^{k+m-1} && \boldsymbol{x} \in \Gamma_{jl},\ l \notin I_m \end{aligned} \tag{3.6}$$

and $\tilde{e}_j^{k+m} < E^k$ in the interior. Defining E^{k+m} as before we find $E^{k+m} \leq \gamma_m E^k$ for some constant $\gamma_m < 1$ independent of k. Now for the next iteration step $k+m+1$ all the errors e_j^{k+m+1} have boundary values less than or equal to $E^{k+m} \leq \gamma_m E^k$, since they come from iteration step $k+m$ in the interior of neighboring subdomains. Defining $\gamma := \gamma_m$ the result follows. ■

Note that the above estimate for γ is quite conservative. In a practical implementation the measured convergence rate is expected to be better, since γ was derived assuming worst case behavior of the error. γ will also depend on the shape of the subdomains, which is hidden in the above argument because of the generality of the domain decomposition employed.

Theorem 3.2 (Linear Convergence) *The waveform relaxation algorithm converges linearly on unbounded time intervals in the infinity norm. The error decays like*

$$\max_j ||e_j^{km}(\cdot,\cdot)||_\infty \leq \gamma^k \max_j ||e_j^0(\cdot,\cdot)||_\infty \tag{3.7}$$

where $\gamma < 1$ as in Lemma 3.1.

Proof The proof follows by induction using Lemma 3.1. ■

Note that the convergence result on unbounded time domains depends on the number of subdomains. The more subdomains one uses, the longer it takes for information to propagate from the outer boundary of Ω to the inner subdomains. This is because the steady state solution is limiting the convergence rate, and the steady state solution does not see the zero initial condition. This is different in the superlinear convergence analysis of the next section.

4 Superlinear Convergence for Bounded Time Domains

We now consider a bounded time interval, $T < \infty$. Like in the unbounded time domain case we are solving at each step $k+1$ of the waveform relaxation iteration the subproblem (3.1) using the boundary information from the neighboring subdomains at step k. We are interested in estimating the decay of the error e_j^{k+1} in (3.2) for short time t. Define the infinity norm of a function $g(\boldsymbol{x}, t)$ on the boundary Γ_j of subdomain Ω_j by

$$||g(\cdot,\cdot)||_{\Gamma_j} := \sup_{\boldsymbol{x}\in\Gamma_j, 0<t<T} |g(\boldsymbol{x}, t)|.$$

We first estimate the decay of the error on the boundary of the subdomains over one step of the iteration.

Lemma 4.1 *In n dimensions, the maximum error e_j^{k+1} on the boundary of all subdomains Ω_j decays in the infinity norm at the rate*

$$\max_j ||e_j^{k+1}||_{\Gamma_j} \leq 2n erfc(\frac{\delta}{2\sqrt{nT}}) \max_j ||e_j^k||_{\Gamma_j}.$$

Proof Define $E_k(t) := \max_j(\sup_{\boldsymbol{x}\in\Gamma_j} |e_j^k(\boldsymbol{x},t)|)$. Because of the overlapping property we are interested in the magnitude of $e_j^k(\boldsymbol{x},t)$ at distance δ from the boundary Γ_j for short time. An upper bound for all e_j^k can be obtained from the decay of $\overline{e}$ which satisfies

$$\begin{array}{rcll} \frac{\partial \overline{e}}{\partial t} & = & \Delta \overline{e} & \boldsymbol{x} \in B(\delta),\ 0 < t < T \\ \overline{e}(\boldsymbol{x},t) & = & E_k(t) & \boldsymbol{x} \in \partial B(\delta),\ 0 < t < T \\ \overline{e}(\boldsymbol{x},0) & = & 0 & \boldsymbol{x} \in B(\delta) \end{array} \tag{4.1}$$

where $B(\delta)$ denotes a ball with radius δ in $\mathbb{R}^n$. To get an explicit bound, inscribe a hypercube $Q(\delta)$ in $\mathbb{R}^n$ with side $\frac{2\delta}{\sqrt{n}}$ into the ball $B(\delta)$ and consider the decay of $\tilde{e}$ in the hypercube $Q(\delta)$,

$$\begin{array}{rcll} \frac{\partial \tilde{e}}{\partial t} & = & \Delta \tilde{e} & \boldsymbol{x} \in Q(\delta),\ 0 < t < T \\ \tilde{e}(\boldsymbol{x},t) & = & E_k(t) & \boldsymbol{x} \in \partial Q(\delta),\ 0 < t < T \\ \tilde{e}(\boldsymbol{x},0) & = & 0 & \boldsymbol{x} \in Q(\delta). \end{array} \tag{4.2}$$

Evaluating $\tilde{e}$ at the center of the hypercube $Q(\delta)$ we obtain an upper bound on the error e_j^k at distance δ from the boundary Γ_j. An upper bound on $\tilde{e}$ at the center of the hypercube $Q(\delta)$ can be obtained by summing the half space solution v of the heat equation for each of the $2n$ faces of the hypercube,

$$\begin{array}{rcll} \frac{\partial v}{\partial t} & = & \Delta v & \boldsymbol{x} \in \mathbb{R}^n,\ x_1 > 0,\ 0 < t < T \\ v(\boldsymbol{x},t) & = & E_k(t) & \boldsymbol{x} \in \mathbb{R}^n,\ x_1 = 0,\ 0 < t < T \\ v(\boldsymbol{x},0) & = & 0 & \boldsymbol{x} \in \mathbb{R}^n,\ x_1 \geq 0, \end{array} \tag{4.3}$$

where x_1 denotes the first component of $\boldsymbol{x} \in \mathbb{R}^n$. The solution of (4.3) is [7]

$$v(\boldsymbol{x},t) = \int_0^t K_x(x_1, t-\tau) E_k(\tau) d\tau, \tag{4.4}$$

with the kernel

$$K_x(x,t) = \frac{x}{2\sqrt{\pi} t^{3/2}} e^{-\frac{x^2}{4t}}. \tag{4.5}$$

Hence an upper bound on e_j^k at distance δ from the boundary Γ_j is given by $2n$ times (4.4). This value is by the formulation of the iteration an upper bound on the boundary values for e_j^{k+1} and thus

$$E_{k+1}(t) := \max_j(\sup_{\boldsymbol{x}\in\Gamma_j}|e_j^{k+1}(\boldsymbol{x},t)|) \leq 2n\int_0^t K_x(\frac{\delta}{\sqrt{n}}, t-\tau)E_k(\tau)d\tau. \tag{4.6}$$

Now take the supremum of $E_k(t)$, $0 < t < T$ out of the integral and apply the variable transform

$$y := \frac{\delta}{2\sqrt{n(t-\tau)}}$$

to the integral. This leads to

$$E_{k+1}(t) \leq 2n\mathrm{erfc}(\frac{\delta}{2\sqrt{nt}})\max_j||e_j^k||_{\Gamma_j}.$$

Since $\mathrm{erfc}(\delta/(2\sqrt{nt}))$ is monotonically increasing with t we can replace t with the upper bound T on the right. Now the bound is independent of t so we can take the supremum over t on the left and the result follows. ■

Note that Lemma 4.1 can be used to derive an arbitrary fast linear convergence rate by shortening the time interval $[0,T)$ appropriately, since $\lim_{x\to\infty}\mathrm{erfc}(x) = 0$. We derive however an upper bound on the decay of the error over k steps of the iteration which leads to a super linear convergence result.

Lemma 4.2 *In n dimensions, the maximum error e_j^k on the boundary of all subdomains Ω_j decays in the infinity norm at the rate*

$$\max_j||e_j^k||_{\Gamma_j} \leq (2n)^k erfc(\frac{k\delta}{2\sqrt{nT}})\max_j||e_j^0||_{\Gamma_j}.$$

Proof By iteration of inequality (4.6) in Lemma 4.1 we get a bound in form of a convolution, namely

$$\begin{aligned} E_k(t) \quad \leq \quad & (2n)^k\int_0^t K_x(\frac{\delta}{\sqrt{n}}, t-s_1)\cdots \\ & \int_0^{s_{k-1}} K_x(\frac{\delta}{\sqrt{n}}, s_{k-1}-s_k)ds_k\cdots ds_1\max_j||e_j^0||_{\Gamma_j}. \end{aligned}$$

To unfold the convolutions, note that the Laplace transform of a convolution is the product of the Laplace transformed kernels. In our case the Laplace transform of the kernel is [1]

$$\int_0^\infty e^{st}K_x(\frac{\delta}{\sqrt{n}}, t)dt = e^{-\frac{\delta}{2\sqrt{n}}\sqrt{s}}$$

and thus the k-fold convolution is the product of identical exponentials in the Laplace transformed domain,

$$e^{-\frac{k\delta}{2\sqrt{n}}\sqrt{s}}.$$

Backtransforming this expression, we find

$$E_k(t) \leq (2n)^k \int_0^t K_x(\frac{k\delta}{\sqrt{n}}, t-\tau)d\tau \max_j ||e_j^0||_{\Gamma_j}.$$

Using a similar variable transform as in Lemma 4.1 the result follows. ■

Defining for bounded functions $g(\boldsymbol{x}, t) : \Omega \times [0, T) \to \mathbb{R}$ the norm

$$||g(\cdot,\cdot)||_T := \sup_{\boldsymbol{x}\in\Omega, 0<t<T} |g(\boldsymbol{x}, t)|$$

we have the following

Theorem 4.3 (Superlinear Convergence) *The waveform relaxation algorithm converges superlinearly on bounded time intervals in the infinity norm. The error decays like*

$$\max_j ||e_j^k(\cdot,\cdot)||_T \leq (2n)^k \, erfc(\frac{k\delta}{2\sqrt{nT}}) \max_j ||e_j^0(\cdot,\cdot)||_T. \tag{4.7}$$

Proof The proof follows from Lemma 4.2 and the maximum principle. ■

5 Generalization

Although we used the linear heat equation as our guiding example, the techniques introduced here can be generalized. The linear convergence rate is derived using a maximum principle. Hence similar results can be obtained for more general equations satisfying a maximum principle, like equations with variable coefficients and convection terms. The superlinear convergence rate relies only on a one dimensional result in our analysis, so that convergence results obtained in [10] for the one dimensional reaction diffusion equation lead to similar results in higher dimensions using the techniques presented here.

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