Influence of Overlap on the Convergence Rate of Waveform Relaxation

Martin J. Gander, Andrew M. Stuart

Abstract

We consider a waveform relaxation method applied to the inhomogeneous heat equation with piecewise continuous initial and boundary conditions and a bounded Hölder continuous forcing function. Traditionally, to obtain a waveform relaxation algorithm, the equation is first discretized in space and the discrete matrix is split. Superlinear convergence on bounded time intervals can be shown but the error bounds are dependent on Lipschitz constants of the splitting which, in the case of the heat equation, typically blow up as Δx goes to zero.

We split the partial differential equation (PDE) directly by using overlapping domain decomposition. We prove linear convergence of the algorithm in the continuous case on an infinite time interval, at a rate depending on the size of the overlap. This result remains valid after discretizing in space, leading to a waveform relaxation algorithm for the spatially discretized heat equation which exhibits linear convergence at a rate independent of the mesh parameter Δx. The algorithm is in the class of waveform relaxation algorithms based on over-lapping splittings.

Numerical results are presented which support the convergence theory.

1 Introduction

The basic ideas underlying waveform relaxation were first suggested in the late 19th century by Picard and Lindelöf ([13], [20]). There has been much recent interest in waveform relaxation as a practical parallel method for the solution of stiff ordinary differential equations (ODE's) after the publication of a paper by Lelarasmee and coworkers [12], and the paper by O'Leary and White [19] which introduced multi-splittings of matrices for the solution of linear systems of equations. Recent work in this field includes papers by Miekkala and Nevanlinna [15], [16], Nevanlinna [17] and Bellen and Zennaro [1].

The standard convergence result for a system of nonlinear ODE's needs the assumption that the splitting function is Lipschitz continuous in both arguments. It states superlinear convergence on any finite time interval $[0, T]$. Specifically the constant relating the error at the n^{th} iteration to the initial error is

$$\frac{(C_1 T)^n}{n!} e^{C_2 T} \tag{1.1}$$

where C_1 and C_2 are the global Lipschitz constants of the splitting function in the first and second argument. This result can be found for example in Bjørhus [4].

For a linear system of ODE's which is asymptotically stable Miekkala and Nevanlinna show in [15] the existence of splittings such that the waveform relaxation algorithm converges linearly on the infinite time interval $[0, \infty)$. Jeltsch and Pohl extended the results for the linear case on bounded time intervals to overlapping splittings and show superlinear convergence in [11]. They also extend the results on unbounded time intervals to overlapping splittings for a certain class of problems. Overlapping splittings lead to natural parallelism in the solution process.

However in all the results mentioned the constants in general depend badly on Δx if the linear ODE arises from a PDE which is discretized in space.

The domain decomposition algorithm was developed by Schwarz in 1869 [21] to show existence of harmonic functions on irregular domains which were compositions of regular domains. The paper by Lions [14] develops a framework for studying domain decomposition methods. Recent interest in domain decomposition as a computational tool has been motivated by the work of Bramble, Pasciak and Shatz [5], Dryja [10], Bjørstad and Widlund [2] and a thorough review may be found in Chan and Mathew [9]. There are two different approaches for domain decomposition. The first one is to divide the domain into overlapping subdomains and then solve the equations iteratively on each subdomain using the data from the adjacent domains as boundary data. This method is known as the Schwarz algorithm. The second approach is to divide the domain into nonoverlapping subdomains and then to solve the equations on the boundaries between subdomains first, before calculating the solution in their interior. This technique is known as the Schur or Poincaré-Steklov approach.

Motivated by the work of Bjørhus [3], we show in this paper how one can use overlapping domain decomposition to obtain a waveform relaxation algorithm for the semi-discrete heat equation which converges at a rate independent of the mesh parameter Δx. In section 2 we consider a decomposition of the domain into two subdomains. We prove linear convergence of the algorithm dependent on the size of the overlap in the continuous case, and we show that the same method of proof can be applied in the semi-discrete case. This section is mainly for illustrative purposes since the analysis can be performed in great detail. In section 3 we generalize this result to an arbitrary number of subdomains. Section 4 shows how our algorithm can be formulated in the framework of waveform relaxation, using overlapping splittings. In section 5 we show numerical experiments which confirm the convergence results, and section 6 contains concluding remarks.

Some ideas similar to those used here were apparently introduced by Nevanlinna at the 4^{th} International Symposium on Domain Decomposition Methods for Partial Differential Equations in Moscow in 1990 [18], but never published.

2 Two Subdomains

2.1 Continuous Case

Consider the one dimensional inhomogeneous heat equation on the interval $[0, L]$,

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + f(x,t) && 0 < x < L,\ t > 0 \\ u(0,t) &= g_1(t) && t > 0 \\ u(L,t) &= g_2(t) && t > 0 \\ u(x,0) &= u_0(x) && 0 < x < L, \end{aligned} \tag{2.1}$$

where we assume $f(x,t)$ to be bounded on the domain $[0, L] \times [0, \infty)$ and uniformly Hölder continuous on each compact subset of the domain. We assume furthermore that the initial data $u_0(x)$ and the boundary data $g_1(t)$, $g_2(t)$ are piecewise continuous. Then (2.1) has a unique bounded solution [8]. Given any function $f(t) : \mathbb{R}^+ \longrightarrow \mathbb{R}$ we define

$$||f(\cdot)||_\infty := \sup_{t>0} |f(t)|$$

Theorem 2.1 (Maximum Principle) *The solution $u(x,t)$ of the heat equation (2.1) with $f(x,t) \equiv 0$ attains its maximum and minimum either on the initial line $t = 0$ or on the boundary at $x = 0$ or $x = L$. If $u(x,t)$ attains its maximum in the interior, then $u(x,t)$ must be constant.*

Proof The proof can be found in [24]. ■

Corollary 2.2 *The solution $u(x,t)$ of the heat equation (2.1) with $f(x,t) \equiv 0$ and $u_0 \equiv 0$ satisfies the inequality*

$$||u(x,\cdot)||_\infty \leq \frac{L-x}{L}||g_1(\cdot)||_\infty + \frac{x}{L}||g_2(\cdot)||_\infty,\ 0 \leq x \leq L. \tag{2.2}$$

Proof Consider $\tilde{u}$ solving

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial t} &= \frac{\partial^2 \tilde{u}}{\partial x^2} && 0 < x < L,\ t > 0 \\ \tilde{u}(0,t) &= ||g_1(\cdot)||_\infty && t > 0 \\ \tilde{u}(L,t) &= ||g_2(\cdot)||_\infty && t > 0 \\ \tilde{u}(x,0) &= \frac{L-x}{L}||g_1(\cdot)||_\infty + \frac{x}{L}||g_2(\cdot)||_\infty && 0 \leq x \leq L \end{aligned} \tag{2.3}$$

The solution $\tilde{u}$ of (2.3) does not depend on t and is given by the steady state solution

$$\tilde{u}(x) = \frac{L-x}{L}||g_1(\cdot)||_\infty + \frac{x}{L}||g_2(\cdot)||_\infty$$

By construction we have $\tilde{u}(x) - u(x,t) \geq 0$ at $t = 0$ and on the boundary $x = 0$ and $x = L$. Since $\tilde{u} - u$ is in the kernel of the heat operator, we have by the maximum principle for the heat equation $\tilde{u}(x) - u(x,t) \geq 0$ on the whole domain $[0, L]$. Hence

$$u(x,t) \leq \frac{L-x}{L}||g_1(\cdot)||_\infty + \frac{x}{L}||g_2(\cdot)||_\infty.$$

Likewise $\tilde{u}(x) + u(x,t) \geq 0$ at $t = 0$, $x = 0$ and $x = L$ and is in the kernel of the heat operator. Hence

$$u(x,t) \geq -\left(\frac{L-x}{L}||g_1(\cdot)||_\infty + \frac{x}{L}||g_2(\cdot)||_\infty\right).$$

Therefore we have

$$|u(x,t)| \leq \frac{L-x}{L}||g_1(\cdot)||_\infty + \frac{x}{L}||g_2(\cdot)||_\infty.$$

Now the right hand side does not depend on t, so we can take the supremum over t, which leads to the desired result. ∎

We decompose the domain $\Omega = [0, L] \times [0, \infty)$ into two overlapping subdomains $\Omega_1 = [0, \beta L] \times [0, \infty)$ and $\Omega_2 = [\alpha L, L] \times [0, \infty)$ where $0 < \alpha < \beta < 1$ as given in figure 1. The solution $u(x,t)$ of (2.1) can now be obtained by composing the solutions

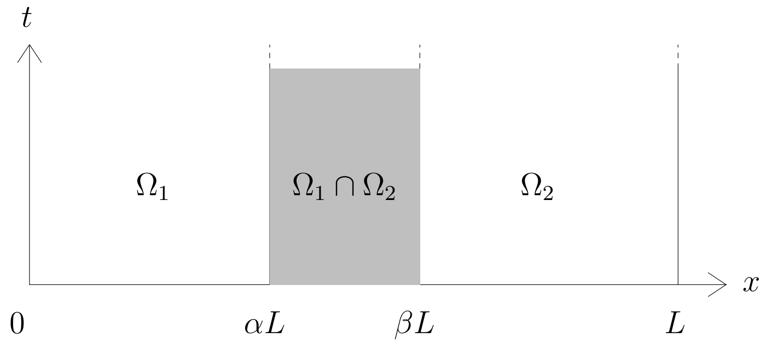

Figure 1: Decomposition into two overlapping subdomains.

$v(x,t)$ on Ω_1 and $w(x,t)$ on Ω_2, which satisfy the equations

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} + f(x,t) && 0 < x < \beta L,\ t > 0 \\ v(0,t) &= g_1(t) && t > 0 \\ v(\beta L, t) &= w(\beta L, t) && t > 0 \\ v(x,0) &= u_0(x) && 0 < x < \beta L \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \frac{\partial w}{\partial t} &= \frac{\partial^2 w}{\partial x^2} + f(x,t) && \alpha L < x < L,\ t > 0 \\ w(\alpha L,t) &= v(\alpha L, t) && t > 0 \\ w(L, t) &= g_2(t) && t > 0 \\ w(x,0) &= u_0(x) && \alpha L < x < L. \end{aligned} \tag{2.5}$$

To see why u can be obtained by composing v and w, note first that $v = u$ on Ω_1 and $w = u$ on Ω_2 are solutions to (2.4) and (2.5). To show that these solutions are unique, assume there is a second pair of solutions $\tilde{v} \neq v$ and $\tilde{w} \neq w$. Then the differences $\phi := v - \tilde{v}$ and $\psi := w - \tilde{w}$ satisfy the equations

$$\begin{array}{rcll} \frac{\partial \phi}{\partial t} & = & \frac{\partial^2 \phi}{\partial x^2} & 0 < x < \beta L,\ t > 0 \\ \phi(0,t) & = & 0 & t > 0 \\ \phi(\beta L, t) & = & \psi(\beta L, t) & t > 0 \\ \phi(x,0) & = & 0 & 0 < x < \beta L \end{array}$$

and

$$\begin{array}{rcll} \frac{\partial \psi}{\partial t} & = & \frac{\partial^2 \psi}{\partial x^2} & \alpha L < x < L,\ t > 0 \\ \psi(\alpha L,t) & = & \phi(\alpha L, t) & t > 0 \\ \psi(L, t) & = & 0 & t > 0 \\ \psi(x,0) & = & 0 & \alpha L < x < L. \end{array}$$

By the maximum principle ϕ attains its maximum ϕ_{max} and minimum ϕ_{min} on the boundary or on the initial line. Since we assumed $\phi \neq 0$ at least one of ϕ_{max}, ϕ_{min} is not equal to zero. Assume without loss of generality that $\phi_{max} > 0$. Then by the maximum principle $\phi_{max} = \sup_{t>0} \psi(\beta L, t)$ and by Corollary 2.2 $\phi(\alpha L, t) \leq \frac{\alpha}{\beta}\phi_{max} < \phi_{max}$ since $0 < \alpha < \beta < 1$. Similar $\psi_{max} = \sup_{t>0} \phi(\alpha L, t)$ and $\psi(\beta L, t) < \psi_{max}$. This implies

$$\phi_{max} = \sup_{t>0} \psi(\beta L, t) < \psi_{max} = \sup_{t>0} \phi(\alpha L, t) < \phi_{max}$$

which is a contradiction. Hence $\phi \equiv 0$. A similar argument shows $\psi \equiv 0$. Hence u is given by v on $[0, \beta L]$ and w on $[\alpha L, L]$. Note that $v \equiv w$ in the overlap $[\alpha L, \beta L]$.

The system (2.4) and (2.5) of equations, which is coupled through the boundary, can be solved using the Schwarz iteration

$$\begin{array}{rcll} \frac{\partial v^{k+1}}{\partial t} & = & \frac{\partial^2 v^{k+1}}{\partial x^2} + f(x,t) & 0 < x < \beta L,\ t > 0 \\ v^{k+1}(0,t) & = & g_1(t) & t > 0 \\ v^{k+1}(\beta L, t) & = & w^k(\beta L, t) & t > 0 \\ v^{k+1}(x,0) & = & u_0(x) & 0 < x < \beta L \end{array}$$

and

$$\begin{array}{rcll} \frac{\partial w^{k+1}}{\partial t} & = & \frac{\partial^2 w^{k+1}}{\partial x^2} + f(x,t) & \alpha L < x < L,\ t > 0 \\ w^{k+1}(\alpha L,t) & = & v^k(\alpha L, t) & t > 0 \\ w^{k+1}(L, t) & = & g_2(t) & t > 0 \\ w^{k+1}(x,0) & = & u_0(x) & \alpha L < x < L. \end{array}$$

Let $d^k(x,t) := v^k(x,t) - v(x,t)$ and $e^k(x,t) := w^k(x,t) - w(x,t)$ and consider the error equations

$$\begin{array}{rcll} \frac{\partial d^{k+1}}{\partial t} & = & \frac{\partial^2 d^{k+1}}{\partial x^2} & 0 < x < \beta L,\ t > 0 \\ d^{k+1}(0,t) & = & 0 & t > 0 \\ d^{k+1}(\beta L, t) & = & e^k(\beta L, t) & t > 0 \\ d^{k+1}(x,0) & = & 0 & 0 < x < \beta L \end{array} \tag{2.6}$$

and

$$\begin{array}{rcll} \frac{\partial e^{k+1}}{\partial t} & = & \frac{\partial^2 e^{k+1}}{\partial x^2} & \alpha L < x < L, \ t > 0 \\ e^{k+1}(\alpha L, t) & = & d^k(\alpha L, t) & t > 0 \\ e^{k+1}(L, t) & = & 0 & t > 0 \\ e^{k+1}(x, 0) & = & 0 & \alpha L < x < L. \end{array} \tag{2.7}$$

Lemma 2.3 *The error of the Schwarz iteration for the heat equation with two subdomains decays on $x = \alpha L$ and $x = \beta L$ at a rate depending on the size of the overlap. Specifically, we have*

$$||d^{k+2}(\alpha L, \cdot)||_\infty \leq \frac{\alpha(1-\beta)}{\beta(1-\alpha)} ||d^k(\alpha L, \cdot)||_\infty, \tag{2.8}$$

$$||e^{k+2}(\beta L, \cdot)||_\infty \leq \frac{\alpha(1-\beta)}{\beta(1-\alpha)} ||e^k(\beta L, \cdot)||_\infty. \tag{2.9}$$

Proof By Corollary 2.2 we have

$$||d^{k+2}(x, \cdot)||_\infty \leq \frac{x}{\beta L} ||e^{k+1}(\beta L, \cdot)||_\infty \quad \forall x \in [0, \beta L]. \tag{2.10}$$

and

$$||e^{k+1}(x, \cdot)||_\infty \leq \frac{L - x}{(1-\alpha)L} ||d^k(\alpha L, \cdot)||_\infty \quad \forall x \in [\alpha L, L]. \tag{2.11}$$

From (2.11) at $x = \beta L$ we get

$$||e^{k+1}(\beta L, \cdot)||_\infty \leq \frac{1-\beta}{1-\alpha} ||d^k(\alpha L, \cdot)||_\infty. \tag{2.12}$$

Likewise from (2.10) at $x = \alpha L$ we obtain

$$||d^{k+2}(\alpha L, \cdot)||_\infty \leq \frac{\alpha}{\beta} ||e^{k+1}(\beta L, \cdot)||_\infty. \tag{2.13}$$

Combining (2.12) and (2.13) gives the desired result. The second inequality (2.9) is obtained in the same way. ∎

Given any function $g(x, t) : [a, b] \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ we define

$$||g(\cdot, \cdot)||_{\infty,\infty} := \sup_{a<x<b, t>0} |g(x, t)|$$

Theorem 2.4 *The Schwarz iteration for the heat equation with two subdomains converges at a rate depending on the size of the overlap. The error on the two subdomains decays at the rate*

$$||d^{2k+1}(\cdot, \cdot)||_{\infty,\infty} \leq \left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^k ||e^0(\beta L, \cdot)||_\infty \tag{2.14}$$

$$||e^{2k+1}(\cdot, \cdot)||_{\infty,\infty} \leq \left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^k ||d^0(\alpha L, \cdot)||_\infty. \tag{2.15}$$

Proof Since the errors d^k and e^k are both in the kernel of the heat operator they obtain, by the maximum principle, their maximum value on the boundary or on the initial line. On the initial line and the exterior boundary both d^k and e^k vanish. Hence

$$\begin{aligned} ||d^{2k+1}(\cdot,\cdot)||_{\infty,\infty} &\leq ||e^{2k}(\beta L,\cdot)||_{\infty} \\ ||e^{2k+1}(\cdot,\cdot)||_{\infty,\infty} &\leq ||d^{2k}(\alpha L,\cdot)||_{\infty}. \end{aligned}$$

Using Lemma 2.3 the result follows. ∎

2.2 Semi-Discrete Case

Consider the heat equation continuous in time, but discretized in space using a centered second order finite difference scheme on a grid with n grid points and $\Delta x = \frac{L}{n+1}$. This gives

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} &= A_{(n)}\boldsymbol{u} + \boldsymbol{f}(t) \qquad t > 0 \\ \boldsymbol{u}(0) &= \boldsymbol{u}_0, \end{aligned} \tag{2.16}$$

where the $n \times n$ matrix $A_{(n)}$ is given by

$$A_{(n)} = \frac{1}{(\Delta x)^2} \begin{bmatrix} -2 & 1 & & 0 \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -2 \end{bmatrix} \tag{2.17}$$

and $\boldsymbol{f}(t) = (f(\Delta x, t) + \frac{1}{(\Delta x)^2} g_1(t), f(2\Delta x, t), \ldots, f((n-1)\Delta x, t), f(n\Delta x, t) + \frac{1}{(\Delta x)^2} g_2(t))^T$, $\boldsymbol{u}_0 = (u_0(\Delta x), \ldots, u_0(n\Delta x))^T$.

We note the following property of $A_{(n)}$ for later use: let $\boldsymbol{p} := (p_1, \ldots, p_n)^T$ where $p_j := j$. Then

$$A_{(n)}\boldsymbol{p} = (0, \ldots, 0, \frac{-(n+1)}{(\Delta x)^2})^T. \tag{2.18}$$

Likewise let $\boldsymbol{q} := (q_1, \ldots, q_n)^T$ where $q_j := n + 1 - j$. Then

$$A_{(n)}\boldsymbol{q} = (\frac{-(n+1)}{(\Delta x)^2}, 0, \ldots, 0)^T \tag{2.19}$$

We use the following notation: let $\boldsymbol{v}(t)$ be a time dependent vector valued function. We define $\boldsymbol{v}(i,t)$ to be the i-th component of $\boldsymbol{v}(t)$. We are choosing this notation to emphasize that the index i in the semi-discrete case plays the role of x in the continuous case. Furthermore if $\boldsymbol{u}(t)$ is another time dependent vector valued function, the notation $\boldsymbol{v}(t) \geq \boldsymbol{u}(t)$ means that the inequality holds in each component.

Theorem 2.5 (**Semi-Discrete Maximum Principle**) *Assume $\boldsymbol{u}(t)$ solves the semi-discrete heat equation (2.16) with $\boldsymbol{f}(t) = (f_1(t), 0, \ldots, 0, f_2(t))^T$ and $\boldsymbol{u}(0) = (u_1(0), \ldots, u_n(0))^T$. If $f_1(t)$ and $f_2(t)$ are non negative for $t \geq 0$ and $u_i(0) \geq 0$ for $i = 1, \ldots, n$ then*

$$\boldsymbol{u}(t) \geq 0, \ \forall t \geq 0.$$

Proof We follow Varga's proof in [22]. By Duhamel's principle the solution $\boldsymbol{u}(t)$ is given by

$$\boldsymbol{u}(t) = e^{A_{(n)}t}\boldsymbol{u}(0) + \int_0^t e^{A_{(n)}(t-s)}\boldsymbol{f}(s)ds \tag{2.20}$$

Now the matrix $e^{A_{(n)}t}$ has only non negative entries. To see this write $A_{(n)} = -2I_{(n)} + J_{(n)}$ where $J_{(n)}$ contains only non negative entries and $I_{(n)}$ is the identity matrix of size $n \times n$. We get

$$\begin{aligned} e^{A_{(n)}t} &= e^{-2I_{(n)}t}e^{J_{(n)}t} \\ &= e^{-2t}e^{J_{(n)}t} \\ &= e^{-2t}\sum_{l=0}^{\infty}\frac{J_{(n)}^l t^l}{l!} \end{aligned}$$

where the last expression has clearly only non negative entries. Since the matrix exponential in (2.20) is applied only to vectors with non-negative entries, it follows that $\boldsymbol{u}(t)$ can not become negative. ∎

Corollary 2.6 *The solution $\boldsymbol{u}(t)$ of the semi-discrete heat equation (2.16) with $\boldsymbol{f}(t) = (\frac{1}{(\Delta x)^2}g_1(t), 0, \ldots, 0, \frac{1}{(\Delta x)^2}g_2(t))^T$ and $\boldsymbol{u}_0 \equiv 0$ satisfies the inequality*

$$||\boldsymbol{u}(j,\cdot)||_\infty \leq \frac{n+1-j}{n+1}||g_1(\cdot)||_\infty + \frac{j}{n+1}||g_2(\cdot)||_\infty,\ 1 \leq j \leq n. \tag{2.21}$$

Proof Consider $\tilde{\boldsymbol{u}}(t)$ solving

$$\begin{aligned} \frac{\partial\tilde{\boldsymbol{u}}}{\partial t} &= A_{(n)}\tilde{\boldsymbol{u}} + \tilde{\boldsymbol{f}}, && t > 0 \\ \tilde{\boldsymbol{u}}(j,0) &= \frac{n+1-j}{n+1}||g_1(\cdot)||_\infty + \frac{j}{n+1}||g_2(\cdot)||_\infty, && 1 \leq j \leq n, \end{aligned} \tag{2.22}$$

with $\tilde{\boldsymbol{f}} = (\frac{1}{(\Delta x)^2}||g_1(t)||_\infty, 0, \ldots, 0, \frac{1}{(\Delta x)^2}||g_2(t)||_\infty)^T$. Using the properties (2.18) and (2.19) of $A_{(n)}$ and the linearity of (2.22) we find that the solution $\tilde{\boldsymbol{u}}$ of (2.22) does not depend on t and is given by the steady state solution

$$\tilde{\boldsymbol{u}}(j) = \frac{n+1-j}{n+1}||g_1(\cdot)||_\infty + \frac{j}{n+1}||g_2(\cdot)||_\infty,\ 1 \leq j \leq n.$$

The difference $\boldsymbol{\phi}(j,t) := \tilde{\boldsymbol{u}}(j) - \boldsymbol{u}(j,t)$ satisfies the equation

$$\begin{aligned} \frac{\partial\boldsymbol{\phi}}{\partial t} &= A_{(n)}\boldsymbol{\phi} + \begin{pmatrix} \frac{1}{(\Delta x)^2}(||g_1(\cdot)||_\infty - g_1(t)) \\ 0 \\ \vdots \\ 0 \\ \frac{1}{(\Delta x)^2}(||g_2(\cdot)||_\infty - g_2(t)) \end{pmatrix}, && t > 0 \\ \boldsymbol{\phi}(j,0) &= \frac{n+1-j}{n+1}||g_1(\cdot)||_\infty + \frac{j}{n+1}||g_2(\cdot)||_\infty, && 1 \leq j \leq n. \end{aligned}$$

and hence the discrete maximum principle applies to $\boldsymbol{\phi}$. We get $\boldsymbol{\phi}(j,t) \geq 0$ for all $t > 0$ and $1 \leq j \leq n$ and thus

$$\boldsymbol{u}(j,t) \leq \frac{n+1-j}{n+1}||g_1(\cdot)||_\infty + \frac{j}{n+1}||g_2(\cdot)||_\infty,\ 1 \leq j \leq n.$$

Likewise $\boldsymbol{\psi}(j,t) := \tilde{\boldsymbol{u}}(j) + \boldsymbol{u}(j,t)$ satisfies the equation

$$\begin{aligned}
\frac{\partial \boldsymbol{\psi}}{\partial t} &= A_{(n)}\boldsymbol{\psi} + \begin{pmatrix} \frac{1}{(\Delta x)^2}(||g_1(\cdot)||_\infty + g_1(t)) \\ 0 \\ \vdots \\ 0 \\ \frac{1}{(\Delta x)^2}(||g_2(\cdot)||_\infty + g_2(t)) \end{pmatrix}, \qquad t > 0 \\
\boldsymbol{\psi}(j,0) &= \frac{n+1-j}{n+1}||g_1(\cdot)||_\infty + \frac{j}{n+1}||g_2(\cdot)||_\infty, \qquad 1 \le j \le n.
\end{aligned}$$

and therefore by discrete maximum principle $\boldsymbol{\psi}(j,t) \geq 0$ for all $t > 0$ and $1 \le j \le n$. Hence

$$\boldsymbol{u}(j,t) \geq -\left(\frac{n+1-j}{n+1}||g_1(\cdot)||_\infty + \frac{j}{n+1}||g_2(\cdot)||_\infty\right), \ 1 \le j \le n.$$

We obtain thus a bound on the modulus of $\boldsymbol{u}$, namely

$$|\boldsymbol{u}(j,t)| \le \frac{n+1-j}{n+1}||g_1(\cdot)||_\infty + \frac{j}{n+1}||g_2(\cdot)||_\infty, \ 1 \le j \le n.$$

Now the right hand side does not depend on t, so we can take the supremum over t, which leads to the desired result. ■

We decompose the domain into two overlapping subdomains Ω_1 and Ω_2 as in figure 2. We assume for simplicity that αL falls on the grid point $i = a$ and βL

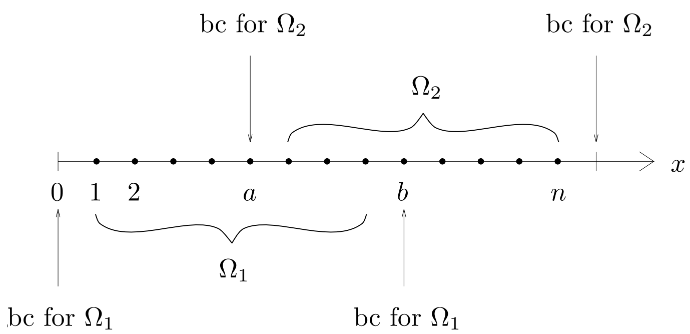

Figure 2: Decomposition in the semi-discrete case.

on the grid point $i = b$. We therefore have $a\Delta x = \alpha L$ and $b\Delta x = \beta L$. Note that the number of grid points in the overlap goes to infinity as Δx goes to zero. As in the continuous case, the solution $\boldsymbol{u}(t)$ of (2.16) can be obtained by composing the solutions $\boldsymbol{v}(t)$ on Ω_1 and $\boldsymbol{w}(t)$ on Ω_2, which satisfy the equations

$$\begin{aligned}
\frac{\partial \boldsymbol{v}}{\partial t} &= A_{(b-1)}\boldsymbol{v} + \boldsymbol{f}^{(w)}(t) \qquad t > 0 \\
\boldsymbol{v}(j,0) &= \boldsymbol{u}_0(j), \qquad 1 \le j < b,
\end{aligned} \tag{2.23}$$

where $\boldsymbol{f}^{(w)} := (\boldsymbol{f}(1,t), \ldots, \boldsymbol{f}(b-2,t), \boldsymbol{f}(b-1,t) + \frac{1}{(\Delta x)^2}\boldsymbol{w}(b-a,t))^T$ and

$$\begin{aligned} \frac{\partial \boldsymbol{w}}{\partial t} &= A_{(n-a)}\boldsymbol{w} + \boldsymbol{f}^{(v)}(t) && t > 0 \\ \boldsymbol{w}(j-a,0) &= \boldsymbol{u}_0(j) && b \le j \le n \end{aligned} \tag{2.24}$$

where $\boldsymbol{f}^{(v)} := (\boldsymbol{f}(a+1,t) + \frac{1}{(\Delta x)^2}\boldsymbol{v}(a,t), \boldsymbol{f}(a+2,t), \ldots, \boldsymbol{f}(n,t))^T$. To see why one can compose $\boldsymbol{v}$ and $\boldsymbol{w}$ to obtain $\boldsymbol{u}$, note first that $\boldsymbol{v}(t) = (\boldsymbol{u}(1,t), \ldots, \boldsymbol{u}(b-1,t))^T$ and $\boldsymbol{w}(t) = (\boldsymbol{u}(a+1,t), \ldots, \boldsymbol{u}(n,t))^T$ is a solution of (2.23) and (2.24). To show that these solutions are unique, assume there is a second pair of solutions $\tilde{\boldsymbol{v}} \neq \boldsymbol{v}$ and $\tilde{\boldsymbol{w}} \neq \boldsymbol{w}$ to reach a contradiction. The differences $\boldsymbol{\phi} := \boldsymbol{v} - \tilde{\boldsymbol{v}}$ and $\boldsymbol{\psi} := \boldsymbol{w} - \tilde{\boldsymbol{w}}$ satisfy the equations

$$\begin{aligned} \frac{\partial \boldsymbol{\phi}}{\partial t} &= A_{(b-1)}\boldsymbol{\phi} + \boldsymbol{f}^{(\psi)}(t) && t > 0 \\ \boldsymbol{\phi}(0) &= \mathbf{0} \end{aligned}$$

where $\boldsymbol{f}^{(\psi)} := (0, \ldots, 0, \frac{1}{(\Delta x)^2}\boldsymbol{\psi}(b-a,t))^T$ and

$$\begin{aligned} \frac{\partial \boldsymbol{\psi}}{\partial t} &= A_{(n-a)}\boldsymbol{\psi} + \boldsymbol{f}^{(\phi)}(t) && t > 0 \\ \boldsymbol{\psi}(0) &= \mathbf{0} \end{aligned}$$

where $\boldsymbol{f}^{(\phi)} := (\frac{1}{(\Delta x)^2}\boldsymbol{\phi}(a,t), 0, \ldots, 0)^T$. By Corollary 2.6 we have

$$||\boldsymbol{\phi}(j,\cdot)||_\infty \le \frac{j}{b}||\boldsymbol{\psi}(b-a,\cdot)||_\infty \tag{2.25}$$

and

$$||\boldsymbol{\psi}(j,\cdot)||_\infty \le \frac{n+1-a-j}{n+1-a}||\boldsymbol{\phi}(a,\cdot)||_\infty. \tag{2.26}$$

Evaluating equation (2.25) at $j = a$ gives

$$||\boldsymbol{\phi}(a,\cdot)||_\infty \le \frac{a}{b}||\boldsymbol{\psi}(b-a,\cdot)||_\infty. \tag{2.27}$$

Similarly evaluating equation (2.26) at $j = b - a$ yields

$$||\boldsymbol{\psi}(b-a,\cdot)||_\infty \le \frac{n+1-b}{n+1-a}||\boldsymbol{\phi}(a,\cdot)||_\infty. \tag{2.28}$$

Combining (2.27) and (2.28) leads to to

$$||\boldsymbol{\psi}(b-a,\cdot)||_\infty \le \frac{a(n+1-b)}{b(n+1-a)}||\boldsymbol{\psi}(b-a,\cdot)||_\infty. \tag{2.29}$$

Now since $1 \le a < b \le n$ we have

$$\begin{aligned} a < b &\iff a(n+1) < b(n+1) \\ &\iff a(n+1) - ab < b(n+1) - ba \\ &\iff \frac{a(n+1-b)}{b(n+1-a)} < 1 \end{aligned}$$

and therefore (2.29) is a contradiction. Hence $\boldsymbol{\psi} \equiv \mathbf{0}$ and by a similar argument $\boldsymbol{\phi} \equiv \mathbf{0}$. Thus the solution of (2.16) is given by $\boldsymbol{u}(j,t) \equiv \boldsymbol{v}(j,t)$ for $1 \leq j < b$ and by $\boldsymbol{u}(j,t) \equiv \boldsymbol{w}(j-a,t)$ for $b \leq j \leq n$. Note that in the overlap, we have $\boldsymbol{v}(j,t) \equiv \boldsymbol{w}(j,t)$ for $a < j < b$.

The system (2.23) and (2.24), which is coupled trough the forcing term coming from the boundary, can be solved using the Schwarz iteration

$$\begin{array}{rcll} \dfrac{\partial \boldsymbol{v}^{k+1}}{\partial t} &=& A_{(b-1)}\boldsymbol{v}^{k+1} + \boldsymbol{f}^{(w^k)} & \quad t > 0 \\ \boldsymbol{v}^{k+1}(j,0) &=& \boldsymbol{u}_0(j) & \quad 1 \leq j < b, \end{array}$$

with $\boldsymbol{f}^{(w^k)} = (\boldsymbol{f}(1,t)+\frac{1}{(\Delta x)^2}g_1(t), \boldsymbol{f}(2,t), \ldots, \boldsymbol{f}(b-2,t), \boldsymbol{f}(b-1,t)+\frac{1}{(\Delta x)^2}\boldsymbol{w}^k(b-a,t))^T$ and

$$\begin{array}{rcll} \dfrac{\partial \boldsymbol{w}^{k+1}}{\partial t} &=& A_{(n-a)}\boldsymbol{w}^{k+1} + \boldsymbol{f}^{(v^k)} & \quad t > 0 \\ \boldsymbol{w}^{k+1}(j-a,0) &=& \boldsymbol{u}_0(j) & \quad b \leq j \leq n, \end{array}$$

with $\boldsymbol{f}^{(v^k)} = (\boldsymbol{f}(a+1,t)+\frac{1}{(\Delta x)^2}\boldsymbol{v}^k(a,t), \boldsymbol{f}(a+2,t), \ldots, \boldsymbol{f}(n-1,t), \boldsymbol{f}(n,t)+\frac{1}{(\Delta x)^2}g_2(t))^T$. Let $\boldsymbol{d}^k(t) := \boldsymbol{v}^k(t) - \boldsymbol{v}(t)$ and $\boldsymbol{e}^k(t) := \boldsymbol{w}^k(t) - \boldsymbol{w}(t)$ and consider the error equations

$$\begin{array}{rcll} \dfrac{\partial \boldsymbol{d}^{k+1}}{\partial t} &=& A_{(b-1)}\boldsymbol{d}^{k+1} + \boldsymbol{f}^{(e^k)} & \quad t > 0 \\ \boldsymbol{d}^{k+1}(0) &=& \mathbf{0} & \end{array} \tag{2.30}$$

with $\boldsymbol{f}^{(e^k)} = (0, \ldots, 0, \frac{1}{(\Delta x)^2}\boldsymbol{e}^k(b-a,t))^T$ and

$$\begin{array}{rcll} \dfrac{\partial \boldsymbol{e}^{k+1}}{\partial t} &=& A_{(n-a)}\boldsymbol{e}^{k+1} + & \quad t > 0 \\ \boldsymbol{e}^{k+1}(0) &=& \mathbf{0} & \end{array} \tag{2.31}$$

with $\boldsymbol{f}^{(d^k)} = (\frac{1}{(\Delta x)^2}\boldsymbol{d}^k(a,t), 0, \ldots, 0)^T$.

Lemma 2.7 *The error $\boldsymbol{d}^k$ and $\boldsymbol{e}^k$ of the Schwarz iteration (2.30), (2.31) for the semi-discrete heat equation with two overlapping subdomains decays on the grid points a and b at a rate depending on the size of the overlap. Specifically, we have*

$$||\boldsymbol{d}^{k+2}(a,\cdot)||_\infty \leq \frac{\alpha(1-\beta)}{\beta(1-\alpha)}||\boldsymbol{d}^k(a,\cdot)||_\infty \tag{2.32}$$

$$||\boldsymbol{e}^{k+2}(b,\cdot)||_\infty \leq \frac{\alpha(1-\beta)}{\beta(1-\alpha)}||\boldsymbol{e}^k(b,\cdot)||_\infty. \tag{2.33}$$

Proof By Corollary 2.6 we have

$$||\boldsymbol{d}^{k+2}(j,\cdot)||_\infty \leq \frac{j}{b}||\boldsymbol{e}^{k+1}(b-a,\cdot)||_\infty, \ 1 \leq j < b \tag{2.34}$$

and

$$||\boldsymbol{e}^{k+1}(j,\cdot)||_\infty \leq \frac{n+1-a-j}{n+1-a}||\boldsymbol{d}^k(a,\cdot)||_\infty, \ 1 \leq j \leq b-a. \tag{2.35}$$

Evaluating (2.35) at $j = b - a$ we get

$$||\boldsymbol{e}^{k+1}(b-a,\cdot)||_\infty \leq \frac{n+1-b}{n+1-a}||\boldsymbol{d}^k(a,\cdot)||_\infty. \tag{2.36}$$

Likewise from (2.34) at $j = a$ we get

$$||\boldsymbol{d}^{k+2}(a,\cdot)||_\infty \leq \frac{a}{b}||\boldsymbol{e}^{k+1}(b-a,\cdot)||_\infty. \tag{2.37}$$

Combining (2.36) and (2.37) we obtain

$$||\boldsymbol{d}^{k+2}(a,\cdot)||_\infty \leq \frac{a(n+1-b)}{b(n+1-a)}||\boldsymbol{d}^k(a,\cdot)||_\infty.$$

Now using $a\Delta x = \alpha L$, $b\Delta x = \beta L$ and $(n+1)\Delta x = L$ we get the desired result. The second inequality (2.33) is obtained in the same way. ■

Given any vector valued function $\boldsymbol{h}(t) : \mathbb{R}^+ \longrightarrow \mathbb{R}^n$ we define

$$||\boldsymbol{h}(\cdot,\cdot)||_{\infty,\infty} := \max_{1<j<n} \sup_{t>0} |\boldsymbol{h}(j,t)|$$

Theorem 2.8 *The Schwarz iteration for the semi-discrete heat equation with two subdomains converges at a rate depending on the size of the overlap. The error on the two subdomains decays at the rate*

$$\begin{aligned}
||\boldsymbol{d}^{2k+1}(\cdot,\cdot)||_{\infty,\infty} &\leq \left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^k ||\boldsymbol{e}^0(b-a,\cdot)||_\infty \\
||\boldsymbol{e}^{2k+1}(\cdot,\cdot)||_{\infty,\infty} &\leq \left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^k ||\boldsymbol{d}^0(a,\cdot)||_\infty.
\end{aligned}$$

Proof Using Corollary 2.6 we have

$$\begin{aligned}
||\boldsymbol{d}^{2k+1}(j,\cdot)||_\infty &\leq ||\boldsymbol{e}^{2k}(b-a,\cdot)||_\infty, \ 1 \leq j < b \\
||\boldsymbol{e}^{2k+1}(j,\cdot)||_\infty &\leq ||\boldsymbol{d}^{2k}(a,\cdot)||_\infty, \ 1 \leq j < b.
\end{aligned}$$

Using Lemma 2.7 the result follows. ■

3 Arbitrary number of subdomains

3.1 Continuous Case

We generalize the two subdomain case described in section 2 to an arbitrary number of subdomains N. This leads to an algorithm which can be run in parallel. Subdomains with even indices depend only on subdomains with odd indices. Hence one can solve on all the even subdomains in parallel in one sweep, and then on all the odd ones in the next one. Boundary information is propagated in between sweeps.

Consider N subdomains Ω_i of Ω, $i = 1, \ldots, N$ where $\Omega_i = [\alpha_i L, \beta_i L] \times [0, \infty)$ and $\alpha_1 = 0$, $\beta_N = 1$ and $\alpha_{i+1} < \beta_i$ for $i = 1, \ldots, N-1$ so that all the subdomains overlap,

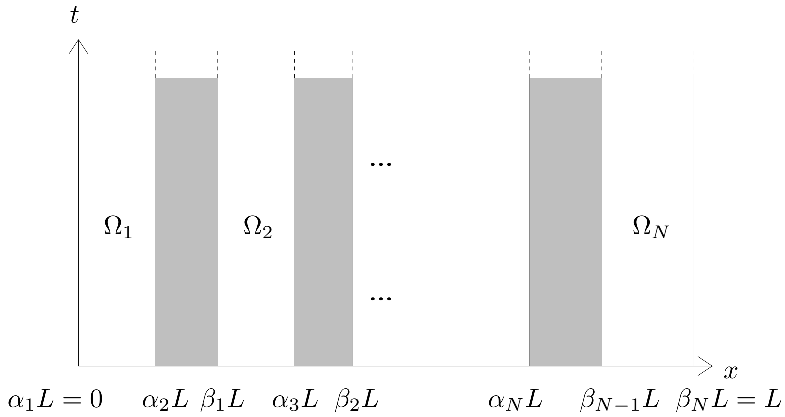

Figure 3: Decomposition into N overlapping subdomains.

as in figure 3. We assume also that $\beta_i \leq \alpha_{i+2}$ for $i = 1, \ldots, N-2$ so that domains which are not adjacent do not overlap. The solution $u(x,t)$ of (2.1) can be obtained as in the case of two subdomains by composing the solutions $v_i(x,t)$, $i = 1, \ldots, N$, which satisfy the equations

$$\begin{array}{rcll} \frac{\partial v_i}{\partial t} &=& \frac{\partial^2 v_i}{\partial x^2} + f(x,t) & \alpha_i L < x < \beta_i L,\ t > 0 \\ v_i(\alpha_i L, t) &=& v_{i-1}(\alpha_i L, t) & t > 0 \\ v_i(\beta_i L, t) &=& v_{i+1}(\beta_i L, t) & t > 0 \\ v(x,0) &=& u_0(x) & \alpha_i L < x < \beta_i L, \end{array} \tag{3.1}$$

where we introduced for convenience of notation the two functions v_0 and v_{N+1} which are constant in x and satisfy the given boundary conditions, namely $v_0(x,t) \equiv g_1(t)$ and $v_{N+1}(x,t) \equiv g_2(t)$. The system (3.1) of equations, which is coupled through the boundary, can be solved using the Schwarz iteration

$$\begin{array}{rcll} \frac{\partial v_i^{k+1}}{\partial t} &=& \frac{\partial^2 v_i^{k+1}}{\partial x^2} + f(x,t) & \alpha_i L < x < \beta_i L,\ t > 0 \\ v_i^{k+1}(\alpha_i L, t) &=& v_{i-1}^k(\alpha_i L, t) & t > 0 \\ v_i^{k+1}(\beta_i L, t) &=& v_{i+1}^k(\beta_i L, t) & t > 0 \\ v_i^{k+1}(x,0) &=& u_0(x) & \alpha_i L < x < \beta_i L, \end{array} \tag{3.2}$$

where again $v_0^k(t) \equiv g_1(t)$ and $v_{N+1}^k(t) \equiv g_2(t)$. Let $e_i^k := v_i^k(x,t) - v_i(x,t)$, $i = 1, \ldots, N$ and consider the error equations (compare figure 4)

$$\begin{array}{rcll} \frac{\partial e_i^{k+1}}{\partial t} &=& \frac{\partial^2 e_i^{k+1}}{\partial x^2} & \alpha_i L < x < \beta_i L,\ t > 0 \\ e_i^{k+1}(\alpha_i L, t) &=& e_{i-1}^k(\alpha_i L, t) & t > 0 \\ e_i^{k+1}(\beta_i L, t) &=& e_{i+1}^k(\beta_i L, t) & t > 0 \\ e_i^{k+1}(x,0) &=& 0 & \alpha_i L < x < \beta_i L, \end{array} \tag{3.3}$$

with $e_0^k(t) \equiv 0$ and $e_{N+1}^k(t) \equiv 0$.

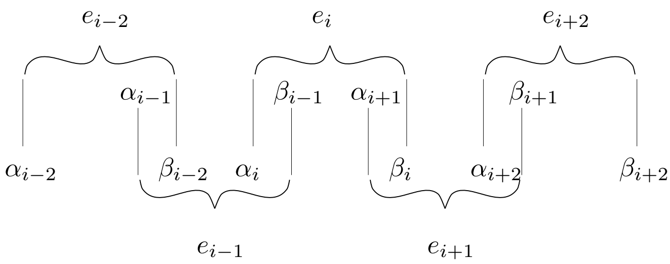

Figure 4: Overlapping subdomains and corresponding error functions e_i

For the following Lemma, we need some additional definitions to facilitate the notation. We define $\alpha_0 = \beta_0 = 0$, $\alpha_{N+1} = \beta_{N+1} = 1$ and the constant functions $e_{-1} \equiv 0$ and $e_{N+2} \equiv 0$.

Lemma 3.1 *The error e_i^{k+2} of the i-th subdomain of the Schwarz iteration (3.3) for the heat equation decays on $x = \beta_{i-1}L$ and $x = \alpha_{i+1}L$ at a rate depending on the size of the overlap. Specifically we have*

$$\begin{aligned} ||e_i^{k+2}(\beta_{i-1}L,\cdot)||_\infty \;\leq\;& r_i r_{i+1}||e_{i+2}^k(\beta_{i+1}L,\cdot)||_\infty + r_i p_{i+1}||e_i^k(\alpha_{i+1}L,\cdot)||_\infty \\ &+p_i q_{i-1}||e_i^k(\beta_{i-1}L,\cdot)||_\infty + p_i s_{i-1}||e_{i-2}^k(\alpha_{i-1}L,\cdot)||_\infty, \end{aligned} \tag{3.4}$$

for $i = 2, \ldots, N$ and

$$\begin{aligned} ||e_i^{k+2}(\alpha_{i+1}L,\cdot)||_\infty \;\leq\;& q_i r_{i+1}||e_{i+2}^k(\beta_{i+1}L,\cdot)||_\infty + q_i p_{i+1}||e_i^k(\alpha_{i+1}L,\cdot)||_\infty \\ &+s_i q_{i-1}||e_i^k(\beta_{i-1}L,\cdot)||_\infty + s_i s_{i-1}||e_{i-2}^k(\alpha_{i-1}L,\cdot)||_\infty, \end{aligned} \tag{3.5}$$

for $i = 1, \ldots, N-1$, where the ratios of the overlaps are given by

$$r_i = \frac{\beta_{i-1}-\alpha_i}{\beta_i-\alpha_i},\; p_i = \frac{\beta_i-\beta_{i-1}}{\beta_i-\alpha_i},\; q_i = \frac{\alpha_{i+1}-\alpha_i}{\beta_i-\alpha_i},\; s_i = \frac{\beta_i-\alpha_{i+1}}{\beta_i-\alpha_i}. \tag{3.6}$$

Proof By Corollary 2.2 we have

$$||e_i^{k+2}(x,\cdot)||_\infty \leq \frac{x-\alpha_i L}{(\beta_i-\alpha_i)L}||e_{i+1}^{k+1}(\beta_i L,\cdot)||_\infty + \frac{\beta_i L - x}{(\beta_i-\alpha_i)L}||e_{i-1}^{k+1}(\alpha_i L,\cdot)||_\infty. \tag{3.7}$$

Since this result holds on all the subdomains Ω_i, we can recursively apply it to the errors on the right in (3.7), namely

$$||e_{i+1}^{k+1}(\beta_i L,\cdot)||_\infty \leq \frac{\beta_i-\alpha_{i+1}}{\beta_{i+1}-\alpha_{i+1}}||e_{i+2}^k(\beta_{i+1}L,\cdot)||_\infty + \frac{\beta_{i+1}-\beta_i}{\beta_{i+1}-\alpha_{i+1}}||e_i^k(\alpha_{i+1}L,\cdot)||_\infty$$

and

$$||e_{i-1}^{k+1}(\alpha_i L,\cdot)||_\infty \leq \frac{\alpha_i-\alpha_{i-1}}{\beta_{i-1}-\alpha_{i-1}}||e_i^k(\beta_{i-1}L,\cdot)||_\infty + \frac{\beta_{i-1}-\alpha_i}{\beta_{i-1}-\alpha_{i-1}}||e_{i-2}^k(\alpha_{i-1}L,\cdot)||_\infty.$$

Substituting these equations back into the right hand side of (3.7) and evaluating (3.7) at $x = \beta_{i-1}L$ leads to inequality (3.4). Evaluation at $x = \alpha_{i+1}$ leads to inequality (3.5). ■

This result is different from the result in the two subdomain case (Lemma 2.3), because we cannot get the error directly as a function of the error at the same location two steps before. The error at a given location depends on the errors at different locations also. This leads to the two independent linear systems of inequalities,

$$\boldsymbol{\xi}^{k+2} \le D\boldsymbol{\xi}^{k} \tag{3.8}$$

and

$$\boldsymbol{\eta}^{k+2} \le E\boldsymbol{\eta}^{k}, \tag{3.9}$$

where the inequality sign here means less than or equal for each component of the vectors $\boldsymbol{\xi}^{k+2}$ and $\boldsymbol{\eta}^{k+2}$. These vectors and the matrices D and E are slightly different if the number of subdomains N is even or odd. We assume in the sequel that N is even. The case where N is odd can be treated in a similar way. For N even, $\boldsymbol{\xi}^k$ is a vector containing the time infinity norm of the point wise errors in the subdomains Ω_i with odd index according to

$$\boldsymbol{\xi}^k = \begin{pmatrix} ||e_1^k(\alpha_2 L,\cdot)||_\infty \\ ||e_3^k(\beta_2 L,\cdot)||_\infty \\ ||e_3^k(\alpha_4 L,\cdot)||_\infty \\ ||e_5^k(\beta_4 L,\cdot)||_\infty \\ ||e_5^k(\alpha_6 L,\cdot)||_\infty \\ \vdots \\ ||e_{N-3}^k(\beta_{N-4} L,\cdot)||_\infty \\ ||e_{N-3}^k(\alpha_{N-2} L,\cdot)||_\infty \\ ||e_{N-1}^k(\beta_{N-2} L,\cdot)||_\infty \\ ||e_{N-1}^k(\alpha_{N} L,\cdot)||_\infty \end{pmatrix}$$

and D is a banded $(N-1)\times(N-1)$ matrix

$$D = \left[\begin{array}{ccccccccc} q_1p_2 & q_1r_2 & & & & & & & \\ p_3s_2 & p_3q_2 & r_3p_4 & r_3r_4 & & & & & \\ s_3s_2 & s_3q_2 & q_3p_4 & q_3r_4 & & & & & \\ & & p_5s_4 & p_5q_4 & r_5p_6 & r_5r_6 & & & \\ & & s_5s_4 & s_5q_4 & q_5p_6 & q_5r_6 & & & \\ & & & \ddots & & \ddots & & & \\ & & & & \ddots & & \ddots & & \\ & & & & p_{N-3}s_{N-4} & p_{N-3}q_{N-2} & r_{N-3}p_{N-2} & r_{N-3}r_{N-2} & \\ & & & & s_{N-3}s_{N-4} & s_{N-3}q_{N-4} & q_{N-3}p_{N-2} & q_{N-3}r_{N-2} & \\ & & & & & & p_{N-1}s_{N-2} & p_{N-1}q_{N-2} & r_{N-1}p_N \\ & & & & & & s_{N-1}s_{N-2} & s_{N-1}q_{N-2} & q_{N-1}p_N \end{array}\right]. \tag{3.10}$$

Similar $\boldsymbol{\eta}^k$ is a vector containing the point wise errors of the subdomains Ω_i with even index according to

$$\boldsymbol{\eta}^k = \begin{pmatrix} ||e_2^k(\beta_1 L,\cdot)||_\infty \\ ||e_2^k(\alpha_3 L,\cdot)||_\infty \\ ||e_4^k(\beta_3 L,\cdot)||_\infty \\ ||e_4^k(\alpha_5 L,\cdot)||_\infty \\ \vdots \\ ||e_{N-2}^k(\beta_{N-3} L,\cdot)||_\infty \\ ||e_{N-2}^k(\alpha_{N-1} L,\cdot)||_\infty \\ ||e_N^k(\beta_{N-1} L,\cdot)||_\infty \end{pmatrix}$$

and E is a banded $(N-1)\times(N-1)$ matrix

$$E = \begin{bmatrix} p_2q_1 & r_2p_3 & r_2r_3 & & & & & & \\ s_2q_1 & q_2p_3 & q_2r_3 & & & & & & \\ & p_4s_3 & p_4q_3 & r_4p_5 & r_4r_5 & & & & \\ & s_4s_3 & s_4q_3 & q_4p_5 & q_4r_5 & & & & \\ & & \ddots & & \ddots & & & & \\ & & & \ddots & & \ddots & & & \\ & & & & p_{N-2}s_{N-3} & p_{N-2}q_{N-3} & r_{N-2}p_{N-1} & r_{N-2}r_{N-1} \\ & & & & s_{N-2}S_{N-3} & s_{N-2}q_{N-3} & q_{N-2}p_{N-1} & q_{N-2}r_{N-1} \\ & & & & & & p_Ns_{N-1} & p_Nq_{N-1} \end{bmatrix}. \tag{3.11}$$

Note that the infinity norm of D and E equals one. This can be seen for example for D by looking at the row sum of interior rows,

$$\begin{aligned} p_is_{i-1}+p_iq_{i-1}+r_ip_{i+1}+r_ir_{i+1} &= p_i(s_{i-1}+q_{i-1})+r_i(p_{i+1}+r_{i+1}) \\ &= p_i+r_i \\ &= 1 \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} s_is_{i-1}+s_iq_{i-1}+q_ip_{i+1}+q_ir_{i+1} &= s_i(s_{i-1}+q_{i-1})+q_i(p_{i+1}+r_{i+1}) \\ &= s_i+q_i \\ &= 1 \end{aligned} \tag{3.13}$$

whereas the boundary rows sum up to a value less than one, namely

$$\begin{aligned} q_1p_2+q_1r_2 &= q_1(p_2+r_2)=q_1<1 \\ p_{N-1}s_{N-2}+p_{N-1}q_{N-2}+r_{N-1}p_N &= p_{N-1}(s_{N-2}+q_{N-2})+r_{N-1}p_N \\ &= p_{N-1}+r_{N-1}p_N<1 \\ s_{N-1}s_{N-2}+s_{N-1}q_{N-2}+q_{N-1}p_N &= s_{N-1}(s_{N-2}+q_{N-2})+q_{N-1}p_N \\ &= s_{N-1}+q_{N-1}p_N<1. \end{aligned} \tag{3.14}$$

A similar result holds for the matrix E. Since the infinity norm of both D and E equals one, convergence is not obvious at first glance. In the special case with two subdomains treated in section 2 the matrices E and D degenerated to the scalar

q_1p_2, which is strictly less than one and convergence followed. In the case of N subdomains the information from the boundary needs to propagate inward to the interior subdomains, before the algorithm exhibits convergence. Hence we expect that the infinity norm of E and D raised to a certain power becomes strictly less than one. We introduce the following Lemmas to prove convergence in the infinity norm.

Lemma 3.2 *Let $\boldsymbol{r}(A) \in I\!R^p$ denote the vector containing the row sums of the $p \times p$ square matrix A. Then*

$$\boldsymbol{r}(A^{n+1}) = A^n \boldsymbol{r}(A).$$

Proof Let $1\!\!1 = (1, 1, \ldots, 1)^T$. Then we have $\boldsymbol{r}(A) = A1\!\!1$ and hence

$$\boldsymbol{r}(A^{n+1}) = A^{n+1}1\!\!1 = A^n A 1\!\!1 = A^n \boldsymbol{r}(A).$$

■

Lemma 3.3 *Let A be a real $p \times q$ matrix with $a_{ij} \geq 0$ and B be a real $q \times r$ matrix with $b_{ij} \geq 0$. Define $I_i(A) := \{k : a_{ik} > 0\}$ and $J_j(A) := \{k : b_{kj} > 0\}$. Then for $C := AB$ we have*

$$I_i(C) = \{k : I_i(A) \cap J_k(B) \neq \emptyset\}$$

Proof We have, since $a_{ik}, b_{kj} \geq 0$

$$\begin{aligned} c_{ij} > 0 \quad &\Longleftrightarrow \quad \sum_{k=1}^{q} a_{ik} b_{kj} > 0 \\ &\Longleftrightarrow \quad \exists k \text{ s.t. } a_{ik} > 0 \text{ and } b_{kj} > 0 \\ &\Longleftrightarrow \quad I_i(A) \cap J_j(B) \neq \emptyset. \end{aligned}$$

Hence for fixed i, $c_{ij} > 0$ if and only if $I_i(A) \cap J_j(B) \neq \emptyset$. ■

Lemma 3.4 *D^k and E^k have strictly positive entries for all integer $k \geq \frac{N-1}{2}$.*

Proof We show the proof for the matrix D, the proof for E is similar. The row index sets $I_i(D)$ are given by

$$\begin{aligned} I_i(D) &= \begin{cases} \{1, \ldots, i+2\} & i \text{ even} \\ \{1, \ldots, i+1\} & i \text{ odd} \end{cases} & 1 \leq i < 4 \\ I_i(D) &= \begin{cases} \{i-1, \ldots, i+2\} & i \text{ even} \\ \{i-2, \ldots, i+1\} & i \text{ odd} \end{cases} & 4 \leq i \leq N-3 \\ I_i(D) &= \begin{cases} \{i-1, \ldots, N-1\} & i \text{ even} \\ \{i-2, \ldots, N-1\} & i \text{ odd} \end{cases} & N-3 < i \leq N-1 \end{aligned}$$

The column index sets are given by

$$\begin{aligned} J_j(D) &= \{1, \ldots, 3\} & 1 \leq i < 3 \\ J_j(D) &= \begin{cases} \{j-1, \ldots, j+2\} & j \text{ odd} \\ \{j-2, \ldots, j+1\} & j \text{ even} \end{cases} & 3 \leq j \leq N-2 \\ J_{N-1}(D) &= \{N-2, N-1\} \end{aligned}$$

We are interested in the growth of the index sets $I_i(D^k)$ as a function of k. Once every index set contains all the numbers $1 \le j \le N-1$, the matrix D^k has strictly positive entries. We show that every multiplication with D enlarges the index sets $I_i(D^k)$ on both sides by two elements, as long as the elements 1 and $N-1$ are not yet reached. The proof is done by induction: For D^2 we have using Lemma 3.3

$$
\begin{array}{rcll}
I_i(D^2) & = & \begin{cases} \{1,\dots,i+4\} & i \text{ even} \\ \{1,\dots,i+3\} & i \text{ odd} \end{cases} & 1 \le i < 6 \\
I_i(D^2) & = & \begin{cases} \{i-3,\dots,i+4\} & i \text{ even} \\ \{i-4,\dots,i+3\} & i \text{ odd} \end{cases} & 6 \le i \le N-5 \\
I_i(D^2) & = & \begin{cases} \{i-3,\dots,N-1\} & i \text{ even} \\ \{i-4,\dots,N-1\} & i \text{ odd} \end{cases} & N-5 < i \le N-1
\end{array}
$$

Now suppose that for k we obtained the sets

$$
\begin{array}{rcll}
I_i(D^k) & = & \begin{cases} \{1,\dots,i+2k\} & i \text{ even} \\ \{1,\dots,i+2k-1\} & i \text{ odd} \end{cases} & 1 \le i < 2+2k \\
I_i(D^k) & = & \begin{cases} \{i-2k+1,\dots,i+2k\} & i \text{ even} \\ \{i-2k,\dots,i+2k-1\} & i \text{ odd} \end{cases} & 2+2k \le i \le N-2k-1 \\
I_i(D^k) & = & \begin{cases} \{i-2k+1,\dots,N-1\} & i \text{ even} \\ \{i-2k,\dots,N-1\} & i \text{ odd} \end{cases} & N-2k-1 < i \le N-1
\end{array}
$$

Then for $k+1$ we have applying Lemma 3.3 again

$$
\begin{array}{rcll}
I_i(D^{k+1}) & = & \begin{cases} \{1,\dots,i+2(k+1)\} & i \text{ even} \\ \{1,\dots,i+2(k+1)-1\} & i \text{ odd} \end{cases} & 1 \le i < 2+2(k+1) \\
I_i(D^{k+1}) & = & \begin{cases} \{i-2(k+1)-1,\dots,i+2(k+1)\} & i \text{ even} \\ \{i-2(k+1),\dots,i+2(k+1)-1\} & i \text{ odd} \end{cases} & 2+2(k+1) \le i \le N-2(k+1)-1 \\
I_i(D^{k+1}) & = & \begin{cases} \{i-2(k+1)-1,\dots,N-1\} & i \text{ even} \\ \{i-2(k+1),\dots,N-1\} & i \text{ odd} \end{cases} & N-2(k+1)-1 < i \le N-1
\end{array}
$$

Hence every row index set $I_i(D^k)$ grows on both sides by 2 when D^k is multiplied by D, as long as the boundary numbers 1 and $N-1$ are not yet reached. Now the index set $I_1(D^k) = \{1,\dots,2k\}$ has to grow most to reach the boundary number $N-1$, so we need for the number of iterations

$$k \ge \frac{N-1}{2}$$

for the matrix D^k to have strictly positive entries. ■

The infinity norm of a vector $\boldsymbol{v}$ in $\mathbb{R}^n$ and a matrix A in $\mathbb{R}^{n\times n}$ is defined by

$$||\boldsymbol{v}||_\infty := \max_{1<j<n} |\boldsymbol{v}(j)|, \quad ||A||_\infty := \max_{1<i<n} \sum_{j=1}^{n} |A_{ij}|.$$

Lemma 3.5 *For all $k > \frac{N}{2}$ there exists $\gamma = \gamma(k) < 1$ such that*

$$
\begin{aligned}
||D^k||_\infty &\le \gamma \\
||E^k||_\infty &\le \gamma
\end{aligned}
$$

Proof We prove the result for D; the proof for E is similar. We have from (3.12), (3.13) and (3.14) that

$$\boldsymbol{r}(D) = \begin{pmatrix} q_1 \\ 1 \\ \vdots \\ 1 \\ p_{N-1} + r_{N-1}p_N \\ s_{N-1} + q_{N-1}p_N \end{pmatrix}.$$

By Lemma 3.4 D^k has strictly positive entries for any $k \geq \frac{N}{2}$. Note also that $||D^k||_\infty \leq 1$ since $||D||_\infty \leq 1$. Now by Lemma 3.2 we have

$$\begin{aligned} ||D^{k+1}||_\infty &= \max_i r_i(D^{k+1}) \\ &= \max_i \sum_j D^k_{ij} r_j(D) \\ &< 1 \end{aligned}$$

since $D^k_{ij} > 0$ for all i, j, $\sum_j D^k_{ij} \leq 1$ for all i, $r_j(D) \in [0, 1]$ and $r_1(D) < 1$, $r_{N-1}(D) < 1$ and $r_N(D) < 1$. ■

Remark: It suffices for each row index set to reach one of the boundaries, either 1 or $N-1$, for the infinity norm to start decaying. Hence is is enough that there are no more index sets $I_i(D^k)$ (compare the proof of Lemma 3.4) such that $2+2k \leq i \leq N-1-2k$ so that the requirement $k \geq \frac{N-1}{2}$ can be relaxed to $k > \frac{N-3}{4}$.

We now fix some $k > \frac{N}{2}$ and set

$$\gamma := \max(||D^k||_\infty, ||E^k||_\infty) < 1. \tag{3.15}$$

Lemma 3.6 *The vectors* $\boldsymbol{\xi}$ *and* $\boldsymbol{\eta}$ *satisfy*

$$\begin{aligned} ||\boldsymbol{\xi}^{2km}||_\infty &\leq \gamma^m ||\boldsymbol{\xi}^0||_\infty & (3.16) \\ ||\boldsymbol{\eta}^{2km}||_\infty &\leq \gamma^m ||\boldsymbol{\eta}^0||_\infty. & (3.17) \end{aligned}$$

Proof We show the result for $\boldsymbol{\xi}$; the proof for $\boldsymbol{\eta}$ is similar. Recall equation (3.8), $\boldsymbol{\xi}^{k+2} \leq D\boldsymbol{\xi}^k$. We want to show

$$\boldsymbol{\xi}^{2k} \leq D^k \boldsymbol{\xi}^0. \tag{3.18}$$

To this end consider

$$\boldsymbol{z}^{k+2} = D\boldsymbol{z}^k \tag{3.19}$$

together with $\boldsymbol{z}^0 \equiv \boldsymbol{\xi}^0$. By iterating (3.19) we obtain

$$\boldsymbol{z}^{k+2} = D^k \boldsymbol{z}^0 \tag{3.20}$$

Now we take the difference of (3.19) and (3.8), namely

$$\boldsymbol{z}^{k+2} - \boldsymbol{\xi}^{k+2} \geq D(\boldsymbol{z}^k - \boldsymbol{\xi}^k). \tag{3.21}$$

We want to show that $\boldsymbol{\xi}^{2k} \leq \boldsymbol{z}^{2k}$ which implies (3.18), since then

$$\boldsymbol{\xi}^{2k} \leq \boldsymbol{z}^{2k} \leq D^k \boldsymbol{z}^0 = D^k \boldsymbol{\xi}^0.$$

Initially we have $\boldsymbol{\xi}^0 = \boldsymbol{z}^0$. To prove the result by induction assume that $\boldsymbol{\xi}^k \leq \boldsymbol{z}^k$. This implies $\boldsymbol{z}^k - \boldsymbol{\xi}^k \geq 0$ in every component. Together with (3.21) using that all the entries of D are non negative we get

$$\boldsymbol{z}^{k+2} - \boldsymbol{\xi}^{k+2} \geq D(\boldsymbol{z}^k - \boldsymbol{\xi}^k) \geq 0.$$

Hence

$$\boldsymbol{\xi}^{k+2} \leq \boldsymbol{z}^{k+2}$$

and we have proven that $\boldsymbol{\xi}^{2k} \leq \boldsymbol{z}^{2k}$. After a similar argument for $\boldsymbol{\eta}$ we arrive at $\boldsymbol{\eta}^{2k} \leq E^k \boldsymbol{\eta}^0$. Taking norms on both sides and applying Lemma 3.5 the result follows. ■

Theorem 3.7 *The Schwarz iteration for the heat equation with N subdomains converges in the infinity norm in time and space. We have*

$$\max_{1 \leq 2i \leq N} ||e_{2i}^{2km+1}(\cdot,\cdot)||_{\infty,\infty} \leq \gamma^m ||\boldsymbol{\xi}^0||_\infty \tag{3.22}$$

$$\max_{1 \leq 2i+1 \leq N} ||e_{2i+1}^{2km+1}(\cdot,\cdot)||_{\infty,\infty} \leq \gamma^m ||\boldsymbol{\eta}^0||_\infty. \tag{3.23}$$

Proof We use again the maximum principle. Since the error e_i^k is in the kernel of the heat operator, by the maximum principle e_i^k attains its maximum on the initial line or on the boundary. On the initial line e_i^k vanishes, therefore

$$\begin{aligned} \max_{1 \leq 2i \leq N} ||e_{2i}^{2km+1}(\cdot,\cdot)||_{\infty,\infty} &\leq ||\boldsymbol{\xi}^{2km}||_\infty \\ \max_{1 \leq 2i+1 \leq N} ||e_{2i+1}^{2km+1}(\cdot,\cdot)||_{\infty,\infty} &\leq ||\boldsymbol{\eta}^{2km}||_\infty. \end{aligned}$$

Using Lemma 3.6 the result follows. ■

Note that the bound for the rate of convergence in Theorem 3.7 is not explicit. This is unavoidable for the level of generality employed. But, if we assume for simplicity that the overlaps are all of the same size then we can get more explicit rates of convergence. We set $r_i = s_i = r \in (0,1)$ and $p_i = q_i = p \in (0,1)$ where $p + r = 1$. The matrix D then simplifies to

$$\tilde{D} = \begin{bmatrix} p^2 & pr & & & & & & & & & \\ pr & p^2 & pr & r^2 & & & & & & & \\ r^2 & pr & p^2 & pr & & & & & & & \\ & & pr & p^2 & pr & r^2 & & & & & \\ & & r^2 & pr & p^2 & pr & & & & & \\ & & & & \ddots & & \ddots & & & & \\ & & & & & \ddots & & \ddots & & & \\ & & & & & & pr & p^2 & pr & r^2 & \\ & & & & & & r^2 & pr & p^2 & pr & \\ & & & & & & & & pr & p^2 & pr \\ & & & & & & & & r^2 & pr & p^2 \end{bmatrix}$$

and E to

$$\tilde{E} = \begin{bmatrix} p^2 & pr & r^2 & & & & & & & & & \\ pr & p^2 & pr & & & & & & & & & \\ & pr & p^2 & pr & r^2 & & & & & & & \\ & r^2 & pr & p^2 & pr & & & & & & & \\ & & & pr & p^2 & pr & r^2 & & & & & \\ & & & & & \ddots & & \ddots & & & & \\ & & & & & & \ddots & & \ddots & & & \\ & & & & & & & pr & p^2 & pr & r^2 \\ & & & & & & & r^2 & pr & p^2 & pr \\ & & & & & & & & & pr & p^2 \end{bmatrix}.$$

In this case we can bound the spectral norm of $\tilde{D}$ and $\tilde{E}$ by an explicit expression less than one. The spectral norm of a vector $\boldsymbol{v}$ in $\mathbb{R}^n$ and a matrix A in $\mathbb{R}^{n\times n}$ is defined by

$$||\boldsymbol{v}||_2 := \sqrt{\sum_{i=1}^{n} \boldsymbol{v}(i)^2}, \quad ||A||_2 := \sup_{||\boldsymbol{v}||_2=1} ||A\boldsymbol{v}||_2.$$

Lemma 3.8 *The spectral norms of $\tilde{D}$ and $\tilde{E}$ are bounded by*

$$\begin{aligned} ||\tilde{D}||_2 &\leq 1 - 4pr\sin^2\frac{\pi}{2(N+1)} \\ ||\tilde{E}||_2 &\leq 1 - 4pr\sin^2\frac{\pi}{2(N+1)}. \end{aligned}$$

Proof We prove the bound for $\tilde{D}$. The bound for $\tilde{E}$ can be obtained similarly. We can estimate the spectral norm of $\tilde{D}$ by letting $\tilde{D} = J + r^2F$ where J is tridiagonal and F has only $O(N)$ nonzero entries and these are equal to 1. In fact $||F||_2 = 1$. Using that the eigenvalues of J are given by

$$\lambda_j(J) = p^2 + 2pr\cos\frac{\pi j}{N+1},$$

the spectral norm of $\tilde{D}$ can be estimated by

$$\begin{aligned} ||\tilde{D}||_2 &\leq ||J||_2 + r^2||F||_2 \\ &= p^2 + 2pr\cos\frac{\pi}{N+1} + r^2 \\ &= p^2 + 2pr + r^2 - 4pr\sin^2\frac{\pi}{2(N+1)} \\ &= 1 - 4pr\sin^2\frac{\pi}{2(N+1)}, \end{aligned}$$

since $p + r = 1$. ∎

Lemma 3.9 *Assume that all the N subdomains overlap at the same ratio $r \in (0, 0.5]$. Then the vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ satisfy*

$$
\begin{aligned}
||\boldsymbol{\xi}^{2k}||_2 &\leq \left(1 - 4r(1-r)\sin^2 \frac{\pi}{2(N+1)}\right)^k ||\boldsymbol{\xi}^0||_2 \\
||\boldsymbol{\eta}^{2k}||_2 &\leq \left(1 - 4r(1-r)\sin^2 \frac{\pi}{2(N+1)}\right)^k ||\boldsymbol{\eta}^0||_2.
\end{aligned}
$$

Proof The proof follows as in Lemma 3.6. ■

Note that $r = 0.5$, which minimizes the upper bound in Lemma 3.9, corresponds to the maximum possible overlap in this setting.

Theorem 3.10 *The Schwarz iteration for the heat equation with N subdomains that overlap at the same ratio $r \in (0, 0.5]$ converges in the infinity norm in time and space. Specifically we have*

$$
\max_{1 \leq 2i \leq N} ||e_{2i}^{2k}(\cdot,\cdot)||_{\infty,\infty} \leq \left(1 - 4r(1-r)\sin^2 \frac{\pi}{2(N+1)}\right)^k ||\boldsymbol{\xi}^0||_2 \quad (3.24)
$$

$$
\max_{1 \leq 2i+1 \leq N} ||e_{2i+1}^{2k}(\cdot,\cdot)||_{\infty,\infty} \leq \left(1 - 4r(1-r)\sin^2 \frac{\pi}{2(N+1)}\right)^k ||\boldsymbol{\eta}^0||_2. \quad (3.25)
$$

Proof From the proof of Theorem 3.7 we have

$$
\begin{aligned}
\max_{1 \leq 2i \leq N} ||e_{2i}^{2k+1}(\cdot,\cdot)||_{\infty,\infty} &\leq ||\boldsymbol{\xi}^{2k}||_\infty \\
\max_{1 \leq 2i+1 \leq N} ||e_{2i+1}^{2k+1}(\cdot,\cdot)||_{\infty,\infty} &\leq ||\boldsymbol{\eta}^{2k}||_\infty.
\end{aligned}
$$

Since $||\boldsymbol{x}||_\infty \leq ||\boldsymbol{x}||_2$ we get

$$
\begin{aligned}
\max_{1 \leq 2i \leq N} ||e_{2i}^{2k+1}(\cdot,\cdot)||_{\infty,\infty} &\leq ||\boldsymbol{\xi}^{2k}||_2 \\
\max_{1 \leq 2i+1 \leq N} ||e_{2i+1}^{2k+1}(\cdot,\cdot)||_{\infty,\infty} &\leq ||\boldsymbol{\eta}^{2k}||_2.
\end{aligned}
$$

Using Lemma 3.9 the result follows. ■

3.2 Semi-Discrete Case

We decompose the domain into N overlapping subdomains as given in figure 5. We assume for simplicity that $\alpha_i L$ falls on the grid points a_i and $\beta_i L$ on the grid points b_i. The solution $\boldsymbol{u}(t)$ of (2.16) can now be obtained as in the case of two subdomains by composing the solutions $\boldsymbol{v}_i(t)$, $i = 1, \ldots, N$ which satisfy

$$
\begin{aligned}
\frac{\partial \boldsymbol{v}_i}{\partial t} &= A_{(b_i - a_{i+1} - 1)} \boldsymbol{v}_i + \boldsymbol{f}^{(v_{i-1}, v_{i+1})} \\
\boldsymbol{v}_i(j - a_i, 0) &= \boldsymbol{u}_0(j) \qquad a_i < j < b_i
\end{aligned}
\quad (3.26)
$$

where $\boldsymbol{f}^{(v_{i-1},v_{i+1})} = (\boldsymbol{f}(a_i+1,t) + \frac{1}{(\Delta x)^2}\boldsymbol{v}_{i-1}(a_i - a_{i-1}, t), \boldsymbol{f}(a_i+2,t), \ldots, \boldsymbol{f}(b_i - 2, t), \boldsymbol{f}(b_i - 1, t) + \frac{1}{(\Delta x)^2}\boldsymbol{v}_{i+1}(b_i - a_{i+1}, t))^T$. These equations can be solved using the

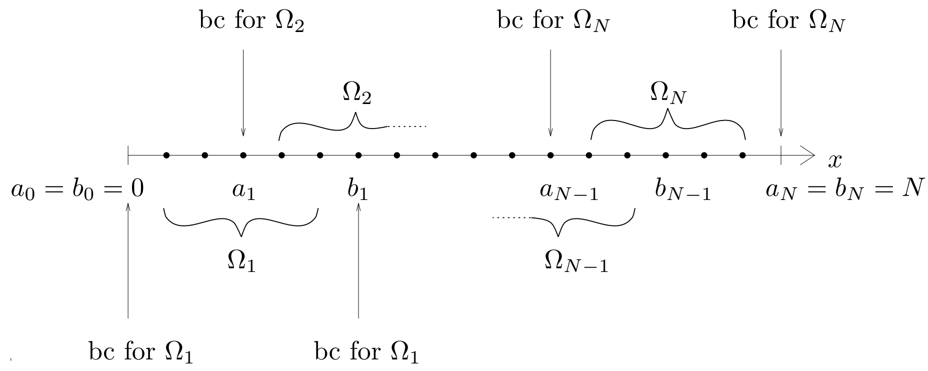

Figure 5: Decomposition into N overlapping subdomains in the semi-discrete case.

Schwarz iteration

$$\begin{aligned} \frac{\partial \boldsymbol{v}_i^{k+1}}{\partial t} &= A_{(b_i - a_{i+1} - 1)} \boldsymbol{v}_i^{k+1} + \boldsymbol{f}^{(v_{i-1}^k, v_{i+1}^k)} \\ \boldsymbol{v}_i^{k+1}(0) &= \boldsymbol{u}_i(0) \qquad\qquad a_i < j < b_i, \end{aligned} \tag{3.27}$$

where $\boldsymbol{f}^{(v_{i-1}^k, v_{i+1}^k)} = (\boldsymbol{f}(a_i+1,t) + \frac{1}{(\Delta x)^2} \boldsymbol{v}_{i-1}^k (a_i - a_{i-1}, t), \boldsymbol{f}(a_i+2,t), \ldots, \boldsymbol{f}(b_i - 2, t), \boldsymbol{f}(b_i - 1, t) + \frac{1}{(\Delta x)^2} \boldsymbol{v}_{i+1}^k (b_i - a_{i+1}, t))^T$. As in the two subdomain case, we form the difference $\boldsymbol{e}_i^k := \boldsymbol{v}_i^k - \boldsymbol{v}$ and consider the error equations

$$\begin{aligned} \frac{\partial \boldsymbol{e}_i^{k+1}}{\partial t} &= A_{(b_i - a_{i+1} - 1)} \boldsymbol{e}_i^{k+1} + \tilde{\boldsymbol{f}}^{(e_{i-1}^k, e_{i+1}^k)} \\ \boldsymbol{e}_i^{k+1}(0) &= \boldsymbol{0}. \end{aligned} \tag{3.28}$$

where $\tilde{\boldsymbol{f}}^{(e_{i-1}^k, e_{i+1}^k)} = (\boldsymbol{e}_{i-1}^k (a_i - a_{i-1}, t), 0, \ldots, 0, \boldsymbol{e}_{i+1}^k (b_i - a_{i+1}, t))^T$. For the next Lemma, we define for notational convenience $a_0 = b_0 = 0$, $a_{N+1} = b_{N+1} = n+1$ and the constant vectors $\boldsymbol{e}_{-1} = \boldsymbol{e}_{N+2} = \boldsymbol{0}$.

Lemma 3.11 *The error $\boldsymbol{e}_i^{k+2}$ of the i-th subdomain of the Schwarz iteration for the semi-discrete heat equation on the grid points $b_{i-1} - a_i$ and $a_{i+1} - a_i$ satisfies the estimates*

$$\begin{aligned} ||\boldsymbol{e}_i^{k+2}(b_{i-1} - a_i, \cdot)||_\infty \leq\ & r_i r_{i+1} ||\boldsymbol{e}_{i+2}^k (b_{i+1} - a_{i+2}, \cdot)||_\infty + r_i p_{i+1} ||\boldsymbol{e}_i^k (a_{i+1} - a_i, \cdot)||_\infty \\ & + p_i q_{i-1} ||\boldsymbol{e}_i^k (b_{i-1} - a_i, \cdot)||_\infty + p_i s_{i-1} ||\boldsymbol{e}_{i-2}^k (a_{i-1} - a_{i-2}, \cdot)||_\infty, \end{aligned} \tag{3.29}$$

for $i = 2, \ldots, N$ and

$$\begin{aligned} ||\boldsymbol{e}_i^{k+2}(a_{i+1} - a_i, \cdot)||_\infty \leq\ & q_i r_{i+1} ||\boldsymbol{e}_{i+2}^k (b_{i+1} - a_{i+2}, \cdot)||_\infty + q_i p_{i+1} ||\boldsymbol{e}_i^k (a_{i+1} - a_i, \cdot)||_\infty \\ & + s_i q_{i-1} ||\boldsymbol{e}_i^k (b_{i-1} - a_i, \cdot)||_\infty + s_i s_{i-1} ||\boldsymbol{e}_{i-2}^k (a_{i_1} - a_{i-2}, \cdot)||_\infty, \end{aligned} \tag{3.30}$$

for $i = 1, \ldots, N-1$, where the ratios of the overlaps are given as in Lemma 3.1 equation (3.6).

Proof By Corollary 2.6 we have

$$||\boldsymbol{e}_i^{k+2}(j,\cdot)||_\infty \leq \frac{j}{b_i - a_i}||\boldsymbol{e}_{i+1}^{k+1}(b_i - a_{i+1},\cdot)||_\infty + \frac{b_i - a_i - j}{b_i - a_i}||\boldsymbol{e}_{i-1}^{k+1}(a_i - a_{i-1},\cdot)||_\infty \tag{3.31}$$

Since this result holds on all the subdomains Ω_i, we can recursively apply it to the errors on the right in (3.7), namely

$$||\boldsymbol{e}_{i+1}^{k+1}(b_i - a_{i+1},\cdot)||_\infty \leq \frac{b_i - a_{i+1}}{b_{i+1} - a_{i+1}}||\boldsymbol{e}_{i+2}^{k}(b_{i+1} - a_{i+2},\cdot)||_\infty + \frac{b_{i+1} - b_i}{b_{i+1} - a_{i+1}}||\boldsymbol{e}_{i}^{k}(a_{i+1} - a_i,\cdot)||_\infty$$

and

$$||\boldsymbol{e}_{i-1}^{k+1}(a_i - a_{i-1},\cdot)||_\infty \leq \frac{a_i - a_{i-1}}{b_{i-1} - a_{i-1}}||\boldsymbol{e}_{i}^{k}(b_{i-1} - a_i,\cdot)||_\infty + \frac{b_{i-1} - a_i}{b_{i-1} - a_{i-1}}||\boldsymbol{e}_{i-2}^{k}(a_{i-1} - a_{i-2},\cdot)||_\infty.$$

Putting those back into the right hand side of (3.31) and evaluating (3.31) at $j = b_{i-1}$ and using $\Delta x a_i = \alpha_i L$ and $\Delta x b_i = \beta_i L$ for $0 \leq i \leq N+1$ leads to inequality (3.29). Evaluation at $j = a_{i+1}$ leads to inequality (3.30). ■

As in the continuous case we are lead to the two independent linear systems of inequalities

$$\boldsymbol{\xi}^{k+2} \leq D\boldsymbol{\xi}^k \tag{3.32}$$

and

$$\boldsymbol{\eta}^{k+2} \leq E\boldsymbol{\eta}^k. \tag{3.33}$$

We assume again that the number of subdomains N is even. The case where N is odd can be treated in a similar way. For N even, $\boldsymbol{\xi}^k$ is a vector containing the errors on the grid points in the subdomains Ω_i with odd index which are the boundaries of the adjacent subdomains according to

$$\boldsymbol{\xi}^k = \begin{pmatrix} ||e_1^k(a_1,\cdot)||_\infty \\ ||e_3^k(b_2 - a_3,\cdot)||_\infty \\ ||e_3^k(a_4 - a_3,\cdot)||_\infty \\ ||e_5^k(b_4 - a_5,\cdot)||_\infty \\ ||e_5^k(a_6 - a_5,\cdot)||_\infty \\ \vdots \\ ||e_{N-3}^k(b_{N-4} - a_{N-3},\cdot)||_\infty \\ ||e_{N-3}^k(a_{N-2} - a_{N-3},\cdot)||_\infty \\ ||e_{N-1}^k(b_{N-2} - a_{N-1},\cdot)||_\infty \\ ||e_{N-1}^k(a_N - a_{N-1},\cdot)||_\infty \end{pmatrix} \tag{3.34}$$

and D is the same banded $(N-1) \times (N-1)$ matrix given in (3.10). Similar $\boldsymbol{\eta}^k$ is a vector containing the errors on the grid points in the subdomains Ω_i with even index according to

$$\boldsymbol{\eta}^k = \begin{pmatrix} ||e_2^k(b_1 - a_2, \cdot)||_\infty \\ ||e_2^k(a_3 - a_2, \cdot)||_\infty \\ ||e_4^k(b_3 - a_4, \cdot)||_\infty \\ ||e_4^k(a_5 - a_4, \cdot)||_\infty \\ \vdots \\ ||e_{N-2}^k(b_{N-3} - a_{N-2}, \cdot)||_\infty \\ ||e_{N-2}^k(a_{N-1} - a_{N-2}, \cdot)||_\infty \\ ||e_N^k(b_{N-1} - a_N, \cdot)||_\infty \end{pmatrix} \tag{3.35}$$

and E is the banded $(N-1) \times (N-1)$ matrix given in 3.11. Defining γ as in (3.15) we get the Theorem

Theorem 3.12 *The Schwarz iteration for the semi-discrete heat equation with N subdomains converges in the infinity norm in time and space. We have*

$$\max_{1 \le 2i \le N} ||\boldsymbol{e}_{2i}^{2km}(\cdot,\cdot)||_{\infty,\infty} \le \gamma^m ||\boldsymbol{\xi}^0||_\infty \tag{3.36}$$

$$\max_{1 \le 2i+1 \le N} ||\boldsymbol{e}_{2i+1}^{2km}(\cdot,\cdot)||_{\infty,\infty} \le \gamma^m ||\boldsymbol{\eta}^0||_\infty, \tag{3.37}$$

where γ is defined as in (3.15).

Proof Using Corollary 2.6 we have

$$\max_{1 \le 2i \le N} ||\boldsymbol{e}_{2i}^{2km+1}(\cdot,\cdot)||_{\infty,\infty} \le ||\boldsymbol{\xi}^{2km}||_\infty$$

$$\max_{1 \le 2i+1 \le N} ||\boldsymbol{e}_{2i+1}^{2km+1}(\cdot,\cdot)||_{\infty,\infty} \le ||\boldsymbol{\eta}^{2km}||_\infty.$$

Using Lemma 3.6 the result follows. ■

Theorem 3.13 *The Schwarz iteration for the heat equation with N subdomains that overlap at the same ratio $r \in (0, 0.5]$ converges in the infinity norm in time and space. We have*

$$\max_{1 \le 2i \le N} ||\boldsymbol{e}_{2i}^{2k}(\cdot,\cdot)||_{\infty,\infty} \le \left(1 - 4r(1-r)\sin^2 \frac{\pi}{2(N+1)}\right)^k ||\boldsymbol{\xi}^0||_2 \tag{3.38}$$

$$\max_{1 \le 2i+1 \le N} ||\boldsymbol{e}_{2i+1}^{2k}(\cdot,\cdot)||_{\infty,\infty} \le \left(1 - 4r(1-r)\sin^2 \frac{\pi}{2(N+1)}\right)^k ||\boldsymbol{\eta}^0||_2. \tag{3.39}$$

Proof The proof follows as in Theorem 3.12. ■

4 The Algorithm in the Framework of Waveform Relaxation

For a linear initial value problem

$$\frac{\partial \boldsymbol{u}(t)}{\partial t} = A\boldsymbol{u}(t) + \boldsymbol{f}(t), \quad \boldsymbol{u}(0) = \boldsymbol{u}_0$$

the standard waveform relaxation algorithm is based on a splitting of the matrix A into $A = M + N$ which yields

$$\frac{\partial \boldsymbol{u}(t)}{\partial t} = M\boldsymbol{u}(t) + N\boldsymbol{u}(t) + \boldsymbol{f}(t), \quad \boldsymbol{u}(0) = \boldsymbol{u}_0.$$

This system of ODE's is solved using an iteration of the form

$$\frac{\partial \boldsymbol{v}^{k+1}}{\partial t} = M\boldsymbol{v}^{k+1} + N\boldsymbol{v}^{k} + \boldsymbol{f}, \quad \boldsymbol{v}^{k+1}(0) = \boldsymbol{u}_0. \tag{4.1}$$

where the starting function $\boldsymbol{v}^0(t)$ is chosen as a constant function $\boldsymbol{v}^0(t) = \boldsymbol{u}_0$. In the case of Block-Jacobi the matrix M is chosen to be block diagonal,

$$M = \begin{bmatrix} D_1 & & & 0 \\ & D_2 & & \\ & & \ddots & \\ 0 & & & D_N \end{bmatrix}, \tag{4.2}$$

and N contains all the remaining blocks. This allows for solving all the subsystems D_i in equation (4.1) in parallel. In the case where A equals $A_{(n)}$ from the semi-discrete heat equation (2.16), the waveform relaxation algorithm with Block-Jacobi splitting computes the same iterates as the Schwarz domain decomposition algorithm presented in subsection 3.2 but with overlap Δx (i.e. one grid point only). More precisely it computes simultaneously the two independent sequences of iterates generated by the Schwarz algorithm starting with the even or odd subdomains. To see this, consider the subsystem D_i, $i = 1, ..., N$ which we define to consist of the equations with number $a_i + 1, \ldots, b_i - 1$. The equation $v_i^k(t)$ satisfies is

$$\begin{aligned} \frac{\partial \boldsymbol{v}_i^{k+1}}{\partial t} &= D_i \boldsymbol{v}_i^{k+1} + \begin{pmatrix} \frac{1}{(\Delta x)^2}\boldsymbol{v}_{i-1}^{k}(a_i - a_{i-1}, t) \\ 0 \\ \vdots \\ 0 \\ \frac{1}{(\Delta x)^2}\boldsymbol{v}_{i+1}^{k}(b_i - a_{i+1}, t) \end{pmatrix} + \begin{pmatrix} \boldsymbol{f}(a_i+1, t) \\ \boldsymbol{f}(a_i+2, t) \\ \vdots \\ \boldsymbol{f}(b_i-2, t) \\ \boldsymbol{f}(b_i-1, t) \end{pmatrix}, \\ \boldsymbol{v}_i^{k+1}(j - a_i, 0) &= \boldsymbol{u}_0(j), \quad a_i < j < b_i, \end{aligned} \tag{4.3}$$

where we define for convenience of notation $\boldsymbol{v}_0 = \boldsymbol{v}_{N+1} = 0$ and $a_0 = b_0 = 0$,$a_{N+1} = b_{N+1} = n + 1$. Note that the matrix $D_i = A_{(b_i - a_i)}$ and hence the equation obtained through the waveform relaxation algorithm (4.3) is identical with the equation obtained from domain decomposition with overlap Δx (3.27). The only difference is the solution strategy employed. Using waveform relaxation traditionally all the subsystems are solved at each step in parallel, whereas in overlapping domain decomposition one may solve even subdomains and odd subdomains alternately. Hence the domain decomposition algorithm computes only one of the independent sequences in figure 6 - the white one if we start with odd subdomains and the grey one if we start with even ones - whereas the waveform relaxation algorithm computes both sequences simultaneously. Thus changing the solution strategy of the waveform relaxation algorithm

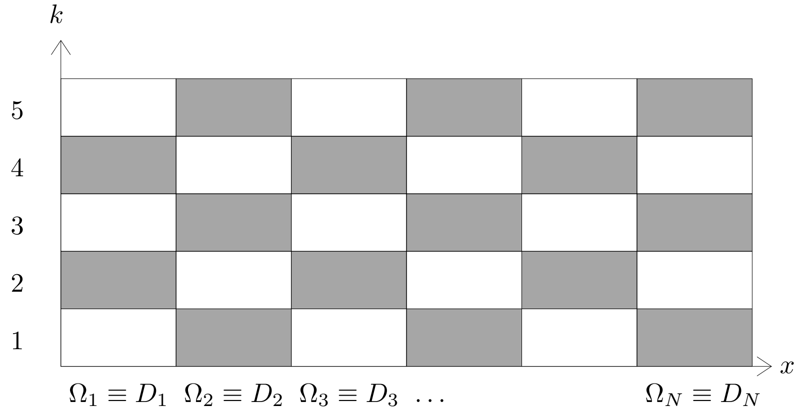

Figure 6: Relation between waveform relaxation and domain decomposition for the semi-discrete heat equation

for the semi-discrete heat equation to solve on even and odd subsystems alternately one can cut the computational cost in half.

To extend this analogy to arbitrary overlaps, the concept of *multi-splittings* is needed, which was first introduced by O'Leary and White in [19] for solving large systems of linear equations on a parallel computer. The idea was generalized to nonlinear problems by White in [23]. Jeltsch and Pohl generalized multi-splittings to linear systems of ODE's and waveform relaxation in [11]. We will need:

Definition 4.1 *Let $N \geq 0$ be a fixed integer. Let A,M_i,N_i and E_i be real $n \times n$ matrices. The set of ordered triples (M_i, N_i, E_i) for $i = 1, \dots, N$ is called a* multi-splitting *of A if*

1. *$A = M_i - N_i \quad$ for $i = 1, \dots, N$*
2. *The matrices E_l are nonnegative diagonal matrices and satisfy*

$$\sum_{i=1}^{N} E_l = I. \tag{4.4}$$

Using the waveform relaxation algorithm, we get N new approximations $\boldsymbol{v}_i^{k+1}$ at each step according to

$$\frac{\partial \boldsymbol{v}_i^{k+1}}{\partial t} = M_i \boldsymbol{v}_i^{k+1}(t) + N_i \boldsymbol{v}_i^k + \boldsymbol{f}_i, \quad \boldsymbol{v}_i^{k+1}(0) = \boldsymbol{u}_0, \tag{4.5}$$

which are combined using the matrices E_i to a new approximation $\boldsymbol{v}^{k+1}$ by

$$\boldsymbol{v}^{k+1} = \sum_{i=1}^{N} E_i \boldsymbol{v}_i^{k+1}.$$

Note that all the equations in (4.5) can be solved in parallel and in addition, components of $\boldsymbol{v}_i^{k+1}$ where E_i has a zero on the diagonal do not have to be computed at all provided they do not couple to other components of $\boldsymbol{v}_i^{k+1}$ where E_i has a non zero diagonal entry. Jeltsch and Pohl prove in [11] that the multi-splitting algorithm converges superlinearly on a finite time interval $[0, T]$ for all splittings and matrices A, and on an infinite time interval linearly if A is an M-matrix and the splitting is an M-splitting. However in the case of the semi-discrete heat equation, the rate of convergence in their analysis may depend badly on Δx since their level of generality includes the Schwarz method with one grid point overlap and spectral radius $1 - O(\Delta x^2)$ - the block Jacobi algorithm (4.3). Jeltsch and Pohl also mention that some overlap appears to be beneficial, a statement that we have substantiated and quantified.

To see this, consider the case where the E_i are chosen in such a way that the domain decomposition algorithm described in the previous sections is recovered. This can be obtained by choosing the N splittings of A according to the N subdomains of the domain decomposition and letting E_i have the value one on the diagonal in the interior of the corresponding subdomain, including the first point of the overlap and some arbitrary distribution in the overlap otherwise, satisfying (4.4). Then the intermediate solutions $\boldsymbol{v}_i^{k+1}$ computed by the multi-splitting algorithm for the heat equation are identical to the solutions computed by the domain decomposition algorithm described in the previous sections, with the only distinction that the multi-splitting algorithm computes again two independent sequences of iterates which are averaged in the overlap after each iteration according to the matrices E_i, whereas the domain decomposition algorithm computes only one of those sequences. This is because the multi-splitting algorithm solves on all the subdomains at every iteration and we have chosen to solve in the domain decomposition algorithm only on even (respectively odd) subdomains, saving half of the computation time. In the terminology of Domain Decomposition our algorithm corresponds to the multiplicative Schwarz algorithm with red black ordering whereas the multi-splitting algorithm corresponds to the additive Schwarz algorithm.

The important point here is that our algorithm converges linearly independent of the mesh size on unbounded time intervals. Hence the multi-splitting algorithm for the semi-discrete heat equation, which computes identical iterates, must converge at the same rate. Thus for certain PDE's the analysis of Jeltsch and Pohl can be refined to give Δx independent rates of convergence if sufficient overlap is used.

5 Numerical Experiments

We perform numerical experiments to measure the actual convergence rate of the algorithm. We consider the example problem

$$
\begin{array}{rcll}
\frac{\partial u}{\partial t} & = & \frac{\partial^2 u}{\partial x^2} + 5e^{-(t-2)^2-(x-\frac{1}{4})^2} & 0 < x < 1, \; 0 < t < 3 \\
u(0,t) & = & 0 & 0 < t < 3 \\
u(1,t) & = & e^{-t} & 0 < t < 3 \\
u(x,0) & = & x^2 & 0 < x < 1.
\end{array}
\tag{5.1}
$$

To solve the semi-discrete heat equation, we use the backward euler method in time. The first experiment is done splitting the domain $\Omega = [0,1] \times [0,3]$ into the two subdomains $\Omega_1 = [0,\alpha] \times [0,3]$ and $\Omega_2 = [\beta,1] \times [0,3]$ for three pairs of values $(\alpha,\beta) \in \{(0.4,0.6),(0.45,0.55),(0.48,0.52)\}$. Figure 7 shows the convergence of the algorithm on the grid point a for $\Delta x = 0.01$ and $\Delta t = 0.01$. The solid line is the predicted convergence rate according to Theorem 2.8 and the dashed line is the measured one. The measured error displayed is the difference between the numerical solution on the

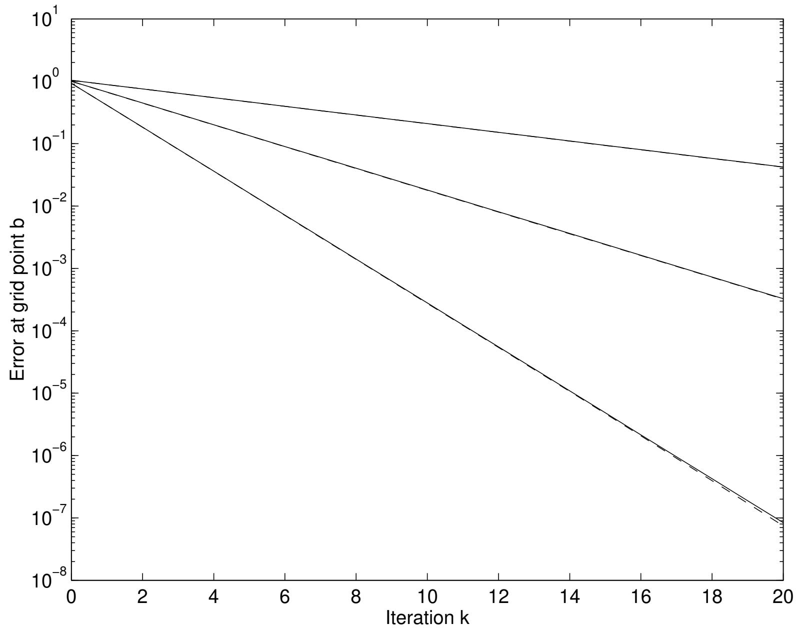

Figure 7: Theoretical and measured decay rate of the error for two subdomains and three different sizes of the overlap

whole domain and the solution obtained from the domain decomposition algorithm, and we used the error after the first iteration as the initial error for the theoretical error decay. We also checked the robustness of the method by refining the time step and obtained similar results.

According to equation (1.1) we should get superlinear convergence after enough iterations are performed. The reason why we do not see superlinear convergence in the previous experiment is that the time interval is too long. Therefore for the second experiment, we shorten the time interval to $[0,0.4]$. We use the second of the splittings of the previous experiment, namely $\alpha = 0.45$ and $\beta = 0.55$. Figure 8 shows the convergence of the algorithm, where again the solid line is the predicted convergence rate and the dashed line is the measured one. We see superlinear convergence of the algorithm.

We solve the same problem (5.1) using eight subdomains which overlap by 35%. The mesh parameter Δx is chosen to be 0.01 and time integration is performed again using Backward Euler and a time step $\Delta t = 0.01$. Figure 9 shows the decay of the infinity norm of $\boldsymbol{\xi}^k$ which is defined in (3.34). The dashed line shows the measured

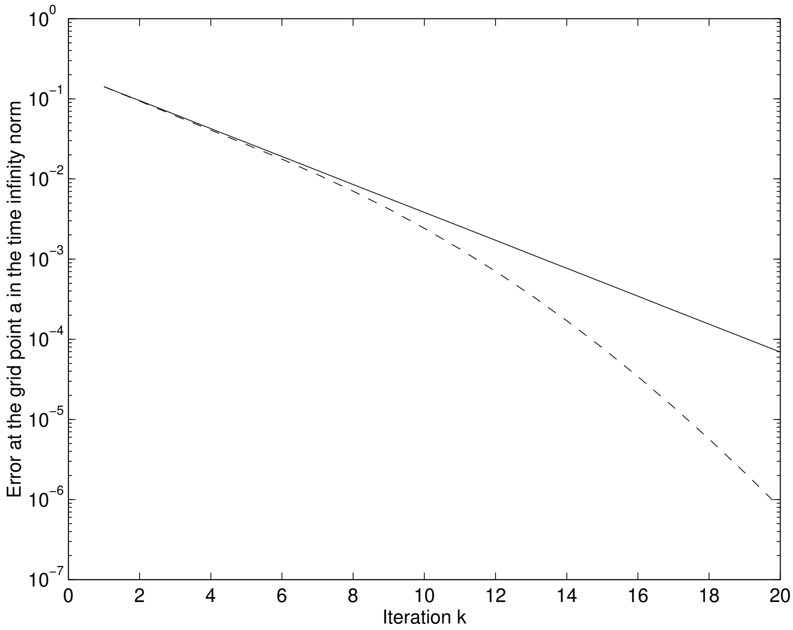

Figure 8: Superlinear decay of the error for a small time interval compared with the linear bound

decay rate and the solid line the predicted one. Note that the measured error decays at the beginning faster than the predicted one. This is due to the fact that in our analysis we assume the worst case by using the infinity norm in time. In reality the smoothing property of the heat equation flattens high peaks in the error immediately if they are surrounded by moderate values.

Hence we construct an example which corresponds to the worst case by starting the iteration with a constant error of 0.5 over the whole domain, which corresponds to setting the first iterate $\boldsymbol{v}_i^0$ equal to the solution on the whole domain plus 0.5. We solve again using eight subdomains with overlap 35%. The decay of the infinity norm of the error vector $\boldsymbol{\xi}^k$ is shown in figure 10. We see that now the measured error decays at the predicted rate from the beginning.

6 Conclusion

We have shown in this paper how to construct a waveform relaxation algorithm for the heat equation using domain decomposition. We proved convergence of the algorithm depending on the size of the overlap, in spite of the unboundedness of the differential operator. This led to a numerical method which converges independent of the mesh parameter Δx.

The one dimensional results given in this paper can be generalized to several dimensions, if a particular splitting is used. For example a rectangular domain $(x, y) \in [0, L] \times [0, W]$ in two dimensions has to be partitioned into strips $[\alpha_i L, \beta_i L] \times [0, W]$,

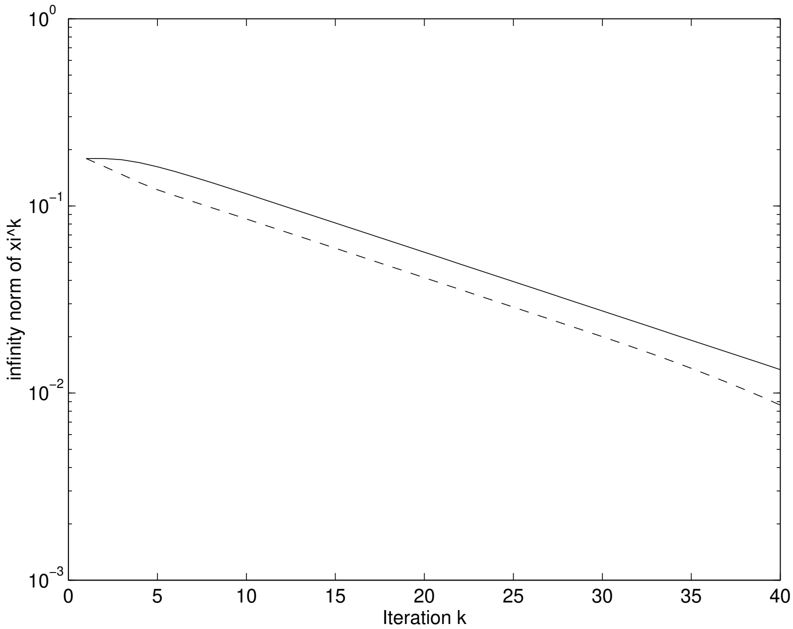

Figure 9: Theoretical and measured decay rate of the error in the case of eight subdomains

so that the subdomains overlap in one direction and extend over the whole domain in the other direction. In [6] the one and two-dimensional heat equation are studied numerically in the framework of multi-splittings; however the overlappings they use are space-discretization dependent and shrink to zero in physical space if the mesh is refined to zero. They show that an overlap by two rows of the submatrices gives a big improvement of the convergence rate in one dimension, whereas in two dimensions there is little improvement. Using the domain decomposition framework we have introduced, this observation can be explained: in one dimension an overlap by two rows corresponds to an increase of the original overlap by $O(\Delta x)$ in physical space. In two dimensions, however, the original overlap is increased only by $O(\Delta x^2)$ in the method used in [6].

Acknowledgements We thank Gene Golub for showing us how to prove Lemma 3.8 and Olavi Nevanlinna, Morten Bjørhus and Sigitas Keras for many interesting discussions.

References

[1] A. Bellen and M. Zennaro, *The Use of Runge-Kutta Formulae in Waveform Relaxation Methods*, Appl. Numer. Math, 11 (1993), pp. 95-114.

[2] P.E. Bjørstad and O.B. Widlund, *Iterative Methods for the Solution of Elliptic Problems on Regions Partitioned into Substructures*, SIAM J. Numer. Anal., 23 (1986), pp. 1093-1120.

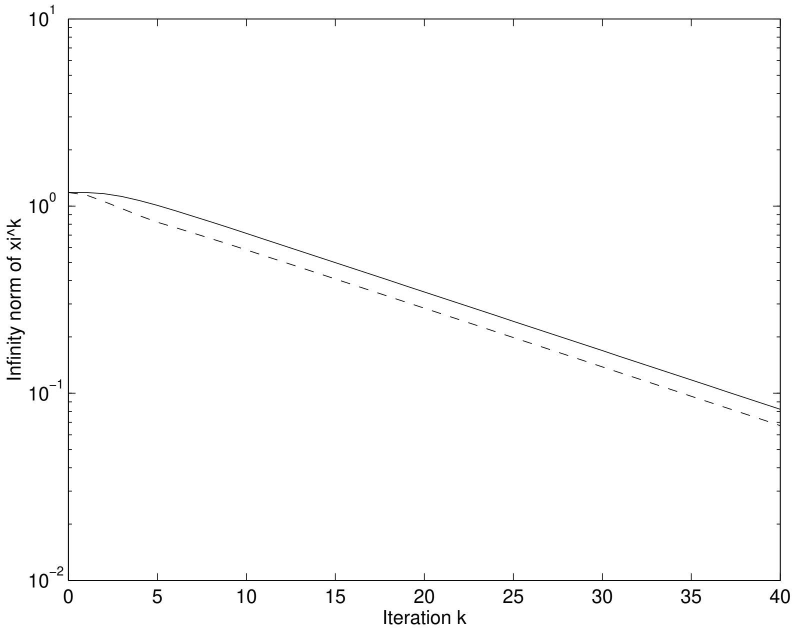

Figure 10: Theoretical and measured decay rate of the error in the case of eight subdomains if the starting value of the iteration has a constant error. This corresponds to the worst case, for which the analysis is done.

[3] M. Bjørhus, *On Domain Decomposition, Subdomain Iteration and Waveform Relaxation*, Dr. Ing. Thesis, Department of Mathematical Sciences, Norwegian Institute of Technology, University of Trondheim, Norway, 1995.

[4] M. Bjørhus, *A Note on the Convergence of Discretized Dynamic Iteration*, BIT, 35 (1995), pp. 291-296.

[5] J.H. Bramble, J.E. Pasciak and A.H. Shatz, *An Iterative Method for Elliptic Problems on Regions Partitioned into Substructures*, Math. Comp. 46 (1986), pp. 361-369.

[6] K. Burrage, Z. Jackiewicz, S.P. Norsett, R.A. Renaut, *Preconditioning waveform relaxation iterations for differential systems*, BIT, 36 no. 1 (1996), pp. 54-76.

[7] K. Burrage, *Parallel and Sequential Methods for Ordinary Differential Equations*, Oxford University Press Inc., New York, 1995.

[8] J. R. Cannon, *The One-Dimensional Heat Equation*, Encyclopedia of Mathematics and its Applications, Addison-Wesley Publishing Company, 1984.

[9] T.F. Chan and T.P. Mathew, *Domain Decomposition Algorithms*, Acta Numerica, 3 (1994), pp. 61-143.

[10] M. Dryja, *A Finite Element-Capacitance Method for Elliptic Problems on Regions Partitioned into Subregions*, Numer. Math., 44 (1984), pp. 153-168.

[11] R. Jeltsch, B. Pohl, *Waveform relaxation with overlapping splittings*, SIAM J. Sci. Comput., 16 no. 1 (1995), pp. 40-49.

[12] E. Lelarasmee, A.E. Ruehli and A.L. Sangiovanni-Vinventelli, *The Waveform Relaxation Method for Time-Domain Analysis of Large Scale Integrated Circuits*, IEEE Trans. on CAD of IC and Syst., 1 (1982), pp. 131-145.

[13] E. Lindelöf, *Sur l'application des méthodes d'approximations successives á l'étude des intégrales réeles des équations différentielles ordinaires. Journal de Mathématiques Pures et Appliqueées*, 10 (1894), pp. 117-128.

[14] P.L. Lions, *On the Schwarz Alternating Method, Proceedings of the First International Symposium on Domain Decomposition Methods for Partial Differential Equations*, Ecole Nationale des Ponts et Chaussées, Paris, France 1987.

[15] U. Miekkala, O. Nevanlinna, *Convergence of Dynamic Iteration Methods for Initial Value Problems*, SIAM J. Sci. Stat. Comput., 8 (1987), pp. 459-482.

[16] U. Miekkala, O. Nevanlinna, *Sets of Convergence and Stability Regions*, BIT, 27 (1987), pp. 554-584.

[17] O. Nevanlinna, *Remarks on Picard-Lindelöf Iterations*, Part I, BIT, 29 (1989), pp. 328-346, Part II, BIT, 29 (1989),pp. 535-562.

[18] O. Nevanlinna, *Domain Decomposition and Iterations in Parabolic Problems, forth international symposium on domain decomposition methods for partial differential equations*, Moscow, 1990.

[19] D. O'Leary, R.E. White, *Multi-Splittings of Matrices and Parallel Solution of Linear Systems*, SIAM J. Alg. Disc. Meth., 6 (1985), pp. 630-640.

[20] E. Picard, *Sur l'application des méthodes d'approximations successives á l'étude de certaines équations différentielles ordinaires*, Journal de Mathématiques Pures et Appliqueées, 9 (1893), pp. 217-271.

[21] H.A. Schwarz, *Über einige Abbildungsaufgaben*, Ges. Math. Abh., 11 (1869), pp. 65-83.

[22] R. S. Varga, *Matrix Iterative Analysis*, Englewood Cliffs, N.J., Prentice-Hall, 1962.

[23] T.E. White, *Parallel Algorithms for Nonlinear Problems*, SIAM J. Alg. Disc. Meth., 7 (1986), pp. 137-149.

[24] E. Zauderer, *Partial Differential Equations of Applied Mathematics*, Second Edition, John Wiley & Sons, New York, 1989.