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OPTIMIZED SCHWARZ METHODS FOR MODEL PROBLEMS WITH CONTINUOUSLY VARIABLE COEFFICIENTS*

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Abstract. Optimized Schwarz methods perform better than classical Schwarz methods because they use more effective transmission conditions between subdomains. These transmission conditions are determined by optimizing the convergence factor, which is obtained by Fourier analysis for simple two subdomain model problems. Such optimizations have been performed for many different types of partial differential equations, but almost exclusively based on the assumption of constant coefficients, because only then Fourier analysis can be applied. We use in this paper the technique of separation of variables to study optimized Schwarz methods for a model problem with a continuously variable reaction term, and a similar analysis could be performed as well for many other problems with variable coefficients. We obtain several new interesting results: first, we show that the technique of separation of variables can successfully decouple the spatial variables and give the convergence factor of subdomain iterations as a function of the eigenvalues of a certain Sturm–Liouville problem that contains the variable coefficient. Second, we introduce a new natural transmission condition involving second order derivatives along the interface, which turns the corresponding optimization problem into a well-studied problem, from which the optimized transmission parameters follow. Finally, we find that for variable coefficient problems, the most important information that enters into the optimized transmission conditions is described by the smallest eigenvalue of the corresponding Sturm–Liouville problem. We illustrate our results with extensive numerical experiments.

Key words. optimized Schwarz methods, optimized transmission conditions, continuously variable coefficients, domain decomposition, parallel computing

AMS subject classifications. 65N55, 65F10

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1. Introduction.

Optimized Schwarz methods (OSMs) are among the most attractive domain decomposition methods, since they greatly enhance the convergence of subdomain iterations by using optimization based transmission conditions [11]. Such optimization has been performed for many different kinds of partial differential equations; for example, see [18, 16, 21] for Helmholtz problems, [4, 42, 39, 38, 8] for Maxwell's equations, [37, 14, 2] for advection diffusion problems, [17] for wave equations, [40] for shallow water equations, and [1] for the primitive equations of the ocean. However, these optimizations are exclusively based on the assumptions of straight interfaces and constant coefficients, because all these results use Fourier analysis (or Laplace analysis for time dependent problems) to decouple the spatial/time variables of the underlying models. The use of Fourier/Laplace transforms limits the applicability of the optimized transmission conditions obtained when in the concrete application the coefficients are variable or the interfaces are curved, even though successful use has been demonstrated by a frozen coefficient approach; see, for example, [11, 18, 35]. Lions noted already in the conclusions of his seminal contribution [28, p. 217] that the convergence properties of the Schwarz methods are influenced by the variable

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coefficients globally, not just locally, which means that some global information involving the variable coefficients should be included in a local transmission strategy. More recently, Gander and Xu considered for a model problem OSMs for overlapping circular domain decompositions, where a much harder optimization problem was obtained and successfully solved to give many optimized transmission conditions [19]. Similar results were also obtained for nonoverlapping circular domain decomposition in [20], where the authors also showed that properly scaled transmission parameters from straight interface analysis [11] are also efficient for circular domain decompositions. However, these analyses are still based on Fourier transforms, and thus cannot be directly used to investigate more general interfaces. More general interfaces were studied asymptotically using spectral analysis for the nonoverlapping case [29, 30], but then the important information on the constants is lost; see, also, [41] for energy estimates.

Actually, many curved interface problems can be transformed into problems with continuously variable coefficients. For example, in log-polar coordinates

$$x = e^{\rho}\cos\theta,\ y = e^{\rho}\sin\theta,\ \rho\in\mathbb{R},\ \theta\in[0,2\pi),$$

the model problem

$$(\Delta-\eta)u = f \tag{1.1}$$

becomes a problem with continuously variable reaction term,

$$\frac{\partial^2 u}{\partial\rho^2}+\frac{\partial^2 u}{\partial\theta^2}-e^{2\rho}\eta u = e^{2\rho}f(\rho,\theta); \tag{1.2}$$

or in elliptic coordinates

$$\begin{aligned} x &= a\cosh\xi\cos\theta, \qquad 0\le\xi,\\ y &= a\sinh\xi\sin\theta, \qquad 0\le\theta<2\pi, \end{aligned}$$

the model problem (1.1) also becomes a problem with continuously variable reaction term,

$$\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\theta^2}-\frac{a^2}{2}(\cosh 2\xi-\cos 2\theta)\eta u = \frac{a^2}{2}(\cosh 2\xi-\cos 2\theta)f(\xi,\theta). \tag{1.3}$$

From these two simple examples, we see that it is of great importance to investigate OSMs for model problems with variable coefficients, which also correspond to problems on heterogeneous media. OSMs have been studied for model problems with discontinuous coefficients, again, however, exclusively based on Fourier/Laplace transforms, where the discontinuities must be aligned with the subdomain interfaces. For analysis at the continuous level, see [9, 7, 32, 33, 34, 13] for elliptic problems and [26, 3, 15] for time dependent problems. For a parabolic problem in one spatial dimension with continuously variable coefficients, the technique of separation of variables was first introduced in [27] to establish the convergence factor for an optimized Schwarz waveform relaxation method, where the corresponding optimization problem was solved numerically to get an approximation of the Robin transmission parameter. More results could so far only be obtained by analysis at the discrete or semidiscrete level; see, for example, [22, 23], where optimized Robin transmission conditions for model problems with continuous coefficients varying parallel to the interface were studied, and optimized transmission parameters were obtained that depend on the eigenvalues of certain matrices; see, also, [10].

From our analysis in section 2, one can see that our approach is also valid for the case where in the model problem (1.1) the reaction term is of the form $\eta(x,y) = \eta_1(x) + \eta_2(y)$, since then the model problem is variable separable. After separation of variables, $\eta_1(x)$ enters the reduced ordinary differential equation and $\eta_2(y)$ acts on the associated Sturm–Liouville problem that is related to the interfaces. When $\eta = \eta_1(x)$, i.e., η varies only in the x direction, the Fourier transform is still applicable and leads to a complicated ordinary differential equation to be analyzed. Such an analysis was performed as mentioned earlier for the model problem (1.2) for a circular domain decomposition in [19, 20], where it was shown that the optimized transmission parameters could be well approximated through a proper scaling by the results from the case where η is a constant and the interfaces are straight. To simplify the analysis and well explain how the information changing along the interfaces affects the performance of the OSMs, we consider in this paper the model problem (1.1) with η varying only in the y direction, i.e., $\eta = \eta(y)$. The case where $\eta(x,y)$ cannot be decoupled into a sum of two functions of just one variable will be discussed numerically as well. The model problem we thus study in detail is

$$\begin{aligned} \Delta u - \eta(y)u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.4}$$

where $\Omega = \{(x,y) | -\infty < x < +\infty, 0 < y < 1\}$ and $\eta(y) \geq 0$ is a nonnegative continuous function. We make the assumption that the domain Ω can be decomposed into the two subdomains $\Omega_1 = \{(x,y) | -\infty < x < L, 0 < y < 1\}$ and $\Omega_2 = \{(x,y) | 0 < x < +\infty, 0 < y < 1\}$, where $L \geq 0$ is the overlap. Thus $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$ and $x = L$ is an artificial interface, which we denote by Γ_1 and $x = 0$ is another artificial interface, which we denote by Γ_2. We note here that our analysis could also be adapted to the case of different boundary conditions than the homogeneous Dirichlet ones.

A parallel Schwarz algorithm for model problem (1.4) is then given by

$$\begin{aligned} \Delta u_1^n - \eta(y)u_1^n &= f \text{ in } \Omega_1, & \Delta u_2^n - \eta(y)u_2^n &= f \text{ in } \Omega_2, \\ u_1^n &= 0 \text{ on } \partial\Omega_1 \backslash \Gamma_1, & u_2^n &= 0 \text{ on } \partial\Omega_2 \backslash \Gamma_2 \end{aligned} \tag{1.5}$$

with the transmission conditions

$$\mathcal{B}_1 u_1^n = \mathcal{B}_1 u_2^{n-1}, \ (x,y) \in \Gamma_1, \qquad \mathcal{B}_2 u_2^n = \mathcal{B}_2 u_1^{n-1}, \ (x,y) \in \Gamma_2, \tag{1.6}$$

where $\mathcal{B}_i, i = 1, 2$, are transmission operators that should be determined such that the Schwarz algorithm is well defined and converges as quickly as possible.

The rest of the paper is organized as follows: in section 2, we apply the technique of separation of variables to the model problem (1.4) to obtain a Sturm–Liouville eigenvalue problem that contains the variable coefficients, and we discuss the properties of the eigenvalues for this Sturm–Liouville problem. We then show a convergence analysis for the classical Schwarz method when applied to our model problem (1.4) in section 3. Optimal transmission conditions are given in section 4, and in section 5 we determine many kinds of optimized transmission conditions that are more practical in real applications, and we show their corresponding convergence rate estimate. In section 6, we present extensive numerical examples to illustrate our theoretical results, and in the last section we draw conclusions.

2. A Sturm–Liouville eigenvalue problem. The parallel Schwarz method (1.5)–(1.6) can be analyzed using Fourier transforms when the coefficient η is a constant, and by linearity it suffices to consider only the homogeneous case, $f = 0$, which corresponds to the error equations, and to analyze convergence to the zero solution. However, in our case, η is not constant, and we thus use for the analysis the technique of separation of variables. To this end, we assume that the solution $u(x, y)$ is variable separable, i.e., $u(x, y) = \phi(x)\psi(y)$ with $\psi(0) = \psi(1) = 0$. Inserting this ansatz into the homogeneous version of (1.4) we obtain

$$\phi^{''}(x)\psi(y) + \phi(x)\psi^{''}(y) - \eta(y)\phi(x)\psi(y) = 0. \tag{2.1}$$

Dividing (2.1) by $\phi(x)\psi(y)$ we get

$$\frac{\phi^{''}(x)}{\phi(x)} = -\frac{\psi^{''}(y)}{\psi(y)} + \eta(y). \tag{2.2}$$

Since the left-hand side of (2.2) depends only on x and the right-hand side depends only on y, a constant α must exist such that

$$\frac{\phi^{''}(x)}{\phi(x)} = -\frac{\psi^{''}(y)}{\psi(y)} + \eta(y) = \alpha. \tag{2.3}$$

Hence (2.1) is equivalent to the two equations

$$\phi^{''}(x) - \alpha\phi(x) = 0 \tag{2.4}$$

and

$$-\psi^{''}(y) + \eta(y)\psi(y) = \alpha\psi(y), \quad \psi(0) = \psi(1) = 0, \tag{2.5}$$

where the constant α is known as an eigenvalue of the Sturm–Liouville eigenvalue problem (2.5). We can thus determine the eigenvalue α by (2.5) and investigate the Schwarz methods in each eigenmode separately. It is well known that the eigenvalues of problem (2.5) are real and positive, and they form an infinite sequence that can be ordered so that

$$\alpha_1 < \alpha_2 < \cdots < \alpha_k < \cdots ,$$

where α_k denotes the kth eigenvalue of (2.5) and satisfies $\alpha_k \to \infty$ as $k \to \infty$ [5]; for a historic review, see [31]. The eigenvalues α_k are all simple, with corresponding linearly independent eigenfunctions, which we denote by $\psi(y; \alpha_k)$. For each k, the eigenfunction $\psi(y; \alpha_k)$ is uniquely determined up to a multiplicative constant, which can be appropriately chosen such that

$$\int_0^1 \psi(y; \alpha_k)\psi(y; \alpha_l)dy = \delta_{kl},$$

where δ_{kl} is the Kronecker delta. The following eigenvalue estimate was proved in [25]; see also [6].

LEMMA 2.1. *Let* $\underline{\eta} := \min_{0\le y\le 1} \eta(y)$ *and* $\bar{\eta} := \max_{0\le y\le 1} \eta(y)$. *The* k*th eigenvalue* α_k *of the Sturm–Liouville problem* (2.5) *satisfies*

$$k^2\pi^2 + \underline{\eta} \le \alpha_k \le k^2\pi^2 + \bar{\eta} \text{ for } k = 1, 2, \ldots . \tag{2.6}$$

Remark 2.2. If $\eta(y)$ degenerates to a constant, we find that $\alpha_k = k^2\pi^2 + \eta$ in our domain decomposition setting, or $\alpha_k = k^2 + \eta$ for $\Omega = \mathbb{R}^2$, which reduces the corresponding analysis to a Fourier analysis [11].

3. Classical Schwarz methods. We now analyze the convergence of the parallel Schwarz method (1.5)–(1.6). To begin with, we set $\mathcal{B}_i = I$, the identity operator, and consider the so-called classical Schwarz method. Without loss of generality, we consider only the homogeneous case $f = 0$ and analyze directly the error equations. We assume that the subdomain solutions are variable separable, $u_i(x, y) = \phi_i(x)\psi(y), i = 1, 2$, with $\psi(0) = \psi(1) = 0$. Inserting this assumption into (1.5) and using the separation procedure described in section 2 we obtain

$$\frac{d^2}{dx^2}\phi_1^n(x) - \alpha\phi_1^n(x) = 0 \text{ for } x < L, \quad \frac{d^2}{dx^2}\phi_2^n(x) - \alpha\phi_2^n(x) = 0 \text{ for } x > 0, \tag{3.1}$$

where α belongs to the set

$$\mathbb{E} := \{\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots\},$$

the eigenvalues of the Sturm–Liouville problem (2.5). Denoting for each α by $\phi_i^n(x; \alpha)$, $i = 1, 2$, the solutions of (3.1) at step n, the subdomain solutions of (1.5) have the general form $u_i^n(x, y) = \sum_{\alpha\in\mathbb{E}} \phi_i^n(x; \alpha)\psi(y; \alpha)$, where $\psi(y; \alpha)$ is the eigenfunction of (2.5) associated with the eigenvalue α defined in the previous section. Hence, the corresponding transmission condition (1.6) is given by

$$\begin{aligned} \textstyle\sum_{\alpha\in\mathbb{E}} \phi_1^n(L; \alpha)\psi(y; \alpha) &= \textstyle\sum_{\alpha\in\mathbb{E}} \phi_2^{n-1}(L; \alpha)\psi(y; \alpha), \\ \textstyle\sum_{\alpha\in\mathbb{E}} \phi_2^n(0; \alpha)\psi(y; \alpha) &= \textstyle\sum_{\alpha\in\mathbb{E}} \phi_1^{n-1}(0; \alpha)\psi(y; \alpha), \end{aligned} \tag{3.2}$$

which yields, because of the orthogonality of $\psi(y; \alpha)$,

$$\phi_1^n(L; \alpha) = \phi_2^{n-1}(L; \alpha), \quad \phi_2^n(0; \alpha) = \phi_1^{n-1}(0; \alpha) \quad \text{for } \alpha \in \mathbb{E}. \tag{3.3}$$

We solve next (3.1) with the requirement that the solutions decay at infinity and arrive at

$$\phi_1^n(x; \alpha) = A^n(\alpha)e^{\sqrt{\alpha}x}, \quad \phi_2^n(x; \alpha) = B^n(\alpha)e^{-\sqrt{\alpha}x}. \tag{3.4}$$

Inserting these solutions into (3.3) and iterating between subdomains Ω_1 and Ω_2, we obtain

$$\phi_1^{2n}(x; \alpha) = \rho_{cla}^n\phi_1^0(x; \alpha), \quad \phi_2^{2n}(x; \alpha) = \rho_{cla}^n\phi_2^0(x; \alpha) \text{ for } \alpha \in \mathbb{E}, \tag{3.5}$$

where the convergence factor ρ_{cla} is given by

$$\rho_{cla} = \rho_{cla}(\alpha, L) := e^{-2\sqrt{\alpha}L}. \tag{3.6}$$

THEOREM 3.1. *The classical Schwarz method converges if and only if the overlap $L > 0$. The corresponding convergence factor $\rho_{cla}(\alpha, L)$ satisfies for $L \to 0$ the estimate*

$$\max_{\alpha\in\mathbb{E}} \rho_{cla}(\alpha, L) = 1 - 2\sqrt{\alpha_{\min}}L + O(L^2), \tag{3.7}$$

where $\alpha_{\min} = \min \mathbb{E} = \alpha_1$ is the smallest eigenvalue of the Sturm–Liouville problem (2.5).

Proof. Since $\alpha > 0$, we have $0 < \rho_{cla}(\alpha, L) < 1$ if and only if $L > 0$. In addition, the convergence factor $\rho_{cla}(\alpha, L)$ clearly attains its maximum in α at $\alpha_{\min}$. Taylor expanding $\rho_{cla}(\alpha_{\min}, L) = e^{-2\sqrt{\alpha_{\min}}L}$ in L for L small gives then the result. ∎

4. Optimal Schwarz methods. We now choose the transmission operators $\mathcal{B}_i$ as $\mathcal{B}_i = \partial_x + \mathcal{S}_i$, $i = 1, 2$, which, together with (1.5), leads to the algorithm

$$\begin{aligned}
&\Delta u_1^n - \eta(y) u_1^n = f \quad \text{in } \Omega_1,\\
&(\partial_x + \mathcal{S}_1) u_1^n(L, y) = (\partial_x + \mathcal{S}_1) u_2^{n-1}(L, y), \quad u_1^n = 0 \text{ on } \partial\Omega_1 \backslash \Gamma_1,\\
&\Delta u_2^n - \eta(y) u_2^n = f \quad \text{in } \Omega_2,\\
&(\partial_x + \mathcal{S}_2) u_2^n(0, y) = (\partial_x + \mathcal{S}_2) u_1^{n-1}(0, y), \quad u_2^n = 0 \text{ on } \partial\Omega_2 \backslash \Gamma_2,
\end{aligned} \tag{4.1}$$

where $\mathcal{S}_i, i = 1, 2$, are linear operators along the interface in the y direction which should be determined to obtain fast convergence of the Schwarz algorithm.

Using the assumption that the solutions are variable separable, $u_i(x, y) = \phi_i(x)\psi(y)$, $i = 1, 2$, with $\psi(0) = \psi(1) = 0$, we get as in section 3 after separation of variables

$$\begin{aligned}
&\frac{d^2}{dx^2}\phi_1^n(x) - \alpha\phi_1^n(x) = 0, \quad x < L,\\
&\left(\frac{d}{dx} + \sigma_1(\alpha)\right)\phi_1^n(x) = \left(\tfrac{d}{dx} + \sigma_1(\alpha)\right)\phi_2^{n-1}(x) \text{ at } x = L,
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
&\frac{d^2}{dx^2}\phi_2^n(x) - \alpha\phi_2^n(x) = 0, \quad x > 0,\\
&\left(\frac{d}{dx} + \sigma_2(\alpha)\right)\phi_2^n(x) = \left(\tfrac{d}{dx} + \sigma_2(\alpha)\right)\phi_1^{n-1}(x) \text{ at } x = 0,
\end{aligned} \tag{4.3}$$

where $\alpha \in \mathbb{E}$ and $\sigma_i(\alpha)$ are symbols of the operators $\mathcal{S}_i, i = 1, 2$, associated with the eigenfunctions $\psi(y; \alpha)$ defined for any smooth function $g(y)$ in $(0, 1)$ by

$$\int_0^1 (\mathcal{S}_i g(y))\, \psi(y; \alpha) dy = \sigma_i(\alpha) \int_0^1 g(y)\psi(y; \alpha) dy.$$

The subdomain solutions are again of the form (3.4), and using the condition on the iterates at infinity and the transmission conditions, we obtain the subdomain solutions for each $\alpha \in \mathbb{E}$,

$$\begin{aligned}
\phi_1^n(x; \alpha) &= \frac{\sigma_1(\alpha) - \sqrt{\alpha}}{\sigma_1(\alpha) + \sqrt{\alpha}} e^{\sqrt{\alpha}(x-L)} \phi_2^{n-1}(L; \alpha),\\
\phi_2^n(x; \alpha) &= \frac{\sigma_2(\alpha) + \sqrt{\alpha}}{\sigma_2(\alpha) - \sqrt{\alpha}} e^{-\sqrt{\alpha}x} \phi_1^{n-1}(0; \alpha).
\end{aligned}$$

Inserting these solutions into algorithm (4.1), we obtain by induction

$$\phi_1^{2n}(0; \alpha) = \rho_{opt}^n \phi_1^0(0; \alpha), \quad \phi_2^{2n}(L; \alpha) = \rho_{opt}^n \phi_2^0(L; \alpha),$$

where the new convergence factor ρ_{opt} is given by

$$\rho_{opt}(\alpha, L, \sigma_1, \sigma_2) := \frac{\sigma_1(\alpha) - \sqrt{\alpha}}{\sigma_1(\alpha) + \sqrt{\alpha}} \frac{\sigma_2(\alpha) + \sqrt{\alpha}}{\sigma_2(\alpha) - \sqrt{\alpha}} e^{-2\sqrt{\alpha}L}. \tag{4.4}$$

From the convergence factor ρ_{opt}, it is easy to see that if we choose $\sigma_1(\alpha) := \sqrt{\alpha}$ and $\sigma_2(\alpha) := -\sqrt{\alpha}$, then the convergence factor ρ_{opt} vanishes and the corresponding Schwarz algorithm, which is known as the optimal Schwarz method, converges in two iterations. However, the choice of $\sigma_1(\alpha) = \sqrt{\alpha}$, $\sigma_2(\alpha) = -\sqrt{\alpha}$ leads to nonlocal transmission conditions that are very expensive to use. In addition, they require as to calculate all eigenvalues of (2.5), which one would also like to avoid.

5. Optimized Schwarz methods. In this section, we would like to find local transmission conditions instead of the optimal nonlocal transmission conditions found in the previous section. To this end, we approximate the optimal symbols $\sigma_i(\alpha), i = 1, 2$, by polynomials in α of the form

$$\sigma_1^{app}(\alpha) = p_1 + q_1\alpha, \quad \sigma_2^{app}(\alpha) = -p_2 - q_2\alpha, \tag{5.1}$$

which correspond to the local transmission operators

$$\mathcal{S}_1 = p_1 - q_1\partial_{yy} + q_1\eta(y), \quad \mathcal{S}_2 = -p_2 + q_2\partial_{yy} - q_2\eta(y).$$

The convergence factor of the Schwarz algorithm (4.1) with approximate symbols (5.1) is then given by

$$\rho(\alpha, L, p_1, p_2, q_1, q_2) := \frac{\sqrt{\alpha} - p_1 - q_1\alpha}{\sqrt{\alpha} + p_1 + q_1\alpha}\frac{\sqrt{\alpha} - p_2 - q_2\alpha}{\sqrt{\alpha} + p_2 + q_2\alpha}e^{-2\sqrt{\alpha}L}. \tag{5.2}$$

THEOREM 5.1. *The optimized Schwarz method* (4.1) *with transmission conditions defined by* (5.1) *converges for all* $p_i > 0, q_i \geq 0$, $i = 1, 2$, *and convergence is faster than for the classical Schwarz method,* $|\rho| < |\rho_{cla}|$ *for all* $\alpha \in \mathbb{E}$.

Proof. This result is evident, noting that $|\rho|$ is defined by a factor that is strictly less than 1 multiplying $|\rho_{cla}|$. ∎

Remark 5.2. The computational domain under consideration is infinite in the x direction, which would not be the case in a real application. When the computational domain is also bounded in the x direction, a similar analysis could however also be performed, and for the influence of geometry on the optimized Schwarz methods, we refer the reader to [12, 43].

It is very important to choose approximate symbols of the form defined in (5.1), since they lead to a convergence factor (5.2) that is very similar to the one used in [11], which helps us to solve the hard optimization problems to determine the optimal transmission parameters. Before we discuss this in detail, we first show parameter choices based on the small eigenvalue approximation, which correspond to the low frequency approximations discussed in [11]. The difference is that we now Taylor expand the optimal symbols around the smallest eigenvalue, instead of $\sqrt{\eta}$, as was done in [11].

5.1. Small eigenvalue approximation. We clearly see that the classical Schwarz method is efficient for large eigenmodes but not for small ones. This can be improved in the optimized Schwarz method using, in the transmission condition, a Taylor expansion of the optimal symbols around the smallest eigenvalue $\alpha_{\min}$,

$$\sigma_1(\alpha) = \sqrt{\alpha_{\min}} + O(\alpha - \alpha_{\min}), \quad \sigma_2(\alpha) = -\sqrt{\alpha_{\min}} + O(\alpha - \alpha_{\min}),$$

which suggests the choice $p_1 = p_2 = \sqrt{\alpha_{\min}}$ and $q_1 = q_2 = 0$, and leads to the so-called Taylor transmission conditions of order 0 ($T0$ for short). Correspondingly, the convergence factor of the Schwarz method is given by

$$\rho_{T0}(\alpha, L) := \left(\frac{\sqrt{\alpha} - \sqrt{\alpha_{\min}}}{\sqrt{\alpha} + \sqrt{\alpha_{\min}}}\right)^2 e^{-2\sqrt{\alpha}L}. \tag{5.3}$$

The nonoverlapping Schwarz method corresponds to the case $L = 0$ in the convergence factor (5.3), and only the factor in front of the exponential term remains unchanged.

The method can however still converge, since $\rho_{T0}(\alpha,0) < 1$ for all finite α. Fortunately, the largest eigenvalue, which we denote by $\alpha_{\max}$, is finite, since in practice when a discretization resulting in N degrees of freedom along the interface is used, we will obtain N eigenvalues corresponding to the discretization of the Sturm–Liouville problem (2.5). Thus, the largest eigenvalue $\alpha_{\max}$ depends on the discretization technique used. An estimate of this largest eigenvalue $\alpha_{\max}$ can be obtained from Lemma 2.1, where we find that the value $\sqrt{\alpha_{\max}}$, which we will frequently use in the rest of the paper, is well approximated by $N\pi$ for N large or, equivalently by π/h for a uniform mesh with grid spacing h for h small. We remark here that the discretization does not necessarily have to be a uniform mesh and we need the largest eigenvalue $\alpha_{\max}$ only for the analysis of nonoverlapping Schwarz methods. Denoting the finite truncation of the eigenvalue set $\mathbb{E}$ of the Sturm–Liouville problem (2.5) by

$$\mathbb{E}_N = \{\alpha_{\min} = \alpha_1, \alpha_2, \ldots, \alpha_{\max} = \alpha_N\},$$

we can then determine the optimized transmission parameters in (5.1) for the nonoverlapping Schwarz methods by an optimization problem over $\mathbb{E}_N$; see subsection 5.2 for details. For the overlapping case, we can use infinity for $\alpha_{\max}$, since large frequencies are effectively damped by the overlap $L > 0$ appearing in the exponential in (5.3), and we do not need to consider any discretization for the analysis.

Remark 5.3. When a certain discretization is used for the Schwarz method (1.5) and (1.6), it implies a corresponding discretization of the Sturm–Liouville eigenvalue problem (2.5). One could thus replace the smallest and the largest eigenvalues $\alpha_{\min}$ and $\alpha_{\max}$ in the rest of our analysis by those from the discrete Sturm–Liouville eigenvalue problem to get accurate predictions for the optimized transmission parameters. We use here however an estimate for the largest eigenvalue $\alpha_{\max}$ based on the continuous formulation, and our results can then be used for a variety of discretizations.

THEOREM 5.4. *With the Taylor transmission conditions of order* 0*, the Schwarz method* (4.1) *converges faster than the classical Schwarz method. When the overlap* $L > 0$ *and* $\alpha_{\max} = \infty$*, the convergence factor satisfies, for* L *tending to* 0*, the asymptotic estimate*

$$\max_{\alpha\in\mathbb{E}} \rho_{T0} = 1 - 4\sqrt{2}\alpha_{\min}^{\frac{1}{4}}\sqrt{L} + O(L). \tag{5.4}$$

Without overlap, i.e., $L = 0$*, and with* $\alpha_{\max}$ *finite, the convergence factor satisfies, for* $\alpha_{\max}$ *going to infinity, the asymptotic estimate*

$$\max_{\alpha\in\mathbb{E}_N} \rho_{T0} = 1 - 4\sqrt{\alpha_{\min}}\alpha_{\max}^{-\frac{1}{2}} + O(\alpha_{\max}^{-1}). \tag{5.5}$$

Proof. We investigate first the overlapping case. Solving the derivative of ρ_{T0} wrt α equal to zero gives the only interior extremal point $\bar{\alpha} = \sqrt{\alpha_{\min}}(\sqrt{\alpha_{\min}}L + 2)/L$. Further investigation on the derivative together with the positivity of the convergence factor ρ_{T0} shows that ρ_{T0} attains its maximum at $\bar{\alpha}$. Then Taylor expanding $\rho_{T0}(\bar{\alpha}, L)$ in L for L small gives the first result (5.4).

We investigate next the nonoverlapping case $L = 0$. It is easy to verify that the derivative of ρ_{T0} wrt α is positive in $(\alpha_{\min}, \alpha_{\max})$, which means that the convergence factor ρ_{T0} is increasing monotonically in α. Together with the fact that $\rho_{T0}(\alpha_{\min}, 0) = 0$, we conclude that ρ_{T0} obtains its maximum at $\alpha_{\max}$. A series expansion of $\rho_{T0}(\alpha_{\max}, 0)$ wrt $\alpha_{\max}$ gives the second result (5.5). ∎

Similarly to the case where $\eta(y)$ is a constant, it is possible to damp the convergence factor for small eigenvalues further by involving the derivatives along the interface in the transmission conditions. To this end, we Taylor expand the optimal symbols $\sigma_i(\alpha), i = 1, 2$, at $\alpha_{\min}$ further and find

$$\sigma_1(\alpha) = \frac{\sqrt{\alpha_{\min}}}{2} + \frac{\alpha}{2\sqrt{\alpha_{\min}}} + o(\alpha - \alpha_{\min}),$$
$$\sigma_2(\alpha) = -\frac{\sqrt{\alpha_{\min}}}{2} - \frac{\alpha}{2\sqrt{\alpha_{\min}}} + o(\alpha - \alpha_{\min}),$$

which suggests the choice $p_1 = p_2 = \sqrt{\alpha_{\min}}/2$ and $q_1 = q_2 = 1/(2\sqrt{\alpha_{\min}})$ and leads to the so-called Taylor transmission conditions of order 2 ($T2$ for short). The convergence factor of the corresponding Schwarz method is then given by

$$\rho_{T2}(\alpha, L) := \left(\frac{\sqrt{\alpha} - \frac{\sqrt{\alpha_{\min}}}{2} - \frac{\alpha}{2\sqrt{\alpha_{\min}}}}{\sqrt{\alpha} + \frac{\sqrt{\alpha_{\min}}}{2} + \frac{\alpha}{2\sqrt{\alpha_{\min}}}} \right)^2 e^{-2\sqrt{\alpha}L}.$$

THEOREM 5.5. *With the Taylor transmission conditions of order* 2*, the Schwarz method* (4.1) *behaves asymptotically similarly to the Taylor transmission conditions of order* 0*. When the overlap* $L > 0$ *and* $\alpha_{\max} = \infty$*, the convergence factor satisfies, for* L *tending to* 0*, the asymptotic estimate*

$$\max_{\alpha \in \mathbb{E}} \rho_{T2} = 1 - 8\alpha_{\min}^{\frac{1}{4}} \sqrt{L} + O(L). \tag{5.6}$$

Without overlap, i.e., $L = 0$*, and with* $\alpha_{\max}$ *finite, the convergence factor satisfies, for* $\alpha_{\max}$ *going to infinity, the asymptotic estimate*

$$\max_{\alpha \in \mathbb{E}_N} \rho_{T2} = 1 - 8\sqrt{\alpha_{\min}}\alpha_{\max}^{-\frac{1}{2}} + O(\alpha_{\max}^{-1}). \tag{5.7}$$

Proof. We omit the proof since it is similar to the proof of Theorem 5.4. ∎

5.2. Optimized transmission conditions. We now impose the following constraints on the free parameters p_i, q_i involved in the approximate optimal symbols $\sigma_i^{app}(\alpha)$, $i = 1, 2$:

OO0: $p_i = p > 0, q_i = 0, i = 1, 2$;

OO2: $p_i = p > 0, q_i = q > 0, i = 1, 2$;

O2s: $p_i > 0, q_i = 0, i = 1, 2$.

To determine the best possible transmission parameters for each case above, we need to minimize the convergence factor ρ in (5.2) over all the eigenvalues contained in $\tilde{\mathbb{E}}$, where $\tilde{\mathbb{E}} = \mathbb{E}$ for the overlapping case and $\tilde{\mathbb{E}} = \mathbb{E}_N$ for the nonoverlapping case. That is to say, we need to solve the optimization problem

$$\min_{p_i, q_i \in O_c} \max_{\alpha \in \tilde{\mathbb{E}}} |\rho(\alpha, L, p_1, p_2, q_1, q_2)|, \tag{5.8}$$

where O_c is one of the constraints OO0, OO2, or O2s. The solution of the optimization problem (5.8) gives for the case OO0 the optimized transmission conditions of order 0 (also known as optimized Robin transmission conditions), for the case OO2 the optimized transmission conditions of order 2, and for the case O2s the optimized two-sided Robin transmission conditions.

We now consider solving the min-max problem (5.8), and give the various optimized transmission conditions. As mentioned earlier, this work benefits from the analysis in [11], by noting that for each case if we set in the corresponding min-max problems in [11] $k^2 = \alpha$ (correspondingly, $k_{\min}^2 = \alpha_{\min}$) and $\eta = 0$, we then arrive at the optimization problem (5.8). Noting that the analysis in [11] is valid as well for $\eta = 0$, the optimized transmission parameters follow, see the following theorems.

THEOREM 5.6 (OO0). *Assume that the constraint OO0 is imposed. When there is an overlap, $L > 0$, and $\alpha_{\max} = \infty$, the min-max problem* (5.8) *is solved by the unique root p^* of the equation*

$$\rho(\alpha_{\min}, L, p^*, p^*, 0, 0) = \rho(\bar\alpha(p^*), L, p^*, p^*, 0, 0), \quad \bar\alpha(p) = p(Lp + 2)/L. \tag{5.9}$$

In addition, for L small, the optimized Robin parameter p^* satisfies asymptotically $p^* = 2^{-\frac{1}{3}} \alpha_{\min}^{\frac{1}{3}} L^{-\frac{1}{3}}$, which leads to the asymptotic convergence factor estimate

$$\max_{\alpha \in \mathbb{E}} |\rho(\alpha, L, p^*, p^*, 0, 0)| = 1 - 2^{\frac{7}{3}} \alpha_{\min}^{\frac{1}{6}} L^{\frac{1}{3}} + O\left(L^{\frac{2}{3}}\right). \tag{5.10}$$

When there is no overlap, $L = 0$, and with $\alpha_{\max}$ finite, the optimized Robin parameter p^* is given by

$$p^* = (\alpha_{\min}\alpha_{\max})^{\frac{1}{4}}, \tag{5.11}$$

which leads for $\alpha_{\max}$ large to the convergence factor estimate

$$\max_{\alpha \in \mathbb{E}_N} |\rho(\alpha, 0, p^*, p^*, 0, 0)| = 1 - 4\alpha_{\min}^{\frac{1}{4}} \alpha_{\max}^{-\frac{1}{4}} + O\left(\alpha_{\max}^{-\frac{1}{2}}\right). \tag{5.12}$$

THEOREM 5.7 (OO2). *Assume that the constraint OO2 is imposed. For the overlap $L > 0$, and $\alpha_{\max} = \infty$, the min-max problem* (5.8) *is solved by the unique solution p^*, q^* of the equioscillation problem*

$$\rho(\alpha_{\min}, L, p^*, p^*, q^*, q^*) = \rho(\bar\alpha_1, L, p^*, p^*, q^*, q^*) = \rho(\bar\alpha_2, L, p^*, p^*, q^*, q^*), \tag{5.13}$$

where the locations of the maxima $\bar\alpha_1$ and $\bar\alpha_2$ are given by

$$\bar\alpha_{1,2}(L, p, q) = \frac{1}{q}\sqrt{\frac{L + 2q - 2Lpq \mp \sqrt{L^2 + 4Lq - 4L^2pq + 4q^2 - 16Lpq^2}}{2L}}.$$

In addition, the optimized parameters have, as $L \to 0$, the asymptotic expressions

$$p^* = 2^{-\frac{3}{5}} \alpha_{\min}^{\frac{2}{5}} L^{-\frac{1}{5}}, \quad q^* = 2^{-\frac{1}{5}} \alpha_{\min}^{-\frac{1}{5}} L^{\frac{3}{5}}, \tag{5.14}$$

which leads to the convergence factor estimate

$$\max_{\alpha \in \mathbb{E}} |\rho(\alpha, L, p^*, p^*, q^*, q^*)| = 1 - 2^{\frac{13}{5}} \alpha_{\min}^{\frac{1}{10}} L^{\frac{1}{5}} + O\left(L^{\frac{2}{5}}\right). \tag{5.15}$$

For the nonoverlapping case, $L = 0$, and with $\alpha_{\max}$ finite, the solution p^*, q^* of the min-max problem (5.8) *is for $\alpha_{\max}$ large given by*

$$\begin{aligned} p^* &= \frac{\sqrt{2}}{2} \frac{(\alpha_{\min}\alpha_{\max})^{\frac{3}{8}}}{(\sqrt{\alpha_{\min}} + \sqrt{\alpha_{\max}})^{\frac{1}{2}}} = 2^{-\frac{1}{2}} \alpha_{\min}^{\frac{3}{8}} \alpha_{\max}^{\frac{1}{8}} + O\left(\alpha_{\max}^{-\frac{3}{8}}\right), \\ q^* &= \frac{\sqrt{2}}{2} \frac{1}{(\alpha_{\min}\alpha_{\max})^{\frac{1}{8}} (\sqrt{\alpha_{\min}} + \sqrt{\alpha_{\max}})^{\frac{1}{2}}} = 2^{-\frac{1}{2}} \alpha_{\min}^{-\frac{1}{8}} \alpha_{\max}^{-\frac{3}{8}} + O\left(\alpha_{\max}^{-\frac{7}{8}}\right), \end{aligned} \tag{5.16}$$

which leads to the convergence factor estimate

$$\max_{\alpha\in\mathbb{E}_N} |\rho(\alpha,0,p^*,p^*,q^*,q^*)| = 1-2^{\frac{5}{2}}\alpha_{\min}^{\frac{1}{8}}\alpha_{\max}^{-\frac{1}{8}} + O\left(\alpha_{\max}^{-\frac{1}{4}}\right). \tag{5.17}$$

THEOREM 5.8 (O2s). *Assume that the constraint O2s is imposed. The optimization problem* (5.8) *is then solved by the parameters*

$$p_1^* = \frac{1-\sqrt{1-4p^*q^*}}{2q^*}, \quad p_2^* = \frac{1+\sqrt{1-4p^*q^*}}{2q^*}. \tag{5.18}$$

When there is overlap, $L>0$, and $\alpha_{\max}=\infty$, p^* and q^* are solutions of (5.13) *with L replaced by $2L$. The optimized parameters p_1^* and p_2^* satisfy, for L small,*

$$p_1^* = 2^{-\frac{4}{5}}\alpha_{\min}^{\frac{2}{5}}L^{-\frac{1}{5}} + O\left(L^{\frac{1}{5}}\right), \quad p_2^* = 2^{-\frac{2}{5}}\alpha_{\min}^{\frac{1}{5}}L^{-\frac{3}{5}} + O\left(L^{-\frac{1}{5}}\right),$$

which leads to the asymptotic convergence factor estimate

$$\max_{\alpha\in\mathbb{E}} |\rho(\alpha,L,p_1^*,p_2^*,0,0)| = 1-2^{-\frac{9}{5}}\alpha_{\min}^{\frac{1}{10}}L^{\frac{1}{5}} + O\left(L^{\frac{2}{5}}\right). \tag{5.19}$$

When there is no overlap, $L=0$, and with $\alpha_{\max}$ finite, p^* and q^* are given by (5.16)*, and asymptotically we have, for $\alpha_{\max}$ large,*

$$p_1^* = 2^{-\frac{1}{2}}\alpha_{\min}^{\frac{3}{8}}\alpha_{\max}^{\frac{1}{8}} + O\left(\alpha_{\max}^{-\frac{1}{8}}\right), \quad p_2^* = 2^{\frac{1}{2}}\alpha_{\min}^{\frac{1}{8}}\alpha_{\max}^{\frac{3}{8}} + O\left(\alpha_{\max}^{\frac{1}{8}}\right),$$

which leads to the convergence factor estimate

$$\max_{\alpha\in\mathbb{E}_N} |\rho(\alpha,L,p_1^*,p_2^*,0,0)| = 1-2^{\frac{3}{2}}\alpha_{\min}^{\frac{1}{8}}\alpha_{\max}^{-\frac{1}{8}} + O\left(\alpha_{\max}^{-\frac{1}{4}}\right). \tag{5.20}$$

6. Numerical experiments.

We perform numerical experiments for our model problem (1.4) on the rectangular domain $\Omega=(-1,1)\times(0,1)$. We decompose the domain Ω into two subdomains $\Omega_1=(-1,L)\times(0,1)$ and $\Omega_2=(0,1)\times(0,1)$, with overlap $L=h$ for the overlapping case and $L=0$ for the nonoverlapping case. We discretize the Laplacian by the classical five-point difference scheme using a uniform mesh with mesh parameter h. Following Lemma 2.1, we estimate $\sqrt{\alpha_{\max}}$ by π/h. The value $\sqrt{\alpha_{\min}}$ however is not necessarily well approximated by π, since it depends on properties of the function $\eta(y)$. We thus need to estimate the value $\sqrt{\alpha_{\min}}$ for good performance of our optimized Schwarz method. There are many ways to do this numerically; see, for example, [36] for a finite difference approach.

We however estimate the smallest eigenvalue $\alpha_{\min}$ of the problem (2.5) using a Fourier spectral approximation as follows: we make the ansatz $\psi(y)=\sum_{j=1}^N c_j\sin j\pi y$, which certainly satisfies $\psi(0)=\psi(1)=0$. Inserting this ansatz into (2.5) and testing by $\sin k\pi y$ gives

$$k^2\pi^2c_k + 2\sum_{j=1}^N\int_0^1\eta(y)c_j\sin j\pi y\sin k\pi y dy = \alpha c_k,\ k=1,2,\dots,N, \tag{6.1}$$

which shows that the smallest eigenvalue of the matrix $M+\pi^2\mathrm{diag}(1^2,2^2,\dots,N^2)$ is a good approximation to $\alpha_{\min}$, where M is a symmetric N by N matrix with entries $M_{jk} = 2\int_0^1\eta(y)\sin j\pi y\sin k\pi y dy$, and N does not necessarily need to be

Table 1
Number of iterations required by the various Schwarz algorithms with overlap $L = h$ for $\eta(y) = 1 + \sin(2\pi\omega y)$.

h	1/32			1/64			1/128			1/256			1/512		
ω	1	5	10	1	5	10	1	5	10	1	5	10	1	5	10
Classical	44	45	44	87	88	87	174	174	173	348	345	346	692	694	693
T0	8	8	9	12	12	11	16	16	17	23	22	23	32	32	32
T0L	8	8	9	12	12	12	16	17	16	23	24	24	33	32	32
T0U	9	8	8	11	12	12	16	16	16	23	23	22	32	31	31
T0A	9	8	8	11	12	12	16	16	16	23	23	23	33	33	32
T2	6	6	6	8	8	8	11	11	11	15	15	15	21	20	21
T2L	6	6	6	8	8	8	11	11	11	15	16	15	21	21	21
T2U	6	6	6	8	8	8	11	10	11	15	15	15	20	20	20
T2A	6	6	6	8	8	8	11	11	11	15	15	15	21	21	20
OO0	7	7	7	8	8	8	11	11	10	13	13	13	16	17	17
OO0L	7	7	7	8	8	8	11	10	11	13	13	13	17	17	16
OO0U	7	7	7	8	8	8	10	10	11	13	13	13	16	16	16
OO0A	7	7	7	8	9	8	10	10	11	13	13	13	16	16	16
OO2	4	5	4	5	5	5	6	6	6	6	7	6	8	7	7
OO2L	5	4	5	5	5	5	6	6	6	6	6	6	7	7	7
OO2U	5	5	4	5	5	5	6	5	6	6	6	7	8	8	7
OO2A	4	5	4	5	5	5	5	6	6	6	6	6	7	7	8
O2s	7	7	7	7	8	8	9	9	9	10	11	10	12	12	12
O2sL	6	7	7	8	8	8	9	9	9	10	10	10	12	12	12
O2sU	7	7	7	8	7	8	9	9	9	10	10	10	12	12	12
O2sA	6	6	6	8	7	8	9	9	9	10	10	10	12	12	12

large because of the fast convergence of the Fourier spectral method [24]. In our application, we use $N = 10$ to get an estimate for $\alpha_{\min}$. In view of inequality (2.6), the smallest eigenvalue $\alpha_{\min}$ can also be approximated by its lower bound $\alpha_{\min}^L = \pi^2 + \underline{\eta}$, by its upper bound $\alpha_{\min}^U = \pi^2 + \bar{\eta}$, as well as the arithmetic mean of both, $\alpha_{\min}^A = \pi^2 + (\underline{\eta} + \bar{\eta})/2$. We indicate the corresponding transmission conditions by ending with capital letters "L" for lower bound approximation, "U" for upper bound approximation, and "A" for arithmetic mean approximation. For example, "OO0L" means the optimized transmission condition of order zero with $\alpha_{\min}$ approximated by its lower bound $\alpha_{\min}^L$. We simulate directly the error equation, $f = 0$, and use a random initial guess on the interface; we refer the readers to [11] for the importance of this. The iteration terminates when the error reduction reaches a tolerance of $1e-6$ and the corresponding number of iterations are reported.

6.1. Coefficients with small amplitude. We consider first the case when the coefficient function $\eta(y)$ varies only with small amplitude. To this end, we choose $\eta(y) = 1 + \sin(2\pi\omega y)$ and investigate how the frequency of oscillation influences the Schwarz methods with various optimized transmission conditions, as well as those obtained by approximations. In this case the smallest eigenvalue $\alpha_{\min}$ is 10.8538 for $\omega = 1$, 10.8691 for $\omega = 5$, and 10.8696 for $\omega = 10$. Note here $\pi^2 \approx 9.8696$, $\underline{\eta} = 0$, and $\bar{\eta} = 2$. Hence both $\pi^2 + \underline{\eta} \approx 9.8696$ and $\pi^2 + \bar{\eta} \approx 11.8696$ approximate well the smallest eigenvalue $\alpha_{\min}$ for all values of $\omega = 1, 5, 10$, especially $\pi^2 + (\underline{\eta} + \bar{\eta})/2 \approx 10.8696$. In Table 1, we show the number of iterations required by the Schwarz methods with various optimized transmission conditions compared to those with approximate $\alpha_{\min}$, with the oscillating frequency ω varying from 1, 5 to 10. Similar results for the non-overlapping case are shown in Table 2. We also show the above results for $\omega = 5$ in Figure 1. We observe in both the overlapping and non-overlapping cases that all the

TABLE 2
Number of iterations required by the various nonoverlapping Schwarz algorithms for $\eta(y) = 1 + \sin(2\pi\omega y)$.

h	1/32			1/64			1/128			1/256			1/512		
ω	1	5	10	1	5	10	1	5	10	1	5	10	1	5	10
T0	82	88	83	172	175	167	342	346	347	701	687	700	1428	1384	1390
T0L	90	88	86	177	177	180	365	362	363	720	718	745	1474	1445	1475
T0U	82	81	77	159	161	169	328	332	324	666	659	666	1330	1330	1346
T0A	83	84	85	167	171	169	347	351	349	700	694	691	1390	1397	1384
T2	22	23	22	44	44	44	88	89	87	178	177	176	350	352	353
T2L	24	23	23	46	46	46	92	92	92	186	184	186	368	372	373
T2U	21	22	21	42	42	42	85	85	83	169	168	166	338	339	336
T2A	22	22	22	44	44	43	89	88	88	177	175	176	350	351	358
OO0	20	20	18	26	27	27	39	37	38	53	53	53	77	77	73
OO0L	19	19	19	27	27	27	36	37	36	51	52	53	73	73	75
OO0U	20	20	20	28	27	28	36	39	37	54	54	54	78	77	76
OO0A	19	19	20	28	27	26	38	38	38	55	54	53	76	77	75
OO2	6	6	6	8	7	7	9	9	9	11	11	11	13	13	13
OO2L	6	6	6	8	8	7	9	9	9	11	11	11	13	13	12
OO2U	7	6	6	8	8	8	9	9	9	11	11	11	13	14	14
OO2A	6	7	7	7	8	7	9	9	9	11	11	10	13	12	13
O2s	12	11	12	14	14	13	17	17	16	20	20	19	25	25	25
O2sL	12	12	11	14	14	14	16	17	17	20	20	21	25	24	25
O2sU	11	12	12	14	14	13	18	17	17	20	20	20	24	24	25
O2sA	11	12	11	14	14	14	17	17	17	20	21	20	26	24	24

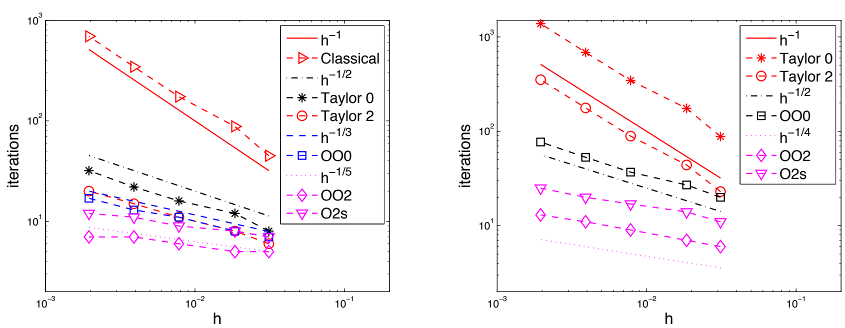

FIG. 1. *Number of iterations required by the various Schwarz methods for model problem* (1.4) *with $\eta(y) = 1 + \sin(10\pi y)$, on the left for the overlapping case and on the right the nonoverlapping case.*

Schwarz methods with the optimized transmission conditions perform as predicted in the asymptotic convergence rates. To see this, one only needs to notice that, for example, for the optimized Robin transmission conditions in the overlapping case, where $\max_{\alpha\in[\alpha_{\min},\alpha_{\max}]} |\rho| = 1 - 2^{\frac{7}{3}}\alpha_{\min}^{\frac{1}{6}}L^{\frac{1}{3}} + O(L^{\frac{2}{3}})$, the number of iterations for reducing the errors to a given tolerance ε behaves like $\frac{\ln\frac{1}{\varepsilon}}{2^{\frac{7}{3}}\alpha_{\min}^{\frac{1}{6}}L^{\frac{1}{3}}} = O(h^{-\frac{1}{3}})$ for $L = h$, and it is similar for the nonoverlapping case by noting that $\alpha_{\max} \propto 1/h^2$. We observe as well that the transmission conditions using the approximate $\alpha_{\min}$ also perform very well, and are comparable to those using the well estimated $\alpha_{\min}$. This confirms that $\alpha_{\min}$ can be well approximated by the lower bound, the upper bound, and the arithmetic mean of the coefficient function $\eta(y)$. In addition, we find that the oscillating frequency ω does not affect the performance of the optimized Schwarz

Table 3
Number of iterations required by the various Schwarz algorithms with overlap $L = h$ for $\eta(y) = 1000 + 1000\sin(2\pi\omega y)$.

h	1/32			1/64			1/128			1/256			1/512		
ω	1	5	10	1	5	10	1	5	10	1	5	10	1	5	10
Classical	13	7	6	26	13	11	50	25	20	99	47	40	196	94	78
T0	5	4	4	7	5	5	9	7	6	13	9	8	17	12	11
T0L	8	7	6	12	11	10	15	16	15	22	22	22	31	32	32
T0U	9	5	5	11	6	5	12	6	6	14	7	7	13	9	10
T0A	8	4	4	8	5	5	9	6	6	9	8	8	11	11	11
T2	5	4	4	6	4	4	7	5	4	9	6	6	11	9	8
T2L	7	7	7	9	9	9	12	12	12	16	16	16	21	22	22
T2U	6	4	4	7	4	4	7	4	4	7	5	5	8	7	7
T2A	5	4	4	5	4	4	6	4	4	6	6	6	8	8	8
OO0	5	4	4	6	5	5	7	6	6	9	7	7	11	9	8
OO0L	6	6	6	8	8	8	10	10	10	13	13	12	16	16	16
OO0U	7	4	4	10	5	5	13	7	6	17	9	8	23	11	9
OO0A	7	4	4	9	5	5	11	6	6	14	7	7	19	9	8
OO2	5	5	5	5	5	5	5	5	5	5	5	5	6	5	5
OO2L	5	5	5	6	6	6	6	6	6	6	6	6	7	7	7
OO2U	7	5	5	9	6	5	11	6	6	11	7	6	15	7	6
OO2A	6	5	5	8	5	5	8	5	5	10	6	5	11	6	5
O2s	5	4	4	6	5	5	7	6	6	8	7	7	9	8	7
O2sL	6	6	6	8	7	7	9	8	8	10	10	10	12	12	12
O2sU	6	4	4	9	5	5	12	6	5	14	7	6	19	9	7
O2sA	6	4	4	7	5	5	9	6	6	12	6	6	13	7	8

method. We note here for the overlapping case that the classical Schwarz method, though converging at the predicted asymptotic rate, requires many more iterations than the optimized variants. Similar observations hold also for the Taylor transmission conditions in the nonoverlapping case.

6.2. Coefficients with large amplitude. We next investigate how the amplitude of the coefficient function $\eta(y)$ influences the performance of the optimized Schwarz methods. To this end, we choose the coefficient function as $\eta(y) = 1000 + 1000\sin(2\pi\omega y)$ and consider simultaneously the influence of the oscillating frequency ω. In this case the smallest eigenvalue $\alpha_{\min}$ is 138.2050 for $\omega = 1$, 647.3858 for $\omega = 5$, and 1009.7520 for $\omega = 10$. Noting that we have in this case $\underline{\eta} = 0$, $\bar{\eta} = 2000$, the smallest eigenvalue $\alpha_{\min}$ for each ω cannot be well approximated by the lower bound approximation $\pi^2 + \underline{\eta} \approx 9.8696$, or the upper bound approximation $\pi^2 + \bar{\eta} \approx 2009.8696$, but the arithmetic mean approximation $\pi^2 + (\underline{\eta} + \bar{\eta})/2 \approx 1009.8696$ is a fairly good approximation, especially for $\omega = 10$. For the overlapping domain decomposition, we show in Table 3 the number of iterations required by the various Schwarz methods compared to those with transmission conditions using the approximate $\alpha_{\min}$. Similar results for the nonoverlapping domain decomposition are shown in Table 4. We find, compared to the results for small amplitude, that the number of iterations required by each Schwarz method is dramatically reduced. In other words, the large amplitude accelerates the convergence of subdomain iterations. In addition, we find as well that the oscillating frequency ω of $\eta(y)$ does affect the performance of each optimized Schwarz method in a surprising way: the faster the function $\eta(y)$ oscillates, the faster the Schwarz method converges. In addition, it is easy to see that the "slow" methods, for example, the classical Schwarz method, are more sensitive to this oscillation. We plot the number of iterations required by each optimized Schwarz method in Figure 2 for $\omega = 5$, which shows that the optimized transmission conditions perform

Table 4
Number of iterations required by various nonoverlapping Schwarz algorithms for $\eta(y) = 1000 + 1000\sin(2\pi\omega y)$.

h	1/32			1/64			1/128			1/256			1/512		
ω	1	5	10	1	5	10	1	5	10	1	5	10	1	5	10
T0	25	11	8	47	22	18	96	44	36	189	86	75	385	178	146
T0L	85	66	62	181	161	143	339	321	330	693	719	691	1471	1405	1432
T0U	15	8	7	14	13	13	26	26	26	53	53	50	102	104	102
T0A	11	9	8	18	17	17	36	36	35	71	72	71	142	142	143
T2	10	6	5	14	7	6	26	12	10	50	24	19	101	46	37
T2L	30	30	31	49	49	48	94	93	94	185	184	184	373	372	365
T2U	8	6	6	8	5	5	8	7	7	14	14	14	27	27	27
T2A	6	6	5	5	6	6	10	10	10	19	19	19	37	38	37
OO0	11	9	9	15	11	11	21	15	15	28	21	20	40	29	27
OO0L	17	15	14	22	23	22	33	33	33	46	46	46	66	65	65
OO0U	21	11	10	29	15	13	40	20	17	55	27	23	78	38	32
OO0A	18	10	9	25	13	11	34	17	15	47	23	20	67	32	27
OO2	5	6	6	6	6	6	7	6	6	8	7	7	9	8	8
OO2L	6	5	5	7	7	6	9	8	8	10	10	10	12	13	12
OO2U	13	8	7	14	8	7	16	9	8	20	10	9	22	12	10
OO2A	10	7	6	12	7	6	13	7	6	16	8	7	18	10	8
O2s	8	7	7	10	8	8	12	10	10	14	12	11	18	14	14
O2sL	11	10	9	14	13	12	16	16	16	20	20	19	25	24	24
O2sU	18	10	9	23	11	10	29	14	12	35	17	14	37	21	17
O2sA	14	7	7	18	9	8	22	11	9	27	13	11	32	16	14

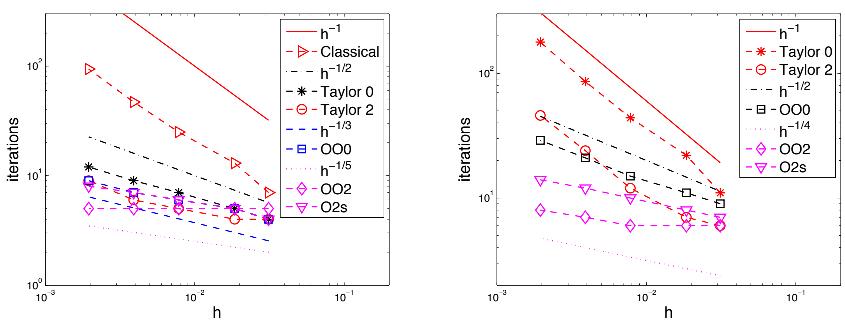

Fig. 2. *Number of iterations required by the various Schwarz methods for model problem* (1.4) *with* $\eta(y) = 1000 + 1000\sin(10\pi y)$, *on the left the overlapping case and on the right the nonoverlapping case.*

as predicted by the asymptotic convergence rates, except for the Taylor transmission condition of order 2 and the optimized second order transmission condition for the overlapping case, where a much more refined mesh would be required to attain the asymptotic regime.

We next investigate how well the continuous analysis predicts the optimal parameters to be used in the numerical setting. To this end, we vary for $\omega = 5$ the Robin parameter p with 51 uniform samples for a fixed problem of mesh size $h = 1/256$ and count for each value p the number of iterations to reach an error reduction $1e-6$, and similarly for other transmission conditions. The results are shown in Figure 3 for the overlapping case and in Figure 4 for the nonoverlapping case. These results show that the analysis predicts very well the optimal parameters. Among all the approximate $\alpha_{\min}$, we find in this case that α_U is better than α_L but $\alpha_{\min}^A$ is the best.

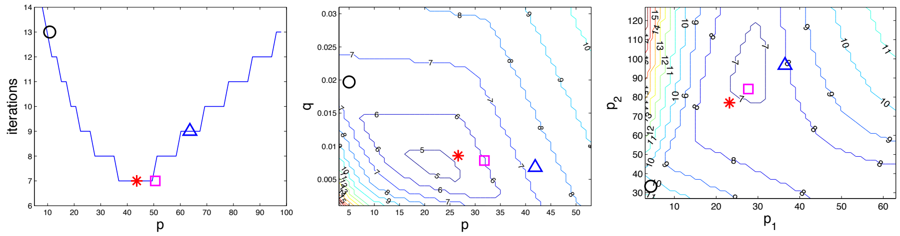

FIG. 3. *Number of iterations required by various overlapping Schwarz methods for model problem* (1.4) *when* $\eta(y) = 1000 + 1000\sin(10\pi y)$*, compared to other parameter values. From left to right,* $OO0$*,* $OO2$*, and* $O2s$ *are shown, where "*$*$*" indicates the optimized parameter, "*$\circ$*" means the lower bound approximation, "*$\triangle$*" means the upper bound approximation, and "*$\square$*" means the arithmetic mean approximation.*

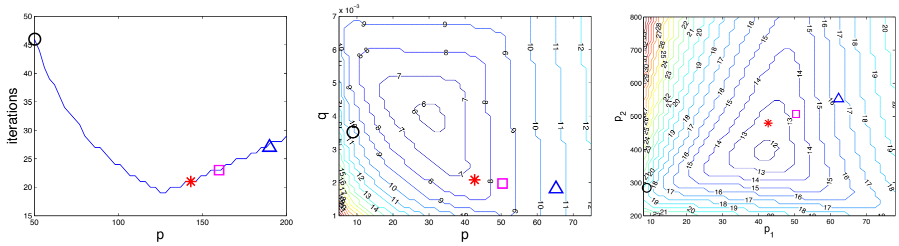

FIG. 4. *Number of iterations required by various nonoverlapping Schwarz methods for model problem* (1.4) *with* $\eta(y) = 1000 + 1000\sin(10\pi y)$*, compared to other parameter values. From left to right,* $OO0$*,* $OO2$*, and* $O2s$ *are shown, where "*$*$*" indicates the optimized parameter, "*$\circ$*" means the lower bound approximation, "*$\triangle$*" means the upper bound approximation, and "*$\square$*" means the arithmetic mean approximation.*

6.3. Coefficients with high contrast. In this experiment we investigate how the optimized Schwarz methods react to the contrast of the coefficient function $\eta(y)$. To this end, we choose $\eta(y) = d - \frac{d}{1+e^{100(y-c)}}$, which describes a coefficient function $\eta(y)$ with contrast d and a transient layer near $y = c$. We then consider first the case that $c = 0$ and the contrast d changes from 500 to 1500. The smallest eigenvalue $\alpha_{\min}$ is then 509.8516 for $d = 500$, 1009.8331 for $d = 1000$, and 1509.8140 for $d = 1500$. Its approximation using the lower bound of $\eta(y)$ is $\pi^2 + \underline{\eta} \approx 9.8696$, for all the cases $d = 500, 1000,$ and 1500; using the upper bound we get $\pi^2 + \bar{\eta} \approx 509.8696, 1009.8696,$ and 1509.8696 for $d = 500, 1000,$ and 1500, and using the arithmetic mean gives $\pi^2 + (\underline{\eta} + \bar{\eta})/2 \approx 259.8696, 509.8696,$ and 759.8696 for $d = 500, 1000,$ and 1500.

In Table 5 we show for the overlapping case the number of iterations required by various Schwarz methods compared to those with transmission conditions using approximate $\alpha_{\min}$. Similar results for the nonoverlapping case are shown in Table 6. We observe first that the higher the contrast is, the faster the Schwarz methods converge, and this phenomenon is more pronounced for Taylor transmission conditions and the classical Schwarz method. In other words, the high contrast will accelerate the convergence of Schwarz methods, especially the "slow" methods. We observe as well that the approximation $\alpha_{\min}^U$ performs as well as the real $\alpha_{\min}$. This is not surprising because $\alpha_{\min}^U$ is quite close to $\alpha_{\min}$.

We also show the above results for the case $d = 1000$ in loglog plots in Figure 5. We see that all the optimized transmission conditions follow the predicted asymptotic

Table 5

Number of iterations required by various Schwarz algorithms with overlap $L = h$ for $\eta(y) = d - \frac{d}{1+e^{100y}}$.

h	1/32			1/64			1/128			1/256			1/512		
d	500	1000	1500	500	1000	1500	500	1000	1500	500	1000	1500	500	1500	1500
Classical	7	6	5	14	10	9	26	19	16	51	37	30	101	72	60
T0	4	4	4	5	5	4	7	6	6	10	8	7	13	11	10
T0L	6	5	5	9	8	7	14	13	12	22	21	19	31	31	29
T0U	4	4	4	5	5	4	7	6	5	9	8	7	13	11	10
T0A	4	4	4	6	5	5	8	7	6	11	9	9	15	13	12
T2	4	4	4	4	4	4	5	4	4	7	6	5	9	7	7
T2L	7	7	6	9	9	9	12	12	12	16	16	16	21	21	21
T2U	4	4	4	4	4	4	5	4	4	7	6	5	9	7	7
T2A	4	4	4	4	4	4	6	5	5	7	6	6	10	9	8
OO0	4	4	4	5	5	4	6	6	5	7	7	6	9	8	8
OO0L	6	5	4	8	7	7	10	9	9	13	12	13	16	16	16
OO0U	4	4	4	5	5	4	6	6	5	7	7	6	9	8	8
OO0A	4	4	4	5	5	5	7	6	6	8	7	7	10	9	9
OO2	5	5	4	5	5	5	5	5	5	5	5	5	5	5	5
OO2L	5	5	5	5	5	6	6	6	6	6	6	6	7	7	7
OO2U	5	5	4	5	5	5	4	5	5	5	5	4	5	5	5
OO2A	5	5	4	4	5	5	4	4	4	5	5	4	5	5	5
O2s	4	4	4	5	5	5	6	6	5	7	6	6	8	7	7
O2sL	5	5	4	7	7	6	9	8	8	10	10	10	12	12	12
O2sU	4	4	4	5	5	5	6	6	5	7	6	6	8	7	7
O2sA	5	4	4	5	5	5	6	6	6	7	7	7	8	8	8

Table 6

Number of iterations required by various Schwarz algorithms without overlap for $\eta(y) = d - \frac{d}{1+e^{100y}}$.

h	1/32			1/64			1/128			1/256			1/512		
d	500	1000	1500	500	1000	1500	500	1000	1500	500	1000	1500	500	1500	1500
T0	11	8	7	25	17	14	49	35	29	100	70	59	200	142	117
T0L	41	28	23	111	82	64	282	227	191	635	556	529	1343	1257	1178
T0U	12	8	7	24	17	14	48	34	28	101	70	56	201	144	118
T0A	15	11	9	33	22	19	67	48	38	140	99	80	280	202	163
T2	5	5	5	7	5	5	14	10	8	27	19	16	52	38	31
T2L	25	27	29	47	49	49	93	92	93	188	182	186	370	371	370
T2U	5	5	5	7	5	5	14	10	8	26	19	16	52	38	31
T2A	5	5	6	10	7	6	19	14	11	37	26	22	73	53	43
OO0	9	8	8	12	10	10	16	14	12	21	18	17	30	25	23
OO0L	14	12	12	22	21	20	32	32	31	46	45	45	66	65	66
OO0U	9	8	8	12	10	10	16	14	13	22	18	17	30	25	23
OO0A	8	7	7	11	9	8	15	13	12	21	18	16	30	25	23
OO2	6	6	6	6	6	6	6	6	6	7	6	6	8	8	7
OO2L	4	4	4	6	5	5	8	8	7	10	10	10	12	12	12
OO2U	6	6	7	6	6	6	6	6	6	7	6	6	8	8	7
OO2A	5	5	6	5	5	5	6	5	5	7	6	6	8	8	7
O2s	7	7	7	8	8	7	10	9	9	12	11	11	15	13	13
O2sL	9	8	8	13	12	11	16	16	15	20	20	19	24	24	24
O2sU	7	7	7	9	8	7	10	9	9	12	11	11	15	14	13
O2sA	7	6	6	9	8	8	11	10	10	13	12	11	16	14	14

convergence rates well, except again the Taylor transmission conditions of order 2 and the optimized second order transmission conditions, where a much more refined mesh would be needed to reach the asymptotic regime.

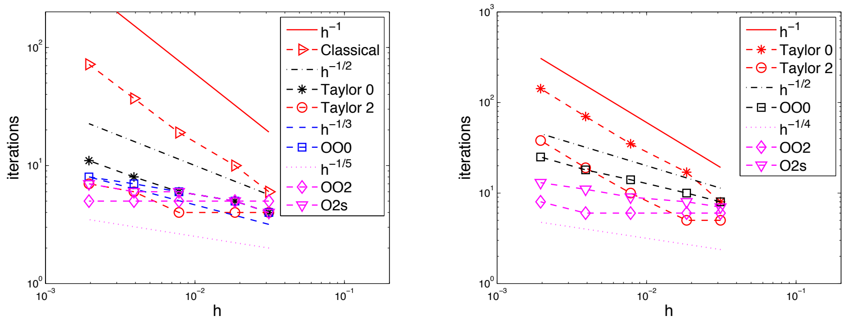

FIG. 5. *Number of iterations required by the various Schwarz methods for model problem* (1.4) *when* $\eta(y) = 1000 - \frac{1000}{1+e^{100(y-0.3)}}$*, on the left the overlapping case and on the right the nonoverlapping case.*

TABLE 7
Number of iterations required by the various Schwarz algorithms with overlap $L = h$ *for* $\eta(y) = 1000 - \frac{1000}{1+e^{100(x-c)}}$.

h	1/32			1/64			1/128			1/256			1/512		
c	0.3	0.6	0.9	0.3	0.6	0.9	0.3	0.6	0.9	0.3	0.6	0.9	0.3	0.6	0.9
Classical	16	29	43	30	58	85	61	114	168	119	228	336	236	453	671
T0	6	7	9	7	10	11	10	14	16	14	18	23	19	26	32
T0L	8	9	8	12	12	12	16	17	16	23	23	22	32	33	33
T0U	9	16	22	10	18	27	11	19	29	12	22	29	12	23	30
T0A	7	12	18	6	13	20	8	15	22	9	15	18	13	14	20
T2	5	6	6	6	8	9	7	10	11	10	13	15	13	17	20
T2L	7	7	6	9	9	8	12	12	12	16	16	15	21	22	22
T2U	6	10	14	6	11	14	7	11	16	6	11	17	8	9	16
T2A	5	7	11	5	8	12	5	7	12	6	9	13	9	9	13
OO0	5	6	7	6	8	9	7	9	10	9	11	13	12	14	16
OO0L	7	7	7	8	9	9	10	11	11	13	13	13	16	16	16
OO0U	7	13	19	10	18	25	13	24	34	16	32	44	23	40	60
OO0A	6	11	17	8	15	22	10	19	28	14	26	39	19	33	49
OO2	5	5	5	5	5	5	5	5	6	5	6	6	6	7	7
OO2L	5	5	5	5	5	5	6	5	6	6	6	6	7	7	7
OO2U	7	12	17	8	15	22	10	17	25	12	20	31	11	24	35
OO2A	6	10	14	7	12	16	7	15	21	9	17	25	10	20	25
O2s	5	6	7	6	7	8	7	8	9	8	9	10	9	11	12
O2sL	7	6	7	7	8	8	9	9	9	10	10	10	12	12	12
O2sU	6	11	16	8	15	22	11	21	28	15	24	38	18	29	44
O2sA	5	9	13	6	14	19	9	17	25	12	22	30	14	26	38

Next, we investigate how the location of the transient layer influences the performance of the various Schwarz methods. We thus fix the high contrast $d = 1000$ and vary the location of the transient layer c from 0.3 to 0.9. The corresponding smallest eigenvalue $\alpha_{\min}$ is given by $94.8798, 25.5064$, and 11.5973 for $c = 0.3, 0.6$, and 0.9, respectively. While $\alpha_{\min}^L \approx 9.8696$ for all the cases $c = 0.3, 0.6$, and 0.9, $\alpha_{\min}^U \approx 1009.8696$ for $c = 0.3$ and 0.6, and $\alpha_{\min}^U \approx 1009.8242$ for $c = 0.9$, $\alpha_{\min}^A \approx 509.8696$ for $c = 0.3$ and 0.6, and $\alpha_{\min}^A \approx 509.8469$ for $c = 0.9$. In Table 7 we show for the overlapping case the number of iterations required by various Schwarz methods as well as those using approximate $\alpha_{\min}$. Similar results for the nonoverlapping case are shown in Table 8. From Tables 7 and 8, we observe that the location of the transient layer

TABLE 8

Number of iterations required by the various Schwarz algorithms without overlap for $\eta(y) = 1000 - \frac{1000}{1+e^{100(x-c)}}$.

h	1/32			1/64			1/128			1/256			1/512		
c	0.3	0.6	0.9	0.3	0.6	0.9	0.3	0.6	0.9	0.3	0.6	0.9	0.3	0.6	0.9
T0	28	57	83	60	113	167	110	225	333	233	441	666	463	916	1360
T0L	84	90	88	178	174	184	356	364	366	719	731	725	1445	1452	1461
T0U	12	22	32	18	23	32	37	37	37	72	73	71	144	144	146
T0A	12	16	23	24	26	26	50	51	52	103	100	103	201	203	207
T2	9	17	23	16	30	43	30	57	85	60	115	172	120	231	337
T2L	26	27	26	48	47	47	94	93	92	186	184	183	370	364	368
T2U	7	12	16	7	11	17	10	12	16	19	19	19	38	37	37
T2A	6	9	13	7	9	13	14	14	14	27	27	27	52	52	52
OO0	12	16	19	16	21	27	23	30	37	32	43	53	44	59	74
OO0L	15	17	19	23	24	25	33	34	35	46	46	51	65	66	74
OO0U	20	38	55	29	54	78	40	76	112	57	107	158	79	152	222
OO0A	18	32	47	24	45	67	34	65	94	48	91	133	67	130	188
OO2	5	6	6	6	7	8	7	8	9	8	10	10	10	11	12
OO2L	6	6	6	7	8	8	8	9	9	10	10	11	12	12	13
OO2U	12	21	31	13	25	36	15	30	42	19	35	49	21	38	59
OO2A	9	17	24	10	20	28	12	23	32	14	27	40	17	27	49
O2s	9	11	11	10	12	14	12	15	17	15	18	20	18	22	24
O2sL	10	11	11	14	13	15	17	17	17	20	20	20	24	24	24
O2sU	17	32	46	21	39	58	25	51	75	32	63	92	40	75	113
O2sA	13	24	38	17	31	45	20	38	57	27	50	72	32	60	89

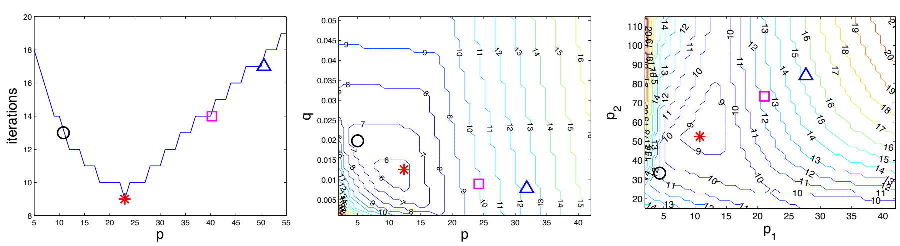

FIG. 6. *Number of iterations required by the various overlapping Schwarz methods for model problem* (1.4) *when* $\eta(y) = 1000 - \frac{1000}{1+e^{100(y-0.3)}}$, *compared to other parameter values. From left to right, OO*0, *OO*2, *and O*2*s are shown, where "∗" indicates the optimized parameter, "○" means the lower bound approximation, "△" means the upper bound approximation, and "□" means the arithmetic mean approximation.*

influences remarkably the performance of the Schwarz methods: the farther right the transient layer is located, the slower the Schwarz methods converge. In addition, this phenomenon is more pronounced for the nonoverlapping Schwarz methods and the "slow" methods.

We finally investigate, for the case of high contrast $\eta(y)$, how well the continuous analysis predicts the optimal parameters to be used in the numerical setting. To this end, we vary for the case $c = 0.3$ the Robin parameter p with 51 uniform samples for a fixed problem of mesh size $h = 1/256$ and count for each value p the number of iterations reaching an error reduction $1e-6$, and similarly for the other transmission conditions. The results are shown in Figure 6 for the overlapping case and in Figure 7 for the nonoverlapping case. These results show that the analysis predicts very well the optimal parameters. Among all the approximate $\alpha_{\min}$, we find, however, that $\alpha^L_{\min}$ is the best in this case.

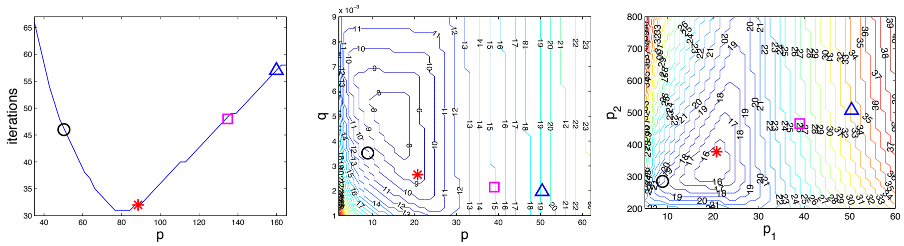

Fig. 7. *Number of iterations required by the various nonoverlapping Schwarz methods for model problem (1.4) when $\eta(y) = 1000 - \frac{1000}{1+e^{100(y-0.3)}}$, compared to other parameter values. From left to right, OO0, OO2, and O2s are shown, where "$*$" indicates the optimized parameter, "$\circ$" means the lower bound approximation, "$\triangle$" means the upper bound approximation, and "$\square$" means the arithmetic mean approximation.*

Table 9

Number of iterations required by the various Schwarz algorithms with overlap $L = h$ for $\eta(x, y) = 1000 + 1000\sin(2\pi\omega y)\cos(\pi\omega x)$.

h	1/32			1/64			1/128			1/256			1/512		
ω	1	5	10	1	5	10	1	5	10	1	5	10	1	5	10
T0	5	4	4	7	5	5	9	7	6	12	9	8	17	12	11
T0_f	6	5	4	9	7	6	14	10	9	18	15	13	27	22	20
T2	5	4	4	6	4	4	7	5	4	9	6	6	11	8	8
T2_f	5	4	4	6	5	5	9	8	7	12	11	10	16	16	14
OO0	5	4	4	6	5	5	7	6	6	9	7	7	11	9	8
OO0_f	6	5	4	7	6	5	9	8	7	12	10	9	13	13	12
OO2	5	5	5	5	5	5	5	4	5	5	5	4	6	5	5
OO2_f	4	4	3	5	4	4	5	5	5	6	6	6	7	7	6
O2s	5	4	4	6	5	5	7	6	6	8	7	6	9	8	7
O2s_f	6	5	4	6	6	5	8	7	7	9	9	8	11	11	9

6.4. Comparison with the frozen coefficient approach. In this subsection, we consider numerically problems where our analysis is not valid, i.e., the case where the reaction coefficients are varying in both the x and y directions, and compare our results to those obtained by the widely used strategy of frozen coefficients. From the previous experiments we know that a small amplitude of the reaction coefficients does not affect the subdomain iterations a lot, and we thus consider in this subsection only the case of large amplitude, and choose $\eta(x, y) = 1000 + 1000\sin(2\pi\omega y)\cos(\pi\omega x)$. On the interface $x = L$, $\eta(L, y)$ is used to determine the smallest eigenvalue $\alpha_{\min}$, and similarly for the interface $x = 0$. We show the number of iterations required by each optimized Schwarz method to reach an error reduction of $1e-6$ for the overlapping case in Table 9, and for the nonoverlapping case in Table 10. On the interface $x = 0$, the function $\eta(0, y) = 1000 + 1000\sin(2\pi\omega y)$ coincides with the case with large amplitude in subsection 6.2, and a comparison with the results in Tables 3 and 4 shows that our optimized transmission parameters require basically the same number of iterations as when the reaction coefficient varies only in the y-direction, which illustrates the efficiency of our predicted transmission parameters.

We then perform a similar experiment, but with the frozen coefficient approach [11, 18, 35]. The results are shown in Tables 9 and 10 for the overlapping and nonoverlapping methods, indicated by a subscript "f" for "frozen." We see that the frozen

Table 10
Number of iterations required by the various Schwarz algorithms for $\eta(x,y) = 1000 + 1000\sin(2\pi\omega y)\cos(\pi\omega x)$, *the nonoverlapping case.*

h	1/32			1/64			1/128			1/256			1/512		
ω	1	5	10	1	5	10	1	5	10	1	5	10	1	5	10
T0	23	11	8	47	23	18	95	45	36	189	83	72	390	180	145
$T0_f$	69	45	58	144	120	119	292	250	222	627	591	520	1340	1292	1219
T2	10	6	5	14	7	5	26	12	10	50	24	19	99	46	38
$T2_f$	16	19	18	41	35	36	85	70	76	180	159	150	344	352	339
OO0	11	9	9	15	11	11	21	15	14	29	25	19	39	28	27
$OO0_f$	14	12	12	17	16	17	30	27	24	43	41	37	61	59	57
OO2	5	6	6	6	5	6	7	6	6	7	7	7	9	8	8
$OO2_f$	5	5	5	6	6	6	8	7	6	9	8	7	11	10	9
O2s	8	7	7	10	8	8	12	10	10	14	12	11	17	14	14
$O2s_f$	9	8	7	11	10	9	15	13	11	19	16	14	22	20	19

coefficient approach also works quite well, but our new methods lead to lower iteration counts, especially in the nonoverlapping case and when the mesh is refined.

7. Conclusion.

In this paper we analyzed optimized Schwarz methods for model problems with coefficients varying continuously parallel to the interface. We decoupled the spatial variables of the model problem using the technique of separation of variables and obtained a convergence factor for the methods as a function of eigenvalues of certain Sturm–Liouville problems containing the variable coefficient. Various optimized transmission condition were then obtained by optimizing the convergence factor over all relevant eigenvalues. This method for analyzing the OSM is promising and can be applied to many other equations with variable coefficients. With this analysis, we obtained the following important new insight: the performance of Schwarz methods and the optimized parameters depend on the smallest eigenvalue of an interface Sturm–Liouville problem, and not locally on the variation of the coefficient along the interface, like one assumed in a frozen coefficient approach. We also saw in the numerical experiments that fast oscillations and high contrast of the reaction term are good for the performance of the Schwarz methods, especially the slow ones.

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