Overlapping Multi-Subdomain Asynchronous Fixed Point Methods for Elliptic Boundary Value Problems

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1 Introduction

We present and analyze asynchronous iterations for the solution of second order elliptic partial differential equations based on an overlapping domain decomposition. Asynchronous iterations constitute a general framework for fixed point methods, where the considered fixed point mapping is defined on a product space. Our general formulation includes standard relaxation algorithms such as Jacobi and Gauss-Seidel and their block versions. Indeed the block versions can be related to the additive and multiplicative Schwarz method respectively. In addition to the standard situations, asynchronous iterations can describe more involved synchronous or asynchronous parallel algorithms. For recent developments for the multi-subdomain multiplicative Schwarz (Gauss-Seidel) method we refer to [1] and [2]. For truly asynchronous methods for overlapping subdomain decompositions we refer to [7] for first partial results.

One important feature of the results presented here is the use of weighted L^∞ norms, which allows us to obtain a stronger convergence property than

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the usual L^∞ one, even for the standard algorithms (see [3] V.2, p. 294), provided that the problem and its decomposition are sufficiently regular (for a particular case with L shaped domain and less regularity see [6]).

Our results are particularly well suited for pseudo stationary problems arising from implicit or semi-implicit schemes for evolution equations. For the repetitive solution of the large scale problems which are usually well conditioned, the asynchronous fixed point method can be competitive to other methods. We refer to [5] for a future paper in this direction.

After the introduction, we present in the second section the problem formulation, introduce the notation used in the sequel and state our basic assumptions. In the third section we define the linear mapping $\mathcal{T}$ which defines the substructured solution process. Then we define the linear fixed point mapping T which is the composition of $\mathcal{T}$ with a suitable restriction operator R. We prove that T is a linear mapping in a suitable function space context. We also study the contraction property of T. We finally introduce an affine mapping whose linear part is T and whose fixed point is the solution of the elliptic partial differential equation. We state in the closing proposition the convergence of asynchronous iterations applied to the approximation of this affine fixed point mapping.

2 Problem Description, Notation and Assumptions

Let Ω be an open bounded domain of $\mathbb{R}^n$ with boundary $\partial\Omega$. We consider the Dirichlet problem

$$Au = f \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega \tag{2.1}$$

with given $f \in \mathbf{L}^p(\Omega)$ and $g \in \mathbf{L}^q(\partial\Omega)$ and we assume that (2.1) has a unique solution in a suitable function space.

To formulate our algorithm, we need an overlapping decomposition of Ω with certain overlap properties. We construct such a decomposition by first decomposing the domain Ω into m non-overlapping open subdomains Ω'_i,

$$\Omega'_i \cap \Omega'_j = \emptyset,\ i \neq j, \quad \bigcup_{i=1}^{m} \overline{\Omega'}_i = \overline{\Omega}. \tag{2.2}$$

The boundary $\partial\Omega_i'$ of Ω_i' consists in general of a part inside Ω, which we call $\Gamma_i' := \partial\Omega_i' \cap \Omega$ and a part coinciding with the boundary of Ω, which we denote by $\gamma_i' := \partial\Omega_i' \cap \partial\Omega$. We now construct from this non-overlapping decomposition of Ω the desired overlapping decomposition which will be used by our algorithm. We enlarge Ω_i' to obtain the overlapping subdomains Ω_i, $\Omega_i' \subset \Omega_i \subset \Omega$, and we define the corresponding boundaries of the overlapping subdomains by

$$\Gamma_i = \partial\Omega_i \cap \Omega, \quad \gamma_i = \partial\Omega_i \cap \partial\Omega. \tag{2.3}$$

We require strict overlap, which means that the closures $\overline{\Omega'}_i$ and $\overline{\Gamma}_i$ satisfy

$$\overline{\Omega'}_i \cap \overline{\Gamma}_i = \emptyset, \quad i = 1, \ldots, m. \tag{2.4}$$

The linear part of the affine mapping we need in order to introduce our algorithm is based on the homogeneous subproblems

$$Av_i = 0 \text{ in } \Omega_i, \quad v_i = 0 \text{ on } \gamma_i, \quad v_i = \varphi_i \text{ on } \Gamma_i,\ \varphi_i \in \mathbf{L}^p(\Gamma_i) \tag{2.5}$$

and we need the following regularity of subdomain solutions

$$v_i \in \left\{ \begin{array}{ll} C^\infty(\overline{\Omega'}_i) & \text{if } \gamma_i' = \emptyset \\ C^1(\overline{\Omega'}_i) & \text{otherwise.} \end{array} \right. \tag{2.6}$$

Assumption 2.1 *The elliptic operator A satisfies both the standard maximum principle with respect to Ω_i and $\partial\Omega_i$ and the Hopf maximum principle with respect to γ_i' (compare [10]).*

For the exchange of information between subdomains, we will also employ the index notation

$$\Gamma_{i,j} = \Gamma_i \cap \Omega_j',\ j \in J(i) \tag{2.7}$$

where the index set J is defined by

$$J(i) = \left\{ j : \Gamma_i \cap \Omega_j' \neq \emptyset, j \neq i \right\}. \tag{2.8}$$

Assumption 2.2 *There exists an indefinitely often continuously differentiable positive Greens function associated with A on each subdomain,*

$$G_i(x, y) > 0, \quad x, y \in \Omega_i,\ x \neq y.$$

To the Greens function corresponds a Poisson kernel $\frac{\partial G_i}{\partial \nu_i}(x, y)$ where $\frac{\partial}{\partial \nu_i}$ denotes the unit outward normal derivative with respect to $\partial\Omega_i$ (compare [8] p. 342 (consequence 5.1)).

Remark 2.3 *Weaker regularity assumptions are be possible for the subdomains Ω_i on the basis of the exterior ball property (cf [9] p. 16) and Hölder regularity of the boundaries.*

Assumption 2.4 *The Poisson Kernel $\frac{\partial G_i}{\partial \nu_i}(x, y)$ is positive and continuously differentiable on $\overline{\Omega}_i \times \partial\Omega_i$, $x \neq y$.*

Assumption 2.5 *There exists a positive eigenfunction $e \in C(\overline{\Omega})$ associated with the smallest eigenvalue λ of A ,*

$$Ae = \lambda e, \;\; \lambda \in \mathbb{R}^+. \tag{2.9}$$

Remark 2.6 *The required regularity (2.6) can be obtained on the basis of Assumptions (2.2) and (2.4).*

3 Linear Mappings Associated with Overlapping Subdomains

3.1 The linear mapping $\mathcal{T}$

We denote all the interior boundaries by $\Gamma = \cup_{i=1}^m \Gamma_i$ and we consider the space $L^1(\Gamma) := \prod_{i=1}^m \prod_{j \in J(i)} L^1(\Gamma_{i,j})$. We will furthermore need the function space $\mathcal{C} := \prod_{i=1}^m C(\overline{\Omega'}_i)$. Note that $\mathcal{C} \neq C(\overline{\Omega})$. We now define the linear mapping

$$\mathcal{T} : L^1(\Gamma) \longrightarrow \mathcal{C}, \quad \mathcal{T} : w \longmapsto v'.$$

For a given function $w \in L^1(\Gamma)$ we first write component-wise

$$w = \{\ldots, w_i, \ldots\}_{i=1,\ldots,m}\,, \quad w_i = \{\ldots, w_{i,j}, \ldots\}_{j \in J(i)} \in L^1(\Gamma_i)$$

and then we compute using w the solutions v_i of the subproblems

$$Av_i = 0 \text{ in } \Omega_i, \quad v_i = w_{i,j} \text{ on } \Gamma_{i,j}, \; j \in J(i), \quad v_i = 0 \text{ on } \gamma_i. \tag{3.1}$$

Now we take the restriction $v_i' = v_i|_{\overline{\Omega'}_i}$ and we define the linear operator $\mathcal{T}_i$ by $\mathcal{T}_i(w) := v_i'$. Finally we set

$$\mathcal{T}(w) := v' = \{\ldots, \mathcal{T}_i(w), \ldots\} \tag{3.2}$$

thus defining the linear mapping $\mathcal{T}$. On using the Greens function and Poisson kernel, we can express v'_i and hence $\mathcal{T}_i$ and $\mathcal{T}$ in analytic form,

$$v'_i(x) = \sum_{j\in J(i)} \int_{\overline{\Gamma}_{i,j}} \frac{\partial G_i}{\partial \nu_j}(x,y) w_{i,j}(y) dy = \int_{\overline{\Gamma}_i} \frac{\partial G_i}{\partial \nu_j}(x,y) w_i(y) dy \quad \forall\, x \in \overline{\Omega'}_i. \tag{3.3}$$

Proposition 3.1 $\mathcal{T} \in \mathcal{L}(L^1(\Gamma), \mathcal{C})$ *is a linear isotone mapping with respect to the natural order.*

3.2 The Linear Mapping T

We first define the space $C_e(\overline{\Omega'}_j)$ containing all the elements of $C(\overline{\Omega'}_j)$ which can be endowed with the norm $|w_j|_{e,\infty} = \max_{x\in\overline{\Omega'}_j} \frac{|w_j(x)|}{e(x)}$ where $e(x)$ denotes the eigenfunction in Assumption 2.5. We define furthermore $C_e = \prod_{j=1}^m C_e(\overline{\Omega'}_j)$ endowed with the norm $|w|_{e,\infty} = \max_{j=1,\ldots,m} |w_j|_{e,\infty}$ where $w = \{\ldots, w_j, \ldots\}$.

Then we define the mapping

$$T : \mathcal{C} \longrightarrow C_e, \quad T : v' \longmapsto \widetilde{v}'$$

on using the mapping $\mathcal{T}$. We first define for the components $v'_j \in C(\overline{\Omega'}_j)$ of $v' \in \mathcal{C}$ the restriction operator $R_{i,j}$, $j \in J(i)$, by

$$w_{i,j} = R_{i,j}(v'_j) = v'_j|_{\Gamma_{i,j}}.$$

Denoting by $w_i = \{\ldots, w_{i,j}, \ldots\}_{j\in J(i)}$ we define the restriction operator R_i by

$$R_i(v') = w_i = \left\{\ldots, R_{i,j}(v'_j), \ldots\right\}_{j\in J(i)}$$

and finally collecting components we get $w = \{\ldots, w_i, \ldots\}$ and we can define the restriction operator R from $\mathcal{C}$ to $\prod_{i=1}^m \prod_{j\in J(i)} C(\overline{\Gamma}_{i,j})$ $(\subset L^1(\Gamma))$ by

$$R(v') := w = \{\ldots, R_i(v'), \ldots\}.$$

Now we can define the mapping T_i as the composition of $\mathcal{T}_i$ and R, $T_i(v') := v'_i = \mathcal{T}_i \circ R(v')$ and collecting the v'_i we find the mapping T to be

$$T(v') := \widetilde{v}' = \mathcal{T} \circ R(v'). \tag{3.4}$$

Proposition 3.2 *$T \in \mathcal{L}(\mathcal{C}, C_e)$ is a linear isotone mapping with respect to the natural order on $\mathcal{C}$ and C_e. The norm of T is given by*

$$||T||_{\mathcal{L}(\mathcal{C},C_e)} = \max_{i=1\ldots m} \max_{x\in\overline{\Omega'}_i} \frac{(T\mathbf{1})_x}{e(x)} = \widetilde{\mu}$$

where $\mathbf{1}$ is the function $\mathbf{1}(x) = 1 \ \forall \ x \in \overline{\Omega}$.

Remark 3.3 *The operator $\mathcal{T}$ is also in $L(\prod_{i=1}^m \prod_{j\in J(i)} C(\overline{\Gamma}_{i,j}), C_e)$, and $\mathcal{T}$ is also an isotone linear mapping with respect to the natural order on $\prod_{i=1}^m \prod_{j\in J(i)} C(\overline{\Gamma}_{i,j})$ and C_e.*

Remark 3.4 *Propositions 3.1 and 3.2 do not assert that the linear mappings $\mathcal{T}$ or T are contractions.*

3.3 Contraction Property of T

On using Assumption (2.5) we consider the subproblems

$$Av_i = 0 \text{ in } \Omega_i, \quad v_i = e \text{ on } \partial\Omega_i \tag{3.5}$$

and also the restriction

$$v'_i = v_i|_{\overline{\Omega'}_i}. \tag{3.6}$$

We define furthermore $e_i := e|_{\overline{\Omega'}_i}$.

Lemma 3.5 *$\frac{v'_i(x)}{e_i(x)}$ is defined and continuous on $\overline{\Omega'}_i$, and $\max_{x\in\overline{\Omega'}_i} \frac{v'_i(x)}{e_i(x)} \leq \mu_i < 1$.*

Proposition 3.6 *$T \in \mathcal{L}(\mathcal{C}, C_e)$ is a contraction with contraction constant $\mu = \max_{i=1,\ldots,m} \mu_i$.*

Remark 3.7 *The proof Proposition 3.6 shows that μ is an upper bound on $\|T\|_{\mathcal{L}(\mathcal{C},C_e)}$ and the bound is reached if $w(x) = e(x)$ and is thus sharp. One furthermore finds that μ is decreasing when the size of overlap increases and μ is decreasing when ε decreases in the case$A = -\varepsilon\Delta + I$. We refer to [5] for numerical experiments and comparison of algorithms.*

3.4 The Fixed Point Mapping

Let us consider an element w such that $w_i - u_i \in C(\overline{\Omega'}_i)$. This implies, with a trivial extension that

$$\delta w = w - u \in \mathcal{C}. \tag{3.7}$$

Let us first solve the subproblems

$$Au_i = f \text{ in } \overline{\Omega}_i, \quad u_i = 0 \text{ on } \Gamma_i, \; u_i = g \text{ on } \gamma_i, \quad i = 1, \ldots, m. \tag{3.8}$$

Restricting u_i to Ω'_i,

$$u'_i = u_i|_{\overline{\Omega'}_i} \tag{3.9}$$

we consider the new subproblems

$$Av_i = f \text{ in } \overline{\Omega}_i, \quad v_i = w \text{ on } \Gamma_i, \quad v_i = g \text{ on } \gamma_i, \quad i = 1, \ldots, m \tag{3.10}$$

and we get the restricted values

$$v'_i = v_i|_{\overline{\Omega'}_i} \tag{3.11}$$

so that the fixed point mapping is given by

$$v'_i = T_i(w) + u'_i. \tag{3.12}$$

Proposition 3.8 *If for w satisfying (3.7), we choose $u^0 = T(w) + u'$ where $u' = \{\ldots, u'_i, \ldots\}$, u'_i defined by (3.8), (3.9) then the asynchronous iterations initialized by u^0, applied to the affine fixed point mapping $F(w) = T(w) + u'$ give rise to a sequence of iterates which converges, with respect to the uniform weighted norm $|\ |_{e,\infty}$ towards the solution u^* of problem (2.1).*

The proof is based on the use of El Tarazi's theorem [4] and on Proposition 3.6. Compare also [7] for previous partial results and numerical experiments.

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