On the limitations of sweeping type preconditioners for wave propagation

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Sweeping type preconditioners have received a lot of attention over the past few years, following the publication by Engquist and Ying [1]. These preconditioners are however not new: they have their roots in optimal and optimized Schwarz methods and AILU preconditioners, see [2] for an overview and the precise relation between various such techniques, and one has to be careful claiming optimality of such methods for wave propagation problems based on numerical experiments.

To illustrate this, we use here the example that arose during the presentation of Laurent Demanet, namely the one dimensional time dependent wave equation

$$\begin{aligned} &\partial_{tt}u = c^2\partial_{xx}u \quad \text{in } (0,L)\times(0,T),\\ &\mathcal{B}_l(u) := (\partial_t - c\partial_x)u(0,t) = 0, \quad \mathcal{B}_r(u) := (\partial_t + c\partial_x)u(L,t) = 0, \end{aligned} \tag{1}$$

with appropriate compactly supported initial conditions. For constant wave speed c, the d'Alembert solution is a sum of a left and a right going wave, $u(x,t) = g_l(x+ct) + g_r(x-ct)$, as one can easily see by just introducing it into the wave equation (1). The boundary conditions in (1) are called the transparent (or exact or non-reflecting) boundary conditions, since the outgoing waves satisfy them exactly, $\mathcal{B}_l(g_l) = 0$ and $\mathcal{B}_r(g_r) = 0$, and thus the solution of (1) coincides with the solution of the problem posed on $\mathbb{R}\times(0,T)$.

An optimal Schwarz waveform relaxation method for this problem was defined and analyzed in [3]. It is based on a decomposition of the spatial domain $\Omega := (0,L)$ into non-overlapping subdomains $\Omega_j := (x_{j-1}, x_j)$, $j = 1,2,\ldots,J$ with $0 =: x_0 < x_1 < \ldots < x_J := L$, and in its alternating (sweeping) version, the method solves at iteration n for a given approximation u_j^{n-1} for $j = 1,2,\ldots,J$ the forward sweep

$$\begin{aligned} &\partial_{tt}u_j^{n-\frac{1}{2}} = c^2\partial_{xx}u_j^{n-\frac{1}{2}} \quad \text{in } \Omega_j\times(0,T),\\ &\mathcal{B}_l(u_j^{n-\frac{1}{2}}) = \mathcal{B}_l(u_{j-1}^{n-\frac{1}{2}}), \quad \mathcal{B}_r(u_j^{n-\frac{1}{2}}) = \mathcal{B}_r(u_{j+1}^{n-1}), \end{aligned} \tag{2}$$

where we defined $u_0^{n-\frac{1}{2}} = 0$ and $u_{J+1}^{n-1} = 0$ to simplify the notation. The forward sweep is followed by the backward sweep which solves for $j = J-1, J-2, \ldots 1$

$$\begin{aligned} &\partial_{tt}u_j^{n} = c^2\partial_{xx}u_j^{n} \quad \text{in } \Omega_j\times(0,T),\\ &\mathcal{B}_l(u_j^{n}) = \mathcal{B}_l(u_{j-1}^{n-\frac{1}{2}}), \quad \mathcal{B}_r(u_j^{n}) = \mathcal{B}_r(u_{j+1}^{n}). \end{aligned} \tag{3}$$

In this alternating version, the algorithm converges in one iteration (a forward sweep followed by a backward sweep) for an arbitrary initial guess u_j^0, as one can see as follows: in the forward sweep, the component of the solution traveling to the right is obtain exactly in each subdomain, since on their left interfaces, no artificial reflections are created. Hence at the end of the forward sweep, $u_J^{\frac{1}{2}} \equiv u$. Now in the backward sweep, this exact solution is just transmitted from right to left, again no reflections are created on the left boundaries of the subdomains, and thus $u_j^1 \equiv u$ also for $j = J-1, J-2, \ldots, 1$. This algorithm is closely related

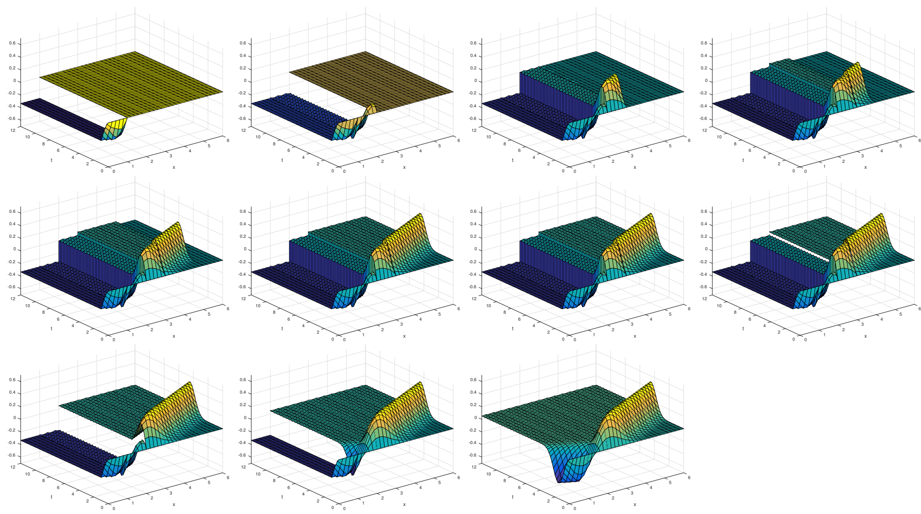

FIGURE 1. The first six panels represent the forward and the last five the backward sweep of one optimal alternating Schwarz waveform relaxation iteration leading to the solution of the problem

to a block LU decomposition, only the right boundary condition in the block LU factorization corresponds to a Dirichlet condition, see [2]. In a parallel version of the algorithm, convergence is achieved in J iterations, see [3], but then it is essential to use the transparent boundary conditions on both subdomain sides.

Discretizing the wave equation (1) using finite differences, we get

$$\begin{array}{ll} \frac{u_{i,k+1}-2u_{i,k}+u_{i,k-1}}{\Delta t^2} = c^2\frac{u_{i+1,k}-2u_{i,k}+u_{i-1,k}}{\Delta x^2} & 0 \le i \le I, \quad 0 < k < K, \\ \frac{u_{0,k+1}-u_{0,k-1}}{2\Delta t} - c\frac{u_{1,k}-u_{-1,k}}{2\Delta x} = 0, & \frac{u_{I,k+1}-u_{I,k-1}}{2\Delta t} + c\frac{u_{I+1,k}-u_{I-1,k}}{2\Delta x} = 0. \end{array} \tag{4}$$

If the discretization parameters are chosen precisely at the limit of the CFL condition, $c\frac{\Delta t}{\Delta x} = 1$, this discretization produces the exact solution, as one can verify by introducing the d'Alembert solution consisting of the forward and backward moving waves into (4). This is thus the ideal situation where we can easily check the influence of using transparent transmission conditions or approximations in the discretized algorithm (2,3).

We first show in Figure 1 how one forward and backward sweep generates the exact solution for an example with $J = 6$ subdomains, $L = 6$, $T = 12$, $c = 1$, and $\Delta x = \Delta t = \frac{1}{5}$ so $c\frac{\Delta t}{\Delta x} = 1$, with initial condition $u(x, 0) = 0$ and $\partial_t u(x, 0) = -2(\frac{3}{2} - x)e^{-(\frac{3}{2}-x)^2}$. We see how the algorithm produces the forward moving wave in the forward sweep and then completes the solution in the backward sweep, converging in one iteration as expected for the transparent transmission conditions.

The key question we want to address now is what happens if the transparent transmission conditions are approximated. To do so, we apply the algorithm to

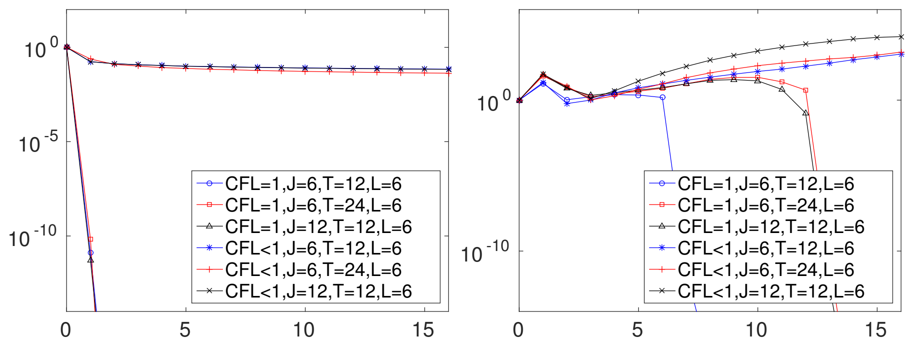

FIGURE 2. Decay of the error as a function of the number of double sweeps when using approximations of the optimal alternating Schwarz waveform relaxation algorithm

the error equations, i.e. with zero initial conditions, and we start with a random initial guess on the interfaces. We show in Figure 2 on the left the influence on the convergence behavior of the algorithm when the discretization parameters are chosen either exactly on the CLF, $c\frac{\Delta t}{\Delta x} = \frac{1/10}{1/10} = 1$, or just a little below, $c\frac{\Delta t}{\Delta x} = \frac{3/35}{1/10} = \frac{6}{7}$. We see that when we are exactly on the CFL, the algorithm converges in one double sweep to machine precision, independently of the number of subdomains or the length of the time interval. If we are slightly below however, the first iteration only brings a small error reduction, and then the algorithm seems to stagnate. The spectacular performance is thus intimately related to having the exact (discrete !) transparent boundary condition at the interfaces. In Figure 2 on the right, we show an inhomogeneous numerical example with constant wave speed per subdomain, $c_1 = \frac{1}{2}$, $c_2 = 1$, $c_3 = \frac{2}{5}$, $c_4 = 1$, $c_5 = \frac{3}{10}$ and $c_6 = 1$ (in the case of $J = 12$ subdomains we just repeat the sequence once more), and we use at the interfaces a discretization of the exact transparent boundary condition based on the neighboring subdomain, not taking into account reflections that will come from subdomains with different wave speed further away. When on the CFL for the fastest wave speeds, $c_{\max}\frac{\Delta t}{\Delta x} = \frac{1/10}{1/10} = 1$, the algorithm now converges in a number of iterations that depends on the number (and size) of subdomains and the length of the time interval[1], which can be understood from the number of reflections that have to be taken into account up to the time of interest T, see [3]. The number of reflections would however be infinity in the time harmonic regime, which corresponds to $T = \infty$. If one is slightly below the CFL for all the subdomains, $c_{\max}\frac{\Delta t}{\Delta x} = \frac{3/35}{1/10} = \frac{6}{7}$, the algorithm seems to diverge used as an

[1]It is essential that each subdomain with a CFL less than one has two neighbors with a CFL equal to one for convergence in a finite number of steps.

iterative solver[2]. This could be masked to some extent using Krylov acceleration, but would certainly manifest itself at some point there as well to indicate that this is not an optimal preconditioner any more.

One thus has to be very careful claiming that an optimal solver has been obtained for the time harmonic case just based on numerical experiments, and such preconditioners should always be tested without Krylov acceleration as well, and with random initial guess[3], so convergence problems are not masked initially by a clever polynomial selection of the Krylov method, or just small frequency content.

References

[1] B. Engquist, L. Ying *Sweeping preconditioner for the Helmholtz equation: Hierarchical matrix representation*, Comm. Pure Appl. Math. **LXIV** (2011), 697–735.

[2] M.J. Gander, H. Zhang, *Iterative Solvers for the Helmholtz Equation: Factorizations, Sweeping Preconditioners, Source Transfer, Single Layer Potentials, Polarized Traces, and Optimized Schwarz Methods*, submitted to SIAM Review (2016).

[3] M.J. Gander, L. Halpern *Optimal Schwarz Waveform Relaxation for the One Dimensional Wave Equation*, SIAM J. Numer. Anal. **41(5)** (2003), 1643–1681.

[4] M.J. Gander, *Schwarz Methods Over the Course of Time*, ETNA **31** (2008), 228–255.

[2]In this time dependent case, super-linear convergence will set in after a very large number of iteration steps related to the number of time steps, namely after $T/(2\Delta t)$ iterations (e.g. 70 in the first example), due to the waveform relaxation nature of the algorithm, but this would not be the case for a time harmonic formulation that corresponds again to $T = \infty$.

[3]See Subsection 5.2, last paragraph in [4] on how to numerically 'prove' optimality of an algorithm which is in fact not optimal, by just using low frequency content.