A CONTINUOUS ANALYSIS OF NEUMANN-NEUMANN METHODS: SCALABILITY AND NEW COARSE SPACES*

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Abstract. We present a new coarse space correction for the iterative Neumann-Neumann method. We describe the method for general elliptic partial differential equations, and perform the analysis for the case of the Poisson and screened Poisson equation (sometimes also called positive definite Helmholtz equation, or Helmholtz equation with the good sign). We prove that the new two-level Neumann-Neumann method converges after one iteration, both at the continuous and discrete level, which means the new coarse space is optimal in the sense of best possible, and it makes the two-level method a direct solver. In two and three space dimensions, the new coarse space is too high dimensional in practice, and we introduce a spectral approximation, which transforms a divergent iterative Neumann-Neumann method into a convergent one. We also identify what the optimized choice of coarse space functions is in the approximation. Our new coarse space thus also addresses convergence or robustness problems of the underlying domain decomposition iteration, similarly to the new coarse spaces GenEO, SHEM, and ACMS, which were designed to treat different convergence difficulties of the underlying domain decomposition method, namely the presence of high contrast media. Several numerical experiments are carried out to demonstrate the performance of this new coarse space correction, also including decompositions with cross points.

Key words. domain decomposition methods, elliptic problems, Neumann-Neumann methods

AMS subject classifications. 65N55, 65F08, 65F10, 65F50

1. Introduction.

We design and analyze a new coarse space correction for domain decomposition solvers of algebraic equations arising from the discretization of second-order elliptic PDEs. Domain decomposition methods are of great interest because of their natural parallelism permitting the use of parallel architectures in order to approximate the solution of partial differential equations. Another important benefit of these methods is that they allow for a better treatment of complex geometries and they are well suited for heterogeneous problems, i.e., problems that have different physics in different parts of the domain, see e.g. [30, 33, 22]. We focus here on a class of non-overlapping domain decomposition methods known in the literature as the Neumann-Neumann methods (NNMs). As most domain decomposition methods, the classical one-level NNMs lack global communication, since the only mechanism for sharing data is through interfaces. A remedy for this is to introduce an additional coarse space correction with a relatively small cost compared to the size of the original problem, see the seminal early contributions [29, 9]. The design of efficient coarse spaces is at the heart of domain decomposition theory and it is in general not straightforward. For the case of algebraic NNM preconditioners, seminal contributions are e.g. [25, 26, 27, 10, 8]. However, the NNM was originally described in [11] as an iteration at the continuous level like the classical Schwarz method, but only for two subdomains. This is because unlike the Schwarz method [24], the convergence of the iterative NNM is not guaranteed for the case of many subdomains, even in the case of a one way decomposition without cross points, when the subdomain aspect

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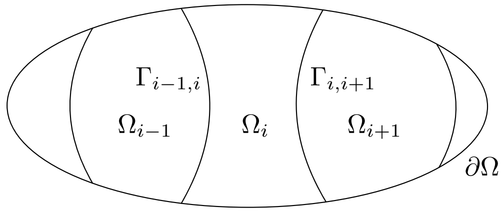

Fig. 1: One way decomposition of the domain Ω.

ratio is unfavorable, see [2, 1]. We are interested here in developing a coarse space correction that leads to a convergent iterative NNM.

An interesting discovery was made in [15], where the authors described a coarse space for the parallel Schwarz method that leads to convergence after one coarse correction step. This became known in the literature as an optimal coarse space; i.e. better convergence cannot be achieved (note optimal here is not to be understood in the sense of scalable). This approach allowed the authors in [14] to describe a new coarse space for optimized Schwarz methods, and led to the new Spectral Harmonically Enriched Multiscale (SHEM) coarse space [19, 18], which was compared to the new ACMS coarse space in [23]. Like GenEO [31], SHEM, and ACMS were developed to handle convergence problems of the underlying domain decomposition method in the presence of high contrast. Our new coarse space here is designed to handle convergence problems for subdomain aspect ratios which create convergence problems of the NNM. It is based on ideas in the short conference paper [3], and we present a complete formulation and detailed analysis here, including the cross point case. We also present coarse space corrections for problems for which the NNM cannot directly be applied due to the problem of floating subdomains.

The paper is organized as follows: in Section 2, we recall the continuous iterative NNM and give a convergence analysis for one-way decompositions in one, two, and three spatial dimensions. We explain in Section 3 how to construct the coarse space correction in one, two, and three spatial dimensions, and also for problems for which the NNM is not well defined due to floating subdomains, and we give corresponding convergence estimates. We show numerical results illustrating the performance of the new method in Section 4, and present the optimal coarse space in the cross point case at the discrete level in Section 5.

2. Analysis of the one-level NNM.

Let Ω be a bounded domain in $\mathbb{R}^d$, $d = 1, 2$. We consider the partial differential equation

$$\begin{aligned} \mathcal{L}u &= f, \text{ in } \Omega, \\ u &= 0, \text{ on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $\mathcal{L}$ is a second order elliptic operator, and $f \in L^2(\Omega)$. We want to solve the problem defined in (2.1) using the NNM. Let $\Omega_1, \ldots, \Omega_N$ be a one way non-overlapping decomposition of Ω as shown in Figure 1 (for decompositions with cross points see Section 5). Let n_i denote the unit outward normal on $\partial\Omega_i$. The iterative one-level NNM is then given by Algorithm 2.1.

2.1. One-dimensional analysis.

We consider a one dimensional decomposition of Ω into N equally-sized subdomains Ω_i as shown in Figure 2, and study the

Algorithm 2.1 One-level NNM

1. Set $g_{i,j}^0$ to zero or any inexpensive initial guess on the interfaces $\partial\Omega_i \cap \partial\Omega_j$
2. Repeat until convergence
   (a) Solve the Dirichlet followed by the Neumann problems

$$\begin{aligned} \mathcal{L}u_i^n &= f \text{ in } \Omega_i, & \mathcal{L}\psi_i^n &= 0 \text{ in } \Omega_i,\\ u_i^n &= g_{i,j}^n \text{ on } \partial\Omega_i \cap \partial\Omega_j, & \partial_{n_i}\psi_i^n &= \left(\partial_{n_i}u_i^n + \partial_{n_j}u_j^n\right)/2 \text{ on } \partial\Omega_i \cap \partial\Omega_j,\\ u_i^n &= 0 \text{ on } \partial\Omega_i \cap \partial\Omega. & \psi_i^n &= 0 \text{ on } \partial\Omega_i \cap \partial\Omega. \end{aligned}$$

   (b) Update the traces

$$g_{i,j}^{n+1} := g_{i,j}^{n+1} - \frac{1}{2}\left(\psi_i^n + \psi_j^n\right) \text{ on } \partial\Omega_i \cap \partial\Omega_j.$$

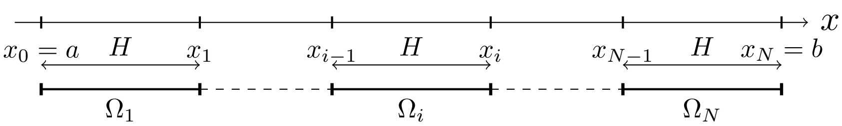

Fig. 2: One-dimensional geometry.

convergence of Algorithm 2.1 for the screened Laplace operator $\mathcal{L} = \eta^2 - \Delta$. By linearity of (2.1), it suffices to set $f = 0$ and study the convergence of $(g_{i,i+1}^n)_{1\le i\le N-1}$ to zero. To simplify the notation, we set here $g_i := g_{i,i+1}$ on the interface $\Gamma_i := \Gamma_{i,i+1} = \partial\Omega_i \cap \partial\Omega_{i+1}$. We first prove a lemma that gives a recurrence relation between the iterates:

LEMMA 2.1. *Let $\boldsymbol{g}^n = \left[g_1^n, g_2^n, \dots, g_{N-1}^n\right]^T \in \mathbb{R}^{N-1}$, then for $N \ge 3$, we have $\boldsymbol{g}^n = T\boldsymbol{g}^{n-1}$, where $T \in \mathbb{R}^{(N-1)\times(N-1)}$ is given by*

$$T := -\frac{1}{4\sinh^2(\eta H)}\begin{bmatrix} 1 & \frac{1}{\cosh(\eta H)} & -1 & 0 & \cdots & \cdots & 0\\ 0 & 2 & 0 & -1 & \ddots & & \vdots\\ -1 & 0 & 2 & 0 & -1 & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & 2 & 0 & -1\\ \vdots & & \ddots & -1 & 0 & 2 & 0\\ 0 & \cdots & \cdots & 0 & -1 & \frac{1}{\cosh(\eta H)} & 1 \end{bmatrix}.$$

Proof. The subdomain solutions are given by

$$u_i^n(x) = g_i^n\frac{\sinh\left(\eta(x - x_{i-1})\right)}{\sinh\left(\eta H\right)} + g_{i-1}^n\frac{\sinh\left(\eta(x_i - x)\right)}{\sinh\left(\eta H\right)},$$

for $i = 1, \dots, N$, where we defined $g_0^n = g_N^n = 0$ for simplicity. Similarly we obtain

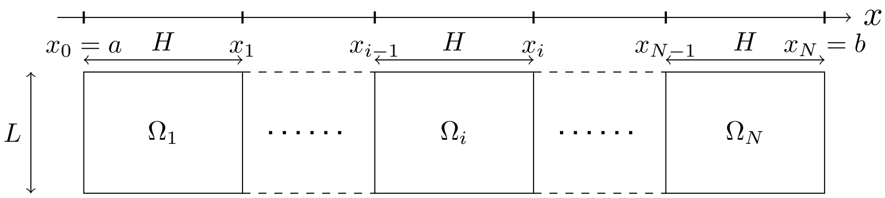

Fig. 3: Two-dimensional geometry.

for the local corrections

$$\psi_i^n = \left(2g_i^n \frac{\cosh(\eta H)}{\sinh(\eta H)} - \frac{g_{i-1}^n}{\sinh(\eta H)} - \frac{g_{i+1}^n}{\sinh(\eta H)}\right)\frac{\cosh(\eta(x-x_{i-1}))}{2\sinh(\eta H)} + \left(2g_{i-1}^n \frac{\cosh(\eta H)}{\sinh(\eta H)} - \frac{g_{i-2}^n}{\sinh(\eta H)} - \frac{g_{i}^n}{\sinh(\eta H)}\right)\frac{\cosh(\eta(x_i-x))}{2\sinh(\eta H)},$$

for $i = 2, \ldots, N-2$, and

$$\psi_1^n = \left(2g_1^n \frac{\cosh(\eta H)}{\sinh H} - \frac{g_2^n}{\sinh(\eta H)}\right)\frac{\sinh(\eta(x-x_0))}{2\cosh(\eta H)},$$
$$\psi_N^n = \left(2g_{N-1}^n \frac{\cosh(\eta H)}{\sinh(\eta H)} - \frac{g_{N-2}^n}{\sinh(\eta H)}\right)\frac{\sinh(\eta(x_N-x))}{2\cosh(\eta H)}.$$

Hence, we get the stated relation. ∎

THEOREM 2.2. *If the width of the subdomains satisfies $H > \frac{\ln(1+\sqrt{2})}{\eta}$, then the one-level NNM for the screened Laplace problem $(\eta^2 - \Delta)u = 0$ in 1D given by Algorithm 2.1 is convergent, and satisfies the convergence estimate*

(2.2) $$\max_{1\le i\le N-1} |g_i^n| \le \frac{1}{\sinh^{2n}(\eta H)} \max_{1\le i\le N-1} |g_i^0|.$$

Proof. By Lemma 2.1, we have that

$$\|T\|_\infty = \max\left\{\frac{1}{\sinh^2(\eta H)}, \frac{1}{2\sinh^2(\eta H)} + \frac{1}{4\sinh^2(\eta H)\cosh(\eta H)}\right\} \le \frac{1}{\sinh^2(\eta H)}.$$

∎

2.2. Two-dimensional analysis. We suppose that the decomposition of the domain Ω is as in Figure 3 and analyze the convergence of Algorithm 2.1 again for $\mathcal{L} = \eta^2 - \Delta$. Since the subdomains are rectangular, the iterates can be expanded in a sine series, i.e.

(2.3) $$u_i^n(x,y) = \sum_{m=1}^{\infty} \hat{u}_i^n(x,m)\sin(\frac{m\pi}{L}y), \quad \psi_i^n(x,y) = \sum_{m=1}^{\infty} \hat{\psi}_i^n(x,m)\sin(\frac{m\pi}{L}y),$$

where $\hat{u}_i^n$ and $\hat{\psi}_i^n$ are the Fourier coefficients of u_i^n and ψ_i^n respectively. As in the one-dimensional case, we obtain a lemma giving the recurrence relation of the Fourier coefficients:

LEMMA 2.3. *Let* $\hat{\boldsymbol{g}}^n(m) = \left[\hat{g}_1^n(m), \hat{g}_2^n(m), \dots, \hat{g}_{N-1}^n(m)\right]^T \in \mathbb{R}^{N-1}$*, then for* $N \geq 3$ *we have* $\hat{\boldsymbol{g}}^n(m) = T_m \hat{\boldsymbol{g}}^{n-1}(m)$*, where* $T_m \in \mathbb{R}^{(N-1)\times(N-1)}$ *is given by*

$$T_m := -\frac{1}{4\sinh^2(k_m H)} \begin{bmatrix} 1 & \frac{1}{\cosh(k_m H)} & -1 & 0 & \cdots & \cdots & 0 \\ 0 & 2 & 0 & -1 & \ddots & & \vdots \\ -1 & 0 & 2 & 0 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 2 & 0 & -1 \\ \vdots & & \ddots & -1 & 0 & 2 & 0 \\ 0 & \cdots & \cdots & 0 & -1 & \frac{1}{\cosh(k_m H)} & 1 \end{bmatrix}$$

where $k_m := \sqrt{\eta^2 + \frac{m^2\pi^2}{L^2}}$.

Proof. For each $m \geq 1$ and $i = 2, \dots, N-1$, $u_i^n(x, m)$ and $\psi_i^n(x, m)$ satisfy

$$\begin{aligned} k_m^2 \hat{u}_i^n - \partial_{xx}\hat{u}_i^n = 0, \qquad & k_m^2 \hat{\psi}_i^n - \partial_{yy}\hat{\psi}_i^n = 0, \\ \hat{u}_i^n(x_{i-1}, m) = \hat{g}_{j-1}^n(m), \quad & \hat{\psi}_i^n(x_{i-1}, m) = (\partial_x \hat{u}_i^n(x_{j-1}, m) - \partial_x \hat{u}_{i-1}^n(x_{j-1}, m))/2, \\ \hat{u}_i^n(x_i, m) = \hat{g}_i^n(m), \qquad & \hat{\psi}_i^n(x_i, m) = (\partial_x \hat{u}_i^n(x_i, m) - \partial_x \hat{u}_{i+1}^n(x_i, m))/2. \end{aligned}$$

The solution of the Dirichlet problems on the interior subdomains are thus

$$\hat{u}_i^n(x, m) = \hat{g}_i^n(m) \frac{\sinh(k_m(x - x_{i-1}))}{\sinh(k_m H)} + \hat{g}_{i-1}^n(m) \frac{\sinh(k_m(x_i - x))}{\sinh(k_m H)},$$

for $i = 1, \dots, N$, where we defined $\hat{g}_0^n = \hat{g}_N^n = 0$ to include the solution to boundary subdomains. Similarly for the Neumann problems on the interior subdomains, we obtain

$$\begin{aligned} \hat{\psi}_i^n(x, m) = & \left(2\,\hat{g}_i^n(m)\frac{\cosh(k_m H)}{\sinh(k_m H)} - \frac{\hat{g}_{i-1}^n(m)}{\sinh(k_m H)} - \frac{\hat{g}_{i+1}^n(m)}{\sinh(k_m H)}\right) \frac{\cosh(k_m(x - x_{j-1}))}{2\sinh(k_m H)} \\ & + \left(2\,\hat{g}_{i-1}^n(m)\frac{\cosh(k_m H)}{\sinh(k_m H)} - \frac{\hat{g}_{i-2}^n(m)}{\sinh(k_m H)} - \frac{\hat{g}_i^n(m)}{\sinh(k_m H)}\right) \frac{\cosh(k_m(x_i - x))}{2\sinh(k_m H)}, \end{aligned}$$

and for the first and last subdomains, we find

$$\begin{aligned} \hat{\psi}_1^n(x, m) &= \left(2\,\hat{g}_1^n(m)\frac{\cosh(k_m H)}{\sinh(k_m H)} - \frac{\hat{g}_2^n(m)}{\sinh(k_m H)}\right) \frac{\sinh(k_m(x - x_0))}{2\cosh(k_m H)}, \\ \hat{\psi}_N^n(x, m) &= \left(2\,\hat{g}_{N-1}^n(m)\frac{\cosh(k_m H)}{\sinh(k_m H)} - \frac{\hat{g}_{N-2}^n(m)}{\sinh(k_m H)}\right) \frac{\sinh(k_m(x_N - x))}{2\cosh(k_m H)}, \end{aligned}$$

from which we obtain the stated formula. ∎

THEOREM 2.4. *If the width of the subdomains satisfies* $H > \frac{\ln(1+\sqrt{2})}{k_1}$, $k_1 = \sqrt{\eta^2 + \frac{\pi^2}{L^2}}$ *with* L *the height of the subdomains, then the one-level NNM for the screened Laplace problem* $(\eta^2 - \Delta)u = 0$ *in 2D given by Algorithm 2.1 is convergent and satisfies the* L^2 *convergence estimate*

(2.4)
$$\max_{1 \leq i \leq N-1} \|g_i^n\|_2 \leq \frac{1}{\sinh^{2n}(k_1 H)} \max_{1 \leq i \leq N-1} \|g_i^0\|_2.$$

Proof. We define the sequences $\Lambda_i^n = \{|\hat{g}_i^n(m)|\}_{m\geq 1}$. By Lemma 2.3, we have

$$\hat{g}_i^{n+1}(m) = \frac{1}{\sinh{(k_m H)}^2}\left(\frac{1}{4}\hat{g}_{i-2}^n(m) - \frac{1}{2}\hat{g}_i^n(m) + \frac{1}{4}\hat{g}_{i+2}^n(m)\right)$$

for each $m \geq 1$. Using then Parseval's identity $\|\Lambda_i^n\|_2^2 = \frac{L}{2}\|g_i^n\|_2^2$, we have for $i = 3, \ldots, N-3$

$$\begin{aligned}
\|\Lambda_i^{n+1}\|_2 &\leq \frac{1}{\sinh^2{(k_1 H)}}\left(\frac{1}{4}\|\Lambda_{i-2}^n + \frac{1}{2}\Lambda_i^n + \frac{1}{4}\Lambda_{i+2}^n\|_2\right),\\
&\leq \frac{1}{\sinh^2{(k_1 H)}}\left(\frac{1}{4}\|\Lambda_{i-2}^n\|_2 + \frac{1}{2}\|\Lambda_i^n\|_2 + \frac{1}{4}\|\Lambda_{i+2}^n\|_2\right),\\
&\leq \frac{1}{\sinh^2{(k_1 H)}}\max_{1\leq i\leq N-1}\|\Lambda_i^n\|_2,
\end{aligned}$$

where we used the triangle inequality, and the monotonicity of $m \mapsto 1/\sinh^2{(k_m H)}$. Similarly one can show that the same bound also holds for the remaining subdomains $i = 1, 2, N-2, N-1$, and hence we get the stated result. ∎

From our analysis, we see that the one-level iterative NNMs only converge provided the subdomain width H is large enough (compared to the height L in 2D). If this is not the case, the iterative method diverges, but could still be used as a preconditioner for a Krylov method, see the numerical experiments in Section 4. Note also that a large η screening parameter helps convergence.

2.3. Three-dimensional analysis. In this part, we suppose that the domain $\Omega = [a, b] \times [0, L] \times [0, L']$ is decomposed into N non-overlapping subdomains $\Omega_i = [x_{i-1}, x_i] \times [0, L] \times [0, L']$, where $i = 1, \ldots, N$. Since the subdomains represent three-dimensional bricks, we can expand the iterates u_i^n and ψ_i^n in a double sine series, i.e.,

$$\begin{aligned}
u_i^n(x,y,z) &= \sum_{m,m'\geq 1}^{\infty}\hat{u}_i^n(x,m,m')\sin(\frac{m\pi}{L}y)\sin(\frac{m'\pi}{L'}z),\\
\psi_i^n(x,y,z) &= \sum_{m,m'\geq 1}^{\infty}\hat{\psi}_i^n(x,m,m')\sin(\frac{m\pi}{L}y)\sin(\frac{m'\pi}{L'}z),
\end{aligned} \tag{2.5}$$

where $\hat{u}_i^n$ and $\hat{\psi}_i^n$ are the Fourier coefficients of u_i^n and ψ_i^n. Following the analysis in the one- and two-dimensional case, we obtain:

LEMMA 2.5. *Let* $\hat{\boldsymbol{g}}^n(m,m') = \left[\hat{g}_1^n(m,m'), \hat{g}_2^n(m,m'), \ldots, \hat{g}_{N-1}^n(m,m')\right]^T \in \mathbb{R}^{N-1}$, *then for* $N \geq 3$ *we have* $\hat{\boldsymbol{g}}^n(m,m') = T_{m,m'}\hat{\boldsymbol{g}}^{n-1}(m,m')$, *where* $T_{m,m'} \in \mathbb{R}^{(N-1)\times(N-1)}$

is given by

$$T_{m,m'} := -\frac{1}{4\sinh^2(k_{m,m'}H)}\begin{bmatrix} 1 & \frac{1}{\cosh(k_{m,m'}H)} & -1 & 0 & \cdots & \cdots & 0 \\ 0 & 2 & 0 & -1 & \ddots & & \vdots \\ -1 & 0 & 2 & 0 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 2 & 0 & -1 \\ \vdots & & \ddots & -1 & 0 & 2 & 0 \\ 0 & \cdots & \cdots & 0 & -1 & \frac{1}{\cosh(k_{m.m'}H)} & 1 \end{bmatrix},$$

with $k_{m,m'} := \sqrt{\eta^2 + \frac{m^2\pi^2}{L^2} + \frac{m'^2\pi^2}{L'^2}}$.

Proof. The proof of Lemma 2.5 is similar to the one of Lemma 2.3. In fact, it suffices to observe that the Fourier coefficients $\hat{u}_i^n(x,m,m')$ and $\hat{\psi}_i^n(x,m,m')$ satisfy

$$\begin{aligned} k_{m,m'}^2\hat{u}_i^n - \partial_{xx}\hat{u}_i^n &= 0, & k_{m,m'}^2\hat{\psi}_i^n - \partial_{yy}\hat{\psi}_i^n &= 0, \\ \hat{u}_i^n(x_{i-1},m,m') &= \hat{g}_{j-1}^n(m,m'), & \hat{\psi}_i^n(x_{i-1},m,m') &= (\partial_x\hat{u}_i^n(x_{j-1},m,m') - \partial_x\hat{u}_{i-1}^n(x_{j-1},m,m'))/2, \\ \hat{u}_i^n(x_i,m,m') &= \hat{g}_i^n(m,m'), & \hat{\psi}_i^n(x_i,m,m') &= (\partial_x\hat{u}_i^n(x_i,m),m' - \partial_x\hat{u}_{i+1}^n(x_i,m,m'))/2, \end{aligned}$$

which are similar to the ones in Lemma 2.3, except that $k_{m,m'}$ includes the frequencies of the Fourier expansion of both the y- and z-direction. Using then the same arguments of Lemma 2.3, we obtain the desired result. ∎

THEOREM 2.6. *If the width of the subdomains in 3D satisfies* $H > \frac{\ln(1+\sqrt{2})}{k_{1,1}}$, $k_{1,1} = \sqrt{\eta^2 + \frac{\pi^2}{L^2} + \frac{\pi^2}{(L')^2}}$ *with* L, L' *the dimensions of the subdomains in* $y-$ *and* $z-$*direction, then the one-level NNM for the screened Laplace problem* $(\eta^2 - \Delta)u = 0$ *in 3D given by Algorithm 2.1 is convergent and satisfies the* L^2 *convergence estimate*

$$\max_{1\le i\le N-1}\|g_i^n\|_2 \le \frac{1}{\sinh^{2n}(k_{1,1}H)}\max_{1\le i\le N-1}\|g_i^0\|_2. \tag{2.6}$$

Proof. Similar to the proof of Theorem 2.4, it suffices to define the sequence $\Lambda_i^n = \{|\hat{g}_i^n(m,m')|\}_{m,m'\ge 1}$. Then, using Lemma 2.5, we obtain

$$\hat{g}_i^{n+1}(m,m') = \frac{1}{\sinh(k_{m,m'}H)^2}\left(\frac{1}{4}\hat{g}_{i-2}^n(m,m') - \frac{1}{2}\hat{g}_i^n(m,m') + \frac{1}{4}\hat{g}_{i+2}^n(m,m')\right)$$

for each $m,m' \ge 1$. Noting that the Parseval identity still holds in 3D, and is given by $\|\Lambda_i^n\|_2^2 = \frac{LL'}{4}\|g_i^n\|_2^2$, we obtain by the triangle inequality that

$$\begin{aligned} \|\Lambda_i^{n+1}\|_2 &\le \frac{1}{\sinh^2(k_{1,1}H)}\left(\frac{1}{4}\|\Lambda_{i-2}^n + \frac{1}{2}\Lambda_i^n + \frac{1}{4}\Lambda_{i+2}^n\|_2\right), \\ &\le \frac{1}{\sinh^2(k_{1,1}H)}\left(\frac{1}{4}\|\Lambda_{i-2}^n\|_2 + \frac{1}{2}\|\Lambda_i^n\|_2 + \frac{1}{4}\|\Lambda_{i+2}^n\|_2\right), \\ &\le \frac{1}{\sinh^2(k_{1,1}H)}\max_{1\le i\le N-1}\|\Lambda_i^n\|_2. \end{aligned}$$

The same bound holds for the subdomains $i = 1,2,N-2,N-1$. This shows that Algorithm 2.1 converges in 3D if the condition $k_{1,1}H > \ln(1+\sqrt{2})$ is satisfied. ∎

3. Analysis of the two-level NNM. In this section, we explain how to add a coarse space correction to Algorithm 2.1. While in the classical approach a coarse space is added to make the method scalable, here we ask the coarse space to do more, namely make the method convergent for all subdomain width H given a subdomain height L. This is similar to the new coarse spaces developed over the last decade like GenEO, ACMS, and SHEM, which were also designed to deal with convergence or robustness problems of the underlying domain decomposition methods. The key idea for the new coarse correction comes from the observation that at convergence of the NNM, the intermediate steps $u_i^{n+\frac{1}{2}} := u_i^n - \psi_i^n$ are continuous in the Dirichlet trace, but during the iteration before convergence they are not, which means that an efficient correction should be able to reduce the jump between neighboring $u_i^{n+\frac{1}{2}}$ as much as possible in order to improve convergence. A good coarse space should thus contain enough harmonic functions[1] that are discontinuous across subdomain boundaries. Thus a complete coarse space X can be defined as

$$X := \left\{ v \in L^2(\Omega) : v_i := v_{|\Omega_i} \in H^1(\Omega_i),\ \mathcal{L}v_i = 0 \text{ in } \Omega_i,\ v_i = 0 \text{ on } \partial\Omega_i \cap \partial\Omega \right\}. \quad (3.1)$$

Moreover, the size of the coarse space X defined in (3.1) can be reduced. In fact, since the iterates $u_i^{n+\frac{1}{2}}$ are already continuous in the normal derivative, it is sufficient that the optimal coarse space is also continuous in the normal derivative across subdomain interfaces. The optimal coarse space is thus given taking all these functions,

$$X_d := \left\{ v \in X : v_i := v_{|\Omega_i},\ \frac{\partial v_i}{\partial n_i} + \frac{\partial v_j}{\partial n_j} = 0 \text{ on } \partial\Omega_i \cap \partial\Omega_j \right\}. \quad (3.2)$$

Now, since the coarse correction must reduce the Dirichlet jump of the iterates $u_i^{n+\frac{1}{2}}$, we choose the correction $U^n \in X_d$ such that it satisfies

$$U^n := \operatorname*{arg\,min}_{v \in \tilde{X}_d \subset X_d} q(u^{n+\frac{1}{2}} + v), \quad (3.3)$$

where $\tilde{X}_d$ is a finite-dimensional subspace of the optimal coarse space X_d, and the quadratic functional q is defined by

$$\begin{aligned} q : X_d &\mapsto \mathbb{R}_+ \\ u &\mapsto \sum_{\partial\Omega_i \cap \partial\Omega_j \neq \emptyset} \int_{\partial\Omega_i \cap \partial\Omega_j} |u_i - u_j|^2 \, \mathrm{d}s. \end{aligned} \quad (3.4)$$

This leads at the continuous level to the new two-level NNM given in Algorithm 3.1.

3.1. One-dimensional analysis. We now analyze the convergence of the two-level NNM for the screened Laplace operator $\mathcal{L} = \eta^2 - \Delta$ and the Laplace operator $\mathcal{L} = -\Delta$ in one dimension. The Laplace operator needs a separate analysis, since the Neumann correction problems are then not well posed for interior subdomains.

3.1.1. Screened Laplace operator $\mathcal{L} = \eta^2 - \Delta$. Since the optimal coarse space X_d in 1D is finite dimensional, it is a practical choice for $\tilde{X}_d$ and makes the Algorithm 3.1 for the problem $\mathcal{L} = \eta^2 - \Delta$ optimal in the sense that it becomes a direct solver:

[1] Here harmonic means satisfying the homogeneous equation $\mathcal{L}u = 0$.

Algorithm 3.1 Two-level NNM

1. Set $g^0_{i,j}$ to zero or any inexpensive initial guess on the interfaces $\partial\Omega_i \cap \partial\Omega_j$
2. Repeat until convergence
   (a) Solve the Dirichlet followed by the Neumann problems

$$\begin{aligned} \mathcal{L}u_i^n &= f \ \text{ in } \Omega_i, & \mathcal{L}\psi_i^n &= 0 \ \text{ in } \Omega_i,\\ u_i^n &= g^n_{i,j} \text{ on } \partial\Omega_i \cap \partial\Omega_j, & \partial_{n_i}\psi_i^n &= \left(\partial_{n_i}u_i^n + \partial_{n_j}u_j^n\right)/2 \text{ on } \partial\Omega_i \cap \partial\Omega_j,\\ u_i^n &= 0 \text{ on } \partial\Omega_i \cap \partial\Omega. & \psi_i^n &= 0 \text{ on } \partial\Omega_i \cap \partial\Omega. \end{aligned}$$

   (b) Set

$$u_i^{n+\frac{1}{2}} := u_i^n - \psi_i^n \text{ in } \Omega_i.$$

   (c) Set

$$\tilde{u}^{n+1} := u^{n+\frac{1}{2}} + U^n$$

   where U^n is as in (3.3).

   (d) Update the traces

$$g_{ij}^{n+1} := \frac{1}{2}\left(\tilde{u}_i^{n+1} + \tilde{u}_j^{n+1}\right) \text{ on } \partial\Omega_i \cap \partial\Omega_j.$$

THEOREM 3.1. *The optimal coarse space X_d is finite dimensional in 1D, and with $\tilde{X}_d := X_d$, Algorithm 3.1 in 1D for $\mathcal{L} = \eta^2 - \Delta$ converges after one iteration.*

Proof. Since the Dirichlet traces at the interfaces are just numbers, the optimal coarse space X_d is finite dimensional, and thus $\tilde{X}_d := X_d$ is a practical choice. Let u_i^0 be the solutions after solving with the initial guess. The function defined by $u_i^{\frac{1}{2}} := u_i^0 - \psi_i^0$ is in X_d. Now in order to compute the correction U^0 in (3.3), we proceed as follows: since the vector space X_d is of finite dimension in 1D, $(\dim(X_d) = N-1)$, it admits a finite basis $\phi_1, \phi_2, \ldots, \phi_{N-1}$. We construct the coarse basis by choosing ϕ_i on each interface $x = x_i$ such that

$$\phi_i := \begin{cases} -\frac{\cosh(\eta(x - x_{i-1}))}{\sinh(\eta H)} & \text{if } x \in (x_{i-1}, x_i),\\ \frac{\cosh(\eta(x_{i+1} - x))}{\sinh(\eta H)} & \text{if } x \in (x_i, x_{i+1}), \end{cases}$$

for $i = 2, \ldots, N-2$, and

$$\phi_1 := \begin{cases} -\frac{\sinh(\eta(x - x_0))}{\cosh(\eta H)} & \text{if } x \in (x_0, x_1),\\ \frac{\cosh(\eta(x_2 - x))}{\sinh(\eta H)} & \text{if } x \in (x_1, x_2), \end{cases}$$

$$\phi_{N-1} := \begin{cases} \frac{\cosh(\eta(x - x_{N-2}))}{\sinh(\eta H)} & \text{if } x \in (x_{N-2}, x_{N-1}),\\ -\frac{\sinh(\eta(x_N - x))}{\cosh(\eta H)} & \text{if } x \in (x_{N-1}, x_N). \end{cases}$$

Since $U^0 \in X_d$, we have that $U^0 = \sum_{i=1}^{N-1} \alpha_i \phi_i$ for some coefficients α_i. Define $u_i^+ := u_{i+1}^{\frac{1}{2}}(x_i)$ and $u_i^- := u_i^{\frac{1}{2}}(x_i)$. The minimization problem in (3.3) reduces to finding

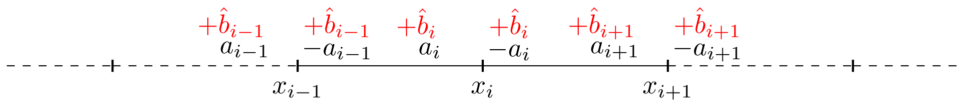

Fig. 4: Auxiliary variables $\hat{b}_i$ in 1D.

$\boldsymbol{\alpha} := [\alpha_1, \dots, \alpha_{N-1}]^\top$ such that

(3.5)
$$\boldsymbol{\alpha} := \arg\min_{\boldsymbol{\alpha}} \sum_{i=1}^{N-1} \left| u_i^+ - u_i^- + \alpha_{i+1}\phi_{i+1}(x_i^+) + \alpha_i(\phi_i(x_i^+) - \phi_i(x_i^-)) - \alpha_{i-1}\phi_{i-1}(x_i^-) \right|^2,$$

which is equivalent to

$$\boldsymbol{\alpha} := \arg\min_{\boldsymbol{\alpha}} \|A\boldsymbol{\alpha} - \boldsymbol{u}\|_2^2, \tag{3.6}$$

where

$$A := \begin{bmatrix} \frac{c}{s} + \frac{s}{c} & -\frac{1}{s} & & & & & \\ -\frac{1}{s} & \frac{2c}{s} & -\frac{1}{s} & & & & \\ & -\frac{1}{s} & \frac{2c}{s} & -\frac{1}{s} & & & \\ & & -\frac{1}{s} & \frac{2c}{s} & -\frac{1}{s} & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -\frac{1}{s} & \frac{2c}{s} & -\frac{1}{s} \\ & & & & & -\frac{1}{s} & \frac{c}{s} + \frac{s}{c} \end{bmatrix}, \quad \boldsymbol{u} := \begin{bmatrix} u_1^- - u_1^+ \\ u_2^- - u_2^+ \\ \vdots \\ u_{N-1}^- - u_{N-1}^+ \end{bmatrix},$$

and $s := \sinh(\eta H)$ and $c := \cosh(\eta H)$. The matrix A is invertible, since it is diagonally dominant, $\frac{c}{s} + \frac{s}{c} > \frac{1}{s}$ and $\frac{2c}{s} > \frac{2}{s}$. Thus the unique minimizer of (3.6) is given by $\boldsymbol{\alpha} = A^{-1}\boldsymbol{u}$. Hence $\widetilde{u}^0$ defined by $\widetilde{u}^0 := u^{\frac{1}{2}} + U^0$ is in X_d and satisfies $q(\widetilde{u}^0) = 0$, and hence is the exact monodomain solution. ∎

3.1.2. Laplace operator $\mathcal{L} = -\Delta$. Note that neither Algorithm 2.1 nor Algorithm 3.1 are well defined in the Laplace case $\mathcal{L} = -\Delta$, because the Neumann problems determine the corrections ψ_i^n in the interior subdomains $i = 2, \dots, N-1$ only up to a constant, and in order to have a well-defined algorithm, the local corrections ψ_i^n need to satisfy the consistency condition

$$\partial_x \psi_i^n(x_i) - \partial_x \psi_i^n(x_{i-1}) = 0, \text{ for } i = 2, \dots, N-1. \tag{3.7}$$

The idea then is to introduce new auxiliary variables $\hat{b}_i$, $i = 1, \dots, N-1$, as shown in Figure 4, such that they satisfy the equations

$$\hat{b}_i - \hat{b}_{i-1} = a_i + a_{i-1}, \tag{3.8}$$

where $a_i := \frac{1}{2}(\partial_x u_i^n(x_i) - \partial_x u_{i+1}^n(x_i))$, $i = 1, \dots, N-1$. The relations in (3.8) form the underdetermined linear system

$$L\hat{\boldsymbol{b}} = \hat{\boldsymbol{a}}, \tag{3.9}$$

where $L \in \mathbb{R}^{N-2 \times N-1}$, and $\hat{\boldsymbol{b}} \in \mathbb{R}^{N-1}$, $\hat{\boldsymbol{a}} \in \mathbb{R}^{N-2}$ are given by

$$L := \begin{bmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{bmatrix}, \quad \hat{\boldsymbol{b}} := \begin{bmatrix} \hat{b}_1 \\ \vdots \\ \hat{b}_{N-1} \end{bmatrix}, \quad \hat{\boldsymbol{a}} := \begin{bmatrix} a_2 + a_1 \\ \vdots \\ a_{N-1} + a_{N-2} \end{bmatrix}. \tag{3.10}$$

LEMMA 3.2 (Properties of the matrix L).

1. *L is surjective i.e.* ($\operatorname{rank}(L) = N-2$).
2. $\ker(L) = \mathbb{R}\left[1, 1, \ldots, 1\right]^T$.
3. *Its pseudoinverse* $L^\dagger \in \mathbb{R}^{N-1\times N-2}$ *is given by*

$$L^\dagger = \frac{1}{N-1}\begin{bmatrix} N-2 & N-3 & \cdots & 1 \\ -1 & N-3 & \cdots & 1 \\ -1 & -2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -2 & \cdots & -(N-2) \end{bmatrix}.$$

Proof. The two first properties are straightforward. In order to compute the explicit formula of $L^\dagger$, we proceed as follows: The action of the pseudo-inverse of L on $\boldsymbol{x} = [x_1, \ldots, x_{N-2}]^\top$ is given by $L^\dagger \boldsymbol{x} = \boldsymbol{v_1} + \boldsymbol{v_2}$ where $L\boldsymbol{v_1} = \boldsymbol{x}$, $L\boldsymbol{v_2} = 0$, and $\|L^\dagger \boldsymbol{x}\|_2$ is minimal. Furthermore, since $\operatorname{rank} L = N-2$, we know by the rank-nullity theorem that the null space of L is spanned by $[1, 1, \ldots, 1, 1]^\top$. Hence there exists $\alpha \in \mathbb{R}$ such that $\boldsymbol{v_2} = \alpha[1, 1 \ldots, 1, 1]^\top$. Moreover, it is straightforward to verify that the vector

$$\boldsymbol{v_1} = \begin{bmatrix} x_1 + \ldots + x_{N-2} \\ x_2 + \ldots + x_{N-2} \\ \vdots \\ x_{N-2} \\ 0 \end{bmatrix}$$

is solution of $L\boldsymbol{v_1} = \boldsymbol{x}$. Therefore, we have

$$L^\dagger \boldsymbol{x} = \begin{bmatrix} x_1 + \ldots + x_{N-2} \\ x_2 + \ldots + x_{N-2} \\ \vdots \\ x_{N-2} \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^{N-1},$$

where α is such that $\|L^\dagger \boldsymbol{x}\|_2$ is minimal. Hence,

$$\alpha = -\frac{1}{N-1}\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 + \ldots + x_{N-2} \\ x_2 + \ldots + x_{N-2} \\ \vdots \\ x_{N-2} \\ 0 \end{bmatrix},$$

and we deduce that

$$
\begin{aligned}
L^\dagger \boldsymbol{x} &= \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & \dots & 1 & 1 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \dots & \dots & 0 & 0 \end{bmatrix} \boldsymbol{x} - \frac{1}{N-1} \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 1 \\ \vdots & \ddots & \ddots & 1 & 1 \\ 1 & \dots & \dots & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & \dots & 1 & 1 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \dots & \dots 0 & 0 & 0 \end{bmatrix} \boldsymbol{x}, \\
&= \frac{1}{N-1} \begin{bmatrix} N-2 & -1 & \dots & -1 & -1 \\ -1 & N-2 & \dots & -1 & -1 \\ \vdots & \ddots & \ddots & N-2 & -1 \\ -1 & \dots & \dots & -1 & N-2 \end{bmatrix} \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & \dots & 1 & 1 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \dots & \dots & 0 & 0 \end{bmatrix} \boldsymbol{x}, \\
&= \frac{1}{N-1} \begin{bmatrix} N-2 & N-3 & \cdots & 1 \\ -1 & N-3 & \cdots & 1 \\ -1 & -2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -2 & \cdots & -(N-2) \end{bmatrix} \boldsymbol{x},
\end{aligned}
$$

which concludes the proof. ∎

The addition of the auxiliary variables $\hat{b}_i$ allows ψ_i^n to be defined but up to a constant. Moreover, the iterates $u_i^{n+\frac{1}{2}}$ have continuous normal derivatives on the interfaces, but discontinuous Dirichlet traces. Hence, the constants on the interior of the subdomains should be chosen such that they minimize the Dirichlet jump on the interfaces. Therefore, we choose $\hat{\boldsymbol{c}} = \left[0, \hat{c}_2, \dots, \hat{c}_{N-1}, 0\right]^T$ such that

$$
\hat{\boldsymbol{c}} := \arg\min_{\boldsymbol{c}} \sum_{i=1}^{N-1} \left| (u_{i+1}^{n+\frac{1}{2}} + c_{i+1}) - (u_i^{n+\frac{1}{2}} + c_i) \right|^2. \tag{3.11}
$$

The NNM for $\mathcal{L} = -\Delta$ in 1D is then given in Algorithm 3.2.

Algorithm 3.2 NNM for $\mathcal{L} = -\Delta$ in 1D

1. Set g_i^0 to zero or any inexpensive initial guess at the interfaces Γ_i
2. Repeat until convergence
   (a) Solve the Dirichlet problems
$$\eta u_i^n - \partial_{xx} u_i^n = f, \quad \text{in } \Omega_i,$$
$$u_i^n(x_{i-1}) = g_{i-1}^n, \quad u_i^n(x_i) = g_i^n.$$
   (b) Solve the Neumann problems
$$-\partial_{xx}\psi_i^n = 0 \quad \text{in } \Omega_i,$$
$$\partial_x \psi_i^n(x_{i-1}) = -a_{i-1} + \hat{b}_{i-1},\ \partial_x \psi_i^n(x_i) = a_i + \hat{b}_i,\ \fint_{\Omega_i} \psi_i^n = 0,$$
   where the $\hat{b}_i$ are defined in (3.9).
   (c) Set
$$u_i^{n+1/2} := u_i^n - \psi_i^n.$$
   (d) Set
$$\tilde{u}_i^{n+\frac{1}{2}} := u_i^{n+\frac{1}{2}} + \hat{c}_i,$$
   where the $\hat{c}_i$ are defined in (3.11).
   (e) Set
$$g_i^{n+1} := (\tilde{u}_{i+1}^{n+\frac{1}{2}}(x_i) + \tilde{u}_i^{n+\frac{1}{2}}(x_i))/2.$$

LEMMA 3.3. *Let $\boldsymbol{g}^n = \left[g_1^n, g_2^n, \ldots, g_{N-1}^n\right]^T \in \mathbb{R}^{N-1}$, then for $N \geq 2$, we have $\boldsymbol{g}^n = \widetilde{T}\boldsymbol{g}^{n-1}$, where $\widetilde{T} \in \mathbb{R}^{(N-1)\times(N-1)}$ is given by*
$$\widetilde{T} := -\frac{1}{4}\begin{bmatrix} \frac{N-2}{(N-1)^2} & 0 & \ldots & 0 & -\frac{N-2}{(N-1)^2} \\ \frac{N-4}{(N-1)^2} & 0 & \ldots & 0 & -\frac{N-4}{(N-1)^2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\frac{N-4}{(N-1)^2} & 0 & \ldots & 0 & \frac{N-4}{(N-1)^2} \\ -\frac{N-2}{(N-1)^2} & 0 & \ldots & 0 & \frac{N-2}{(N-1)^2} \end{bmatrix}.$$
Moreover, we have for all $N \geq 2$
$$\rho(\widetilde{T}) = \|\widetilde{T}\|_\infty = \frac{N-2}{2(N-1)^2} < 1.$$

Proof. The subdomain solutions in the Laplace case are
$$u_i^n(x) = g_{i-1}^n + \frac{g_i^n - g_{i-1}^n}{H}(x - x_{i-1}) \text{ for } i = 1, \ldots, N,$$
where we set $g_0^n := g_N^n := 0$ for simplicity. Then, the a_i are given by
$$a_i := \frac{2g_i^n - g_{i-1}^n - g_{i+1}^n}{2H}, \quad \text{for } i = 1, \ldots, N-1.$$
It thus follows that $\hat{\boldsymbol{b}}$ is given by
$$\hat{\boldsymbol{b}} := L^\dagger D\boldsymbol{g},$$

where

$$D := \frac{1}{2H}\begin{bmatrix} 1 & 1 & -1 & 0 & \dots & \dots & 0 \\ -1 & 1 & 1 & -1 & 0 & \dots & 0 \\ 0 & -1 & 1 & 1 & -1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 & 1 & 1 & -1 \\ 0 & \dots & \dots & 0 & -1 & 1 & 1 \end{bmatrix}.$$

Using Lemma 3.2, we have

$$L^\dagger D = \frac{1}{2H}\begin{bmatrix} \frac{1}{N-1} & 1 & 0 & 0 & \dots & -\frac{1}{N-1} \\ -\frac{N-2}{N-1} & 0 & 1 & 0 & \dots & \vdots \\ \frac{1}{N-1} & -1 & 0 & 1 & \dots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & -\frac{1}{N-1} \\ \frac{1}{N-1} & \vdots & \ddots & -1 & 0 & \frac{N-2}{N-1} \\ \frac{1}{N-1} & \dots & \dots & 0 & -1 & -\frac{1}{N-1} \end{bmatrix},$$

and hence

$$\hat{b}_i := \frac{1}{2H}\left(\tfrac{1}{N-1}g_1^n + g_{i+1}^n - g_{i-1}^n - \tfrac{1}{N-1}g_{N-1}^n\right).$$

The formula for the local corrections ψ_i^n therefore becomes

$$\psi_i^n(x) = \frac{1}{2H}\left(\tfrac{1}{N-1}g_1^n + 2g_i^n - 2g_{i-1}^n - \tfrac{1}{N-1}g_{N-1}^n\right)\left(x - \frac{x_i + x_{i-1}}{2}\right),\ i = 2, \dots, N-1,$$

and for the first and last subdomains

$$\begin{aligned} \psi_1^n(x) &= \frac{1}{2H}\left(\tfrac{2N-1}{N-1}g_1^n - \tfrac{1}{N-1}g_{N-1}^n\right)(x - x_0), \\ \psi_N^n(x) &= \frac{1}{2H}\left(-\tfrac{1}{N-1}g_1^n + \tfrac{2N-1}{N-1}g_{N-1}^n\right)(x_N - x). \end{aligned}$$

We then have

$$\begin{aligned} u_i^{n+\frac{1}{2}}(x_i) &= \frac{1}{4}\left(-\tfrac{1}{N-1}g_1^n + 2g_i^n + 2g_{i-1}^n + \tfrac{1}{N-1}g_{N-1}^n\right), \\ u_i^{n+\frac{1}{2}}(x_{i-1}) &= \frac{1}{4}\left(\tfrac{1}{N-1}g_1^n + 2g_i^n + 2g_{i-1}^n - \tfrac{1}{N-1}g_{N-1}^n\right), \end{aligned}$$

and for the first and last subdomains

$$\begin{aligned} u_1^{n+\frac{1}{2}}(x_1) &= \frac{1}{2}\left(-\tfrac{1}{N-1}g_1^n + \tfrac{1}{N-1}g_{N-1}^n\right), \\ u_N^{n+\frac{1}{2}}(x_{N-1}) &= \frac{1}{2}\left(\tfrac{1}{N-1}g_1^n - \tfrac{1}{N-1}g_{N-1}^n\right), \end{aligned}$$

which gives

$$u_{i+1}^{n+\frac{1}{2}}(x_i) - u_i^{n+\frac{1}{2}}(x_i) = \frac{1}{2}\left(\tfrac{1}{N-1}g_1^n + g_{i+1}^n - g_{i-1}^n - \tfrac{1}{N-1}g_{N-1}^n\right),\ i = 2, \dots, N-2,$$

and for the first and last subdomains, we have

$$u_2^{n+\frac{1}{2}}(x_1) - u_1^{n+\frac{1}{2}}(x_1) = \frac{1}{4}\left(\tfrac{2N+1}{N-1}g_1^n + 2g_2^n - \tfrac{3}{N-1}g_{N-1}^n\right),$$
$$u_N^{n+\frac{1}{2}}(x_{N-1}) - u_{N-1}^{n+\frac{1}{2}}(x_{N-1}) = \frac{1}{4}\left(\tfrac{3}{N-1}g_1^n - 2g_{N-2}^n - \tfrac{2N+1}{N-1}g_{N-1}^n\right).$$

Since $\boldsymbol{c}$ is given by

$$c_i = -\frac{N-i}{N-1}\sum_{k=1}^{i-1}\left(u_{k+1}^{n+\frac{1}{2}}(x_k) - u_k^{n+\frac{1}{2}}(x_k)\right) + \frac{i-1}{N-1}\sum_{k=i}^{N-1}\left(u_{k+1}^{n+\frac{1}{2}}(x_k) - u_k^{n+\frac{1}{2}}(x_k)\right),$$

and

$$\begin{aligned}\sum_{k=1}^{i-1}\left(u_{k+1}^{n+\frac{1}{2}}(x_k) - u_k^{n+\frac{1}{2}}(x_k)\right) &= u_2^{n+\frac{1}{2}}(x_1) - u_1^{n+\frac{1}{2}}(x_1) + \sum_{k=2}^{i-1}\left(u_{k+1}^{n+\frac{1}{2}}(x_k) - u_k^{n+\frac{1}{2}}(x_k)\right)\\
&= \frac{1}{4}\left(\tfrac{2N+1}{N-1}g_1^n + 2g_2^n - \tfrac{3}{N-1}g_{N-1}^n\right)\\
&+ \frac{1}{2}\sum_{k=2}^{i-1}\left(\tfrac{1}{N-1}g_1^n + g_{k+1}^n - g_{k-1}^n - \tfrac{1}{N-1}g_{N-1}^n\right)\\
&= \frac{1}{4}\left(\tfrac{2N+1}{N-1}g_1^n + 2g_2^n - \tfrac{3}{N-1}g_{N-1}^n\right)\\
&+ \frac{1}{2}\left(\tfrac{i-2}{N-1}g_1^n + g_i^n + g_{i-1}^n - g_2^n - g_1^n - \tfrac{i-2}{N-1}g_{N-1}^n\right)\\
&= \frac{1}{4}\left(\tfrac{2i-1}{N-1}g_1^n + 2g_i^n + 2g_{i-1}^n - \tfrac{2i-1}{N-1}g_{N-1}^n\right),\end{aligned}$$

and similarly,

$$\begin{aligned}\sum_{k=i}^{N-1}\left(u_{k+1}^{n+\frac{1}{2}}(x_k) - u_k^{n+\frac{1}{2}}(x_k)\right) &= \sum_{k=i}^{N-2}\left(u_{k+1}^{n+\frac{1}{2}}(x_k) - u_k^{n+\frac{1}{2}}(x_k)\right) + u_N^{n+\frac{1}{2}}(x_{N-1}) - u_{N-1}^{n+\frac{1}{2}}(x_{N-1})\\
&= \frac{1}{2}\sum_{k=i}^{N-2}\left(\tfrac{1}{N-1}g_1^n + g_{k+1}^n - g_{k-1}^n - \tfrac{1}{N-1}g_{N-1}^n\right)\\
&+ \frac{1}{4}\left(\tfrac{3}{N-1}g_1^n - 2g_{N-2}^n - \tfrac{2N+1}{N-1}g_{N-1}^n\right)\\
&= \frac{1}{2}\left(\tfrac{N-i-1}{N-1}g_1^n + g_{N-1}^n + g_{N-2}^n - g_i^n - g_{i-1}^n - \tfrac{N-i-1}{N-1}g_{N-1}^n\right)\\
&+ \frac{1}{4}\left(\tfrac{3}{N-1}g_1^n - 2g_{N-2}^n - \tfrac{2N+1}{N-1}g_{N-1}^n\right)\\
&= \frac{1}{4}\left(\tfrac{2(N-i)+1}{N-1}g_1^n - 2g_i^n - 2g_{i-1}^n - \tfrac{2(N-i)+1}{N-1}g_{N-1}^n\right),\end{aligned}$$

we deduce for the components of $\boldsymbol{c}$ the formula

$$c_i = -\frac{1}{4}\left(\tfrac{N-2i+1}{(N-1)^2}g_1^n + 2g_{i-1}^n + 2g_i^n - \tfrac{N-2i+1}{(N-1)^2}g_{N-1}^n\right),\ i = 2,\ldots,N-2,$$

and hence

$$g_i^{n+1} := -\frac{1}{4}\left(\tfrac{N-2i}{(N-1)^2}g_1^n - \tfrac{N-2i}{(N-1)^2}g_{N-1}^n\right),\ i = 2,\ldots,N-2,$$

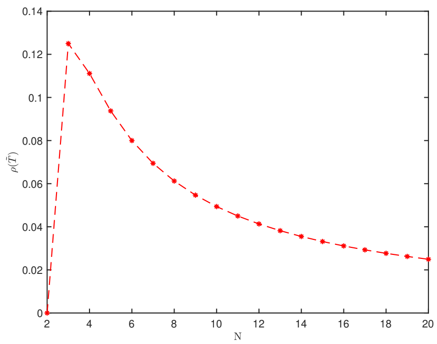

Fig. 5: Curve of the convergence factor in (3.12) with respect to N for $a=-1$, $b=1$

which concludes the proof for the iteration matrix $\tilde{T}$. To prove that $\rho(\widetilde{T}) = \|\widetilde{T}\|_\infty$, we define the vector $\boldsymbol{v} = \begin{bmatrix}1 & \frac{N-2}{N-4} & \dots & -\frac{N-2}{N-4} & -1\end{bmatrix}^T$ (i.e. $v_k = \frac{N-2k}{N-2}$, $k = 1, \dots, N-1$). We clearly see that $\boldsymbol{v}$ is an eigenvector of $\widetilde{T}$ associated to $\|\widetilde{T}\|_\infty$. ∎

THEOREM 3.4. *The two-level NNM for the Laplace problem in 1D given by Algorithm 3.2 is convergent, and satisfies the convergence estimate*

$$\max_{1\le i\le N-1} |g_i^n| \le \left(\frac{H(b-a-2H)}{2(b-a-H)^2}\right)^n \max_{1\le i\le N-1} |g_i^0|. \tag{3.12}$$

Proof. By Lemma 3.3, we have that $\rho(\widetilde{T}) < 1$ for all $N \geq 2$, which shows that the algorithm is convergent. To prove the estimate, it suffice to take the infinity norm of $\boldsymbol{g}^n$ and note that $N = (b-a)/H$. ∎

We see from Theorem 3.4 that a coarse space with piecewise constant functions leads to a well posed and scalable algorithm. Indeed, Figure 5 shows that the convergence factor in (3.12) does not deteriorate for increasing N. This shows also that the constants are not sufficient for an optimal coarse space, i.e., to get a direct solver. In order to obtain a two-level algorithm that converges after the first coarse correction, one would need to choose the optimal coarse space X_d that consists of piecewise linear functions in the 1D Laplace case.

3.2. Two-dimensional analysis. We now analyze the convergence of the two-level NNM in 2D, both for the screened Laplace operator $\mathcal{L} = \eta^2 - \Delta$ and for the Laplacian $\mathcal{L} = -\Delta$.

3.2.1. The screened Laplace operator $\mathcal{L} = \eta^2 - \Delta$. We use for this case Dirichlet boundary conditions imposed on all boundaries of the original domain. In 2D, the optimal coarse space X_d is now infinite dimensional, since the interface traces are now functions. The optimal coarse space is too big to be used in practice, and we need to choose finite dimensional approximation. In order to determine which functions to use in the approximation, we recall that in Subsection 2.2 we have expanded the subdomain solutions in a sine series based on the separation of variables approach. These sine functions are the eigenfunctions of the interface eigenvalue problem

$$\begin{aligned} -\partial_{xx}\psi_i &= \lambda\psi_i \text{ in } \Gamma_i,\\ \psi_i &= 0 \text{ on } \partial\Gamma_i, \end{aligned} \tag{3.13}$$

the eigenpairs being $(\psi_i, \lambda) = \left(\sin(\frac{m\pi}{L}y), \frac{m^2\pi^2}{L^2}\right)$, $m \geq 1$. Note that interface eigenvalue problems of the form (3.13) were also the essential ingredient in the construction of SHEM [19, 18, 21]. We now determine an optimized approximate coarse space using the interface eigenvalue problems in (3.13) for a decomposition as in Figure 3. Since the lowest eigenmode of (3.13) corresponds precisely to the most slowly converging mode of the NNM as we have seen in Theorem 2.4, we should use in an optimized coarse space the lowest modes of (3.13). One can then expect that the more modes we add, the more the convergence improves. This leads to our definition of the optimized spectral coarse space

$$\widetilde{X}_d := \left\{ v \in X_d : \forall i, \partial_x v_i(x_i, y) \in \operatorname{span}\left\{ \sin(\frac{m\pi}{L}y), m = 1, \ldots, J \right\} \right\}, \tag{3.14}$$

where $J \geq 1$ can be suitably chosen to get a richer and richer coarse space, and the optimal one in the limit.

THEOREM 3.5. *The two-level NNM for the screened Laplace problem* $(\eta^2 - \Delta)u = 0$ *in 2D given by Algorithm 3.1 with the optimized coarse space* $\widetilde{X}_d$ *defined in (3.14) satisfies the error bound*

$$\max_{1\leq i\leq N-1} \|g_i^n\|_2 \leq \frac{1}{\sinh^{2n}(k_{J+1}H)} \max_{1\leq i\leq N-1} \|g_i^0\|_2. \tag{3.15}$$

Proof. At each iteration n, the intermediate solution $u_i^{n+1/2}$ can be written as

$$u_i^{n+1/2}(x, y) = \sum_{m=1}^{\infty} \hat{u}_i^{n+1/2}(x, m) \sin(k_m y), \text{ for} i = 1, \ldots, N.$$

Using Parseval's identity, we have for $\nu \in \widetilde{X}_d$

$$\int_0^L \left|(u_{i+1}^{n+\frac{1}{2}}+\nu)(x_i^+,y)-(u_i^{n+\frac{1}{2}}+\nu)(x_i^-,y)\right|^2 dy=\frac{L}{2}\sum_{m=1}^{J}\left|(\hat{u}_{i+1}^{n+\frac{1}{2}}+\hat{\nu})(x_i^+,m)-(\hat{u}_i^{n+\frac{1}{2}}+\hat{\nu})(x_i^-,m)\right|^2 + \frac{L}{2}\sum_{m=J+1}^{\infty}\left|\hat{u}_{i+1}^{n+\frac{1}{2}}(x_i^+,m)-\hat{u}_i^{n+\frac{1}{2}}(x_i^-,m)\right|^2,$$

and hence

$$U^n = \arg\min_{\nu\in\widetilde{X}_d} \sum_{i=1}^{N-1} \sum_{m=1}^{J} \left|(\hat{u}_{i+1}^{n+\frac{1}{2}} + \hat{\nu})(x_i^+, m) - (\hat{u}_i^{n+\frac{1}{2}} + \hat{\nu})(x_i^-, m)\right|^2.$$

Since we are minimizing a finite dimensional quadratic, the quantity can be made zero. Hence, $g_i^{n+1}(m) = 0$ for $m \leq J$, and following the proof of Theorem 2.4, we get the stated bound. ∎

Theorem 3.5 shows that no matter the value of H, we can obtain a convergent NNM by adding enough coarse space components, and thus turn a divergent NNM due to bad aspect ratio of the subdomains into a convergent one. Furthermore, the more we increase the number of coarse space components J, the faster the NNM becomes, and there are no other, more effective coarse space components to add, which explains the term optimized coarse space. Note also that the more modes we add, the better our approximation of the theoretically optimal coarse space X_d becomes, and in the discrete setting, we can actually reach the optimal coarse space, which is then finite dimensional as well.

3.2.2. The Laplace operator $\mathcal{L} = -\Delta$ with Neumann boundary conditions. If we have Neumann boundary conditions in 2D all around the original domain, then the local corrections ψ_i^n are not well defined as in the one-dimensional case, unless they satisfy the 2D consistency conditions

$$\int_0^L \partial_x \psi_i^n(x_i, y)\,\mathrm{d}y - \int_0^L \partial_x \psi_i^n(x_{i-1}, y)\,\mathrm{d}y = 0, \text{ for } i = 2, \dots, N-1. \tag{3.16}$$

To obtain a well defined NNM, we generalize the ideas of Subsection 3.1.2, and introduce new auxiliary functions $\hat{b}_i(y)$ along the interfaces $(0, L)$ such that (3.16) is satisfied, which is equivalent to

$$\int_0^L \hat{b}_{i-1}(y)\,\mathrm{d}y - \int_0^L \hat{b}_i(y)\,\mathrm{d}y = \int_0^L a_{i-1}(y)\,\mathrm{d}y + \int_0^L a_i(y)\,\mathrm{d}y, \quad i = 1, \dots, N-1, \tag{3.17}$$

where, $a_i(y) := \frac{1}{2}(\partial_x u_i^n(x_i, y) - \partial_x u_{i+1}^n(x_i, y))$ on $(0, L)$, $i = 1, \dots, N-1$, and hence any function satisfying this condition is candidate for correcting Algorithm 2.1.

Note that we can choose freely the $\hat{b}_i(y)$ as long as they satisfy (3.17). If chosen carefully, they can accelerate the convergence of the algorithm, but we will restrict ourselves first here to the case where all the auxiliary functions are kept constant along $(0, L)$. Then, using (3.17), we obtain the linear system

$$L\hat{\boldsymbol{b}} = \hat{\boldsymbol{a}} \tag{3.18}$$

where $L \in \mathbb{R}^{N-2 \times N-1}$ is as in Subsection 3.1.2, and $\hat{\boldsymbol{b}} \in \mathbb{R}^{N-1}$, $\hat{\boldsymbol{a}} \in \mathbb{R}^{N-2}$ are given by

$$\hat{\boldsymbol{b}} := \begin{bmatrix} \hat{b}_1 \\ \vdots \\ \hat{b}_{N-1} \end{bmatrix}, \ \hat{\boldsymbol{a}} := \begin{bmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_{N-1} \end{bmatrix},$$

and

$$\hat{a}_i = \frac{1}{L}\left(\int_0^L a_i(y)\,dy + \int_0^L a_{i+1}(y)\,dy\right), \ i = 1, \dots, N-2. \tag{3.19}$$

Again, as in the one dimensional case, the solutions of ψ_i are not unique, and we need to choose the most suitable constants. Following the ideas in Subsection 3.1.2, we choose $\hat{\boldsymbol{c}} = \left[0, \hat{c}_2, \dots, \hat{c}_{N-1}, 0\right]^T$ such that

$$\hat{\boldsymbol{c}} := \arg\min_{\boldsymbol{c}} \sum_{i=1}^{N-1} \int_0^L \left|(u_{i+1}^{n+\frac{1}{2}}(x_i, y) + c_{i+1}) - (u_i^{n+\frac{1}{2}}(x_i, y) + c_i)\right|^2 \,dy. \tag{3.20}$$

We thus obtain the two-level NNM for the Laplace case in 2D in Algorithm 3.3.

LEMMA 3.6. *Let $\hat{\boldsymbol{g}}^n(m) = \left[\hat{g}_1^n(m), \hat{g}_2^n(m), \dots, \hat{g}_{N-1}^n(m)\right]^T \in \mathbb{R}^{N-1}$. Then for $N \geq 3$ we have*

$$\begin{aligned} \hat{\boldsymbol{g}}^n(0) &= \widetilde{T}\hat{\boldsymbol{g}}^{n-1}(0), && \text{for } m = 0, \\ \hat{\boldsymbol{g}}^n(m) &= T_m \hat{\boldsymbol{g}}^{n-1}(m), && \text{for } m \geq 1, \end{aligned} \tag{3.22}$$

where $\widetilde{T}$ and T_m are defined as in Lemmas 2.1 and 2.3.

Algorithm 3.3 NNM for $\mathcal{L} = -\Delta$ in 2D

1. Set g_i^0 to zero or any inexpensive initial guess on the interfaces Γ_i
2. Repeat until convergence
   (a) Solve the Dirichlet problems

$$\begin{aligned} -\Delta u_i^n &= f, \quad \text{in } \Omega_i, \\ u_i^n(x_{i-1}, y) = g_{i-1}^n(y), u_i^n(x_i, y) &= g_i^n(y) \quad \text{on } (0, L), \\ \partial_y u_i^n(x, 0) = \partial_y u_i^n(x, L) &= 0, \quad \text{on } (a, b). \end{aligned} \tag{3.21}$$

   (b) Solve the Neumann problems

$$\begin{aligned} -\Delta \psi_i^n &= 0, \quad \text{in } \Omega_i, \\ \partial_x \psi_i^n(x_{i-1}, y) &= -a_{i-1}(y) + \hat{b}_{i-1}(y), \quad \text{on } (0, L), \\ \partial_x \psi_i^n(x_i, y) &= a_i(y) + \hat{b}_i(y), \quad \text{on } (0, L), \\ \partial_y \psi_i^n(x, 0) = \partial_y \psi_i^n(x, L) &= 0, \quad \text{on } (a, b), \\ \fint_{\Omega_i} \psi_i^n &= 0, \end{aligned}$$

   where the $\hat{b}_i(y)$ are defined in (3.17).
   (c) Set for $i = 1, \ldots, N$

$$\tilde{u}_i^{n+\frac{1}{2}} := u_i^{n+\frac{1}{2}} + \hat{c}_i,$$

   where the $\hat{c}_i$ are defined in (3.20).
   (d) Set for $i = 1, \ldots, N$

$$g_i^{n+1}(y) := \left(\tilde{u}_{i+1}^{n+\frac{1}{2}}(x_i, y) + \tilde{u}_i^{n+\frac{1}{2}}(x_i, y) \right) / 2, \quad \text{on } (0, L).$$

Proof. In the case of Neumann conditions, u_i^n and ψ_i^n can be expanded in a cosine series,

$$u_i^n(x, y) = \sum_{m=0}^{\infty} \hat{u}_i^n(x, m) \cos(k_m y), \ \psi_i^n(x, y) = \sum_{m=0}^{\infty} \hat{\psi}_i^n(x, m) \cos(k_m y),$$

and we thus have as before a sequence of one dimensional problems for each mode m. For the case $m = 0$, the Fourier coefficients $\hat{u}_i^n(x, 0)$, $\hat{\psi}_i^n(x, 0)$ satisfy

$$\begin{array}{rclrcl} -\partial_{xx} \hat{u}_i^n(x, 0) &=& 0, & -\partial_{xx} \hat{\psi}_i^n(x, 0) &=& 0, \\ \hat{u}_i^n(x_{i-1}, 0) &=& \hat{g}_{i-1}^n(0), & \hat{\psi}_i^n(x_{i-1}, 0) &=& a_{i-1}(0), \\ \hat{u}_i^n(x_i, 0) &=& \hat{g}_i^n(0), & \hat{\psi}_i^n(x_i, 0) &=& a_i(0), \end{array}$$

where $a_i(m) = (\partial_x \hat{u}_i^n(x_i, m) - \partial_x \hat{u}_{i+1}^n(x_i, m))/2$, $m \geq 0$, $i = 1, \ldots, N-1$. From (3.19), we find that $\hat{a}_i = a_i(0) + a_{i+1}(0)$, $i = 1, \ldots, N-2$, and since $\int_{\Omega_i} u_i(x, y)\, dxdy = \int_{x_{i-1}}^{x_i} \int_0^L u_i(x, y) = 0$, we deduce that $\int_{x_{i-1}}^{x_i} \hat{u}_i^n(x, 0) dx = 0$, $i = 2, \ldots, N-1$. Moreover, we remark that $\hat{\boldsymbol{c}}$ satisfies

$$\begin{aligned}\hat{\boldsymbol{c}}=&\operatorname*{argmin}_{\boldsymbol{c}}\sum_{i=1}^{N-1}\int_0^L\left|(u_{i+1}^{n+\frac12}(x_i,y)+c_{i+1})-(u_i^{n+\frac12}(x_i,y)+c_i)\right|^2dy,\\ =&\operatorname*{argmin}_{\boldsymbol{c}}\sum_{i=1}^{N-1}L\left|(\hat{u}_{i+1}^{n+\frac12}(x_i,0)+c_{i+1})-(\hat{u}_i^{n+\frac12}(x_i,0)+c_i)\right|^2+\frac{L}{2}\sum_{m=1}^{\infty}\left|\hat{u}_{i+1}^{n+\frac12}(x_i,m)-\hat{u}_i^{n+\frac12}(x_i,m)\right|^2,\\ =&\operatorname*{argmin}_{\boldsymbol{c}}\sum_{i=1}^{N-1}\left|(\hat{u}_{i+1}^{n+\frac12}(x_i,0)+c_{i+1})-(\hat{u}_i^{n+\frac12}(x_i,0)+c_i)\right|^2.\end{aligned}$$

Hence, the Fourier coefficients $\hat{u}_i^n(x,0)$ and $\hat{\psi}_i^n(x,0)$ have the same iterates as in the 1D Algorithm 3.2 for Laplace equation. Using then Lemma 3.6, we get the first recurrence relation of (3.22). For $m\geq 1$, the iterates $\hat{u}_i^n(x,m)$ and $\hat{\psi}_i^n(x,m)$ are the same as in the 1D Algorithm 2.1 for the screened Laplace equation, and using Lemma 2.3, we get the second recurrence relation of (3.22). ∎

THEOREM 3.7. *If the ratio of subdomain width H and height L satisfies $\frac{H}{L}>\frac{\ln(1+\sqrt{2})}{\pi}$, then the two-level NNM for the Laplace problem with Neumann boundary conditions in 2D given by Algorithm 3.3 converges, and satisfies the error estimate*

$$\max_{1\leq i\leq N-1}\|g_i^n\|_2\leq\max\left\{\frac{H(b-a-2H)}{2(b-a-H)^2},\frac{1}{\sinh^2(k_1H)}\right\}^n\max_{1\leq i\leq N-1}\|g_i^0\|_2. \tag{3.23}$$

Proof. From Lemma 3.6, we have that Algorithm 3.3 converges iff $\rho(\widetilde{T})<1$, and $\rho(\widetilde{T}_m)<1$ for $m\geq 1$. Since we already know that $\rho(\widetilde{T})<1$ as shown in Lemma 3.3, it suffices that $\rho(\widetilde{T}_m)\leq\|T_m\|_\infty\leq\|T_1\|_\infty<1$, which is satisfied if $\sinh(k_1H)>1$, or equivalently $k_1H>\ln(1+\sqrt{2})$. To show the error bound, it suffices to use again Parseval's identity $\|g_i^n\|_2^2=L\hat{g}_i(0)^2+\frac{L}{2}\sum_{m=1}^\infty\hat{g}_i(m)^2$, and follow the same steps as in the proof of Theorem 2.4. ∎

Theorem 3.7 shows that piecewise constant functions are sufficient to have a well defined and convergent iterative NNM provided an assumption on the aspect ratio of the subdomain geometry is verified.

3.3. Three-dimensional analysis.

In this part, we analyze the convergence of the two-level NNM for the three-dimensional screened Laplace operator $\mathcal{L}=\eta^2-\Delta$. As in the two-dimensional case, the optimal coarse space X_d defined in (3.2) is of infinite dimension. We propose to approximate this coarse space using the eigenvalue problem defined in Equation (3.13), except that now the interfaces Γ_i represent surfaces in the yz-plane. The solution of this problem is given by the eigenpairs $(\psi_i,\lambda)=\left(\sin\left(\frac{m\pi}{L}y\right)\sin\left(\frac{m'\pi}{L'}z\right),\frac{m^2\pi^2}{L^2}+\frac{m'^2\pi^2}{L'^2}\right)$. This yields the definition of the optimized coarse space $\tilde{X}_d$ in 3D,

$$\widetilde{X}_d:=\left\{v\in X_d:\forall i,\partial_x v_i(x_i,y,z)\in\operatorname{span}\left\{\sin(\frac{m\pi}{L}y)\sin(\frac{m'\pi}{L'}z),(m,m')\in\mathcal{I}_{J,J'}\right\}\right\}, \tag{3.24}$$

where $\mathcal{I}_{J,J'}:=\{1,\ldots,J\}\times\{1,\ldots,J'\}$, and $J,J'\geq 1$ are positive integers that are used to enrich the approximate coarse space.

THEOREM 3.8. *The two-level NNM for the screened Laplace problem $(\eta^2-\Delta)u=0$ in 3D given by Algorithm 3.1 with the optimized coarse space $\widetilde{X}_d$ defined in (3.24)* *satisfies the error bound*

$$\max_{1\le i\le N-1}\|g_i^n\|_2 \le \frac{1}{\sinh^{2n}(k^* H)}\max_{1\le i\le N-1}\|g_i^0\|_2, \tag{3.25}$$

where $k^* := \min_{(m,m')\in\mathbb{Z}^2_{>0}\setminus\mathcal{I}_{J,J'}} k_{m,m'}$.

Proof. We proceed as in the proof of Theorem 3.5. The intermediate solution $u_i^{n+1/2}$ can be written as

$$u_i^{n+1/2}(x,y,z) = \sum_{m,m'\ge 1}\hat{u}_i^{n+1/2}(x,m)\sin(\frac{m\pi}{L}y)\sin(\frac{m'\pi}{L'}z),\quad \text{for } i=1,\dots,N.$$

Using Parseval's identity, we have for $\nu\in\widetilde{X}_d$

$$\begin{aligned}\iint_{[0,L]\times[0,L']}&\left|(u_{i+1}^{n+\frac12}+\nu)(x_i^+,y,z)-(u_i^{n+\frac12}+\nu)(x_i^-,y,z)\right|^2 dy\,dz=\\ &=\frac{LL'}{4}\sum_{(m,m')\in\mathcal{I}_{J,J'}}\left|(\hat{u}_{i+1}^{n+\frac12}+\hat{\nu})(x_i^+,m,m')-(\hat{u}_i^{n+\frac12}+\hat{\nu})(x_i^-,m,m')\right|^2\\ &\quad+\frac{LL'}{4}\sum_{(m,m')\in\mathbb{Z}^2_{>0}\setminus\mathcal{I}_{J,J'}}\left|\hat{u}_{i+1}^{n+\frac12}(x_i^+,m,m')-\hat{u}_i^{n+\frac12}(x_i^-,m,m')\right|^2,\end{aligned}$$

and hence

$$U^n = \arg\min_{\nu\in\widetilde{X}_d}\sum_{i=1}^{N-1}\sum_{(m,m')\in\mathcal{I}_{J,J'}}\left|(\hat{u}_{i+1}^{n+\frac12}+\hat{\nu})(x_i^+,m,m')-(\hat{u}_i^{n+\frac12}+\hat{\nu})(x_i^-,m,m')\right|^2.$$

Since we are minimizing a finite dimensional quadratic, the quantity can be made zero. Hence, $g_i^{n+1}(m,m')=0$ for $m\le J$, $m'\le J'$, and following the proof of Theorem 2.6, we get the stated bound. ∎

4. Numerical experiments without cross points. We illustrate now our convergence results with numerical experiments. We start with the one dimensional case of the screened Poisson problem $(\eta^2-\Delta)u=f$ on the domain $\Omega:=(-1,1)$ with $\eta=\sqrt{2}$ and $f(x)=1$. We use centered finite differences with mesh size $\Delta x=10^{-4}$, and run the one-level algorithm with $N=20$ subdomains. Figure 6 (left) shows that without coarse correction the algorithm fails to converge, and with the optimal coarse space we get convergence after one coarse correction on the interfaces, which implies convergence after the second subdomain solve in volume in our implementation. In Figure 6 (right) we show the convergence behavior for the Poisson case, where the NNM needs already a constant coarse space to be well posed. This method is also convergent, but we see that the constants are not sufficient to have convergence after the one coarse correction: the optimal coarse space consists of linear functions, and leads to convergence after one coarse correction (dashed curve).

In 2D, we decompose the domain $\Omega:=(-1,1)\times(0,1)$ into $N=10$ subdomains, and run Algorithm 3.1 on the screened Poisson problem $\eta u-\Delta u=1$ with $\eta=2$, discretized by centered finite differences using the mesh size $\Delta x=\Delta y=5\cdot 10^{-3}$. We can see in Figure 7 (left) that NNM without coarse correction is divergent. Our proposed approximations of the optimal coarse space make the iterative NNM convergent, and

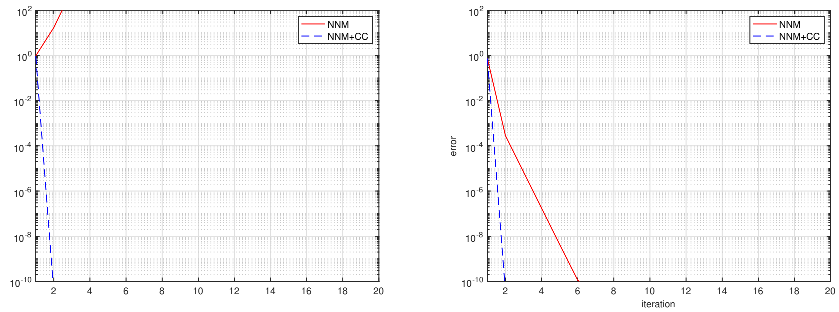

Fig. 6: 1D case: error as a function of iteration for the one and new optimal two-level NNM for the screened Poisson problem with $\eta = 2$ (left), and for the classical two-level NNM with constant coarse space and the new optimal two-level NNM for the Poisson problem (right)

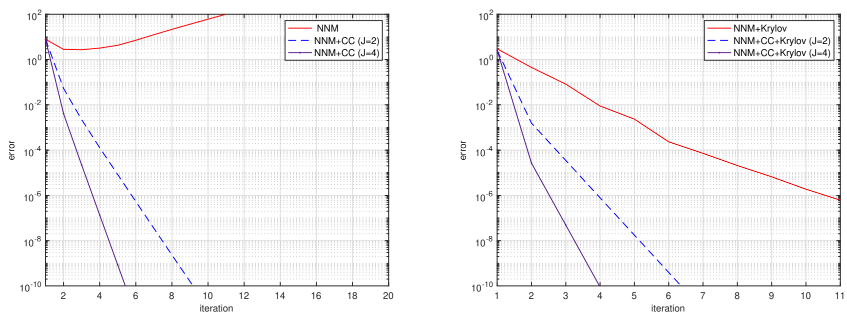

Fig. 7: Left: Convergence of NNM and coarse corrected NNM in 2D. Right: Same but using Krylov (GMRES)

the more we enrich the optimized coarse space $\tilde{X}_d$, the faster the convergence becomes. On the right in Figure 7 we show the corresponding results with Krylov acceleration. We see that GMRES also makes the one-level NNM convergent, but our new coarse corrected NNM still performs better, both as iterative solver and as preconditioner. We further observe that with enough enrichment of the coarse space, our new method as an iterative solver is almost as fast as when used as a preconditioner. In 3D, we decompose the domain $\Omega := (0,1) \times (0,1) \times (0,1)$ into $N = 10$ bricks, and run Algorithm 3.1 on the screened Poisson problem $\eta u - \Delta u = 1$ with $\eta = 2$, discretized by centered finite differences using the mesh size $\Delta x = \Delta y = \Delta z = 5 \cdot 10^{-2}$. We can see from Figure 8 (left) that NNM without coarse correction is divergent. The addition of the coarse correction makes the method convergent. Moreover, enriching the coarse space makes the method faster. As in the two-dimensional case, we next use the one- and two-level methods as preconditioners for a Krylov subspace method (GMRES). We observe from Figure 8 (right) that GMRES makes the one-level method converge, and improves the convergence of the two-level method. Moreover, we see that in both experiments, the two-level method is faster than the one-level method, and the more we increase the size of the coarse space, the faster the two-level method becomes.

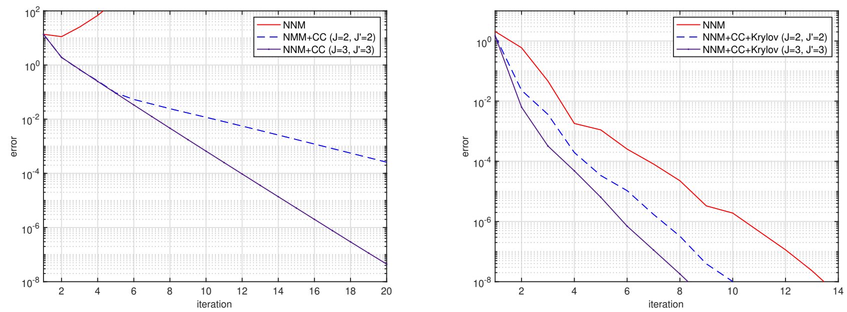

Fig. 8: Left: Convergence of NNM and coarse corrected NNM in 3D. Right: Same but using Krylov (GMRES)

We note also that this approach of constructing the coarse space is not restricted only to problems with constant coefficients. Indeed, the same coarse correction can be used for problems with non-constant coefficients provided that we choose the appropriate coarse functions. In general, the latter are obtained by solving an interface eigenvalue problem on the interfaces; cf. [19, 18]. For the Laplacian example, it can be shown that these eigenfunctions are exactly the sine functions; cf. [21].

5. Discrete analysis of the two-level NNM including cross points. We now present an optimal coarse space correction at the discrete level. Let $\mathcal{T}_h$ be a triangulation of Ω such that $\mathcal{T}_{h,i} := \mathcal{T}_h \cap \overline{\Omega}_i$ is also a triangulation of Ω_i. We denote by $x_{i,j,k}$ the nodes shared by $\mathcal{T}_{h,i} \cap \mathcal{T}_{h,j}$. We define the discrete harmonic basis $\varphi_{i,j,k}$ for i, j such that $\Omega_i \cap \Omega_j \neq \emptyset$ as the unique functions such that

1. their support is in the adjacent subdomains, supp $\varphi_{i,j,k} \subset \Omega_i \cup \Omega_j$,
2. their normal derivatives are continuous, $\varphi_{i,k} := \varphi_{i,j,k|\Omega_i}$ and $\varphi_{j,k} := \varphi_{i,j,k|\Omega_j}$ verify $\frac{\partial \varphi_{i,k}}{\partial n_i}(x_{i,j,k'}) = \delta_{k,k'}$, $\frac{\partial \varphi_{j,k}}{\partial n_j}(x_{i,j,k'}) = -\delta_{k,k'}$,
3. $\varphi_{i,k}$ and $\varphi_{j,k}$ verify the discrete variational formulation of (2.1) in Ω_i and Ω_j respectively.

The discrete optimal coarse space X_h is then defined as

$$X_h := \text{span}\{\varphi_{i,j,k} : \text{ for all } i, j \text{ s.t. } \Omega_i \cap \Omega_j \neq \emptyset\}. \tag{5.1}$$

We now prove the optimality of this discrete coarse space, i.e convergence of the two-level NNM at the discrete level after one coarse correction.

THEOREM 5.1. *Algorithm 3.1 with the discrete coarse space X_h defined in (5.1) converges after one coarse correction.*

Proof. We proceed using an argument of dimension: consider the map

$$\begin{aligned} T\colon X_h &\mapsto \mathbb{R}^{d_1+\cdots+d_N} \\ u &\mapsto (u_i(x_{i,j,k}) - u_j(x_{i,j,k})), \end{aligned}$$

where $d_i := \#\{k : x_{i,j,k} \in \Omega_i \cap \Omega_j \neq \emptyset\}$, $u_i := u_{|\Omega_i}$ and $u_j := u_{|\Omega_j}$. We claim that T is a one-to-one correspondence. In fact, since the $\varphi_{i,j,k}$ are linearly independent, we have that $\dim(X_h) = d_1 + \cdots + d_N$. Moreover, if $v \in Ker(T)$, then $v_i := v_{|\Omega_i}$ are continuous for both the Neumann and Dirichlet trace at the discrete level and they

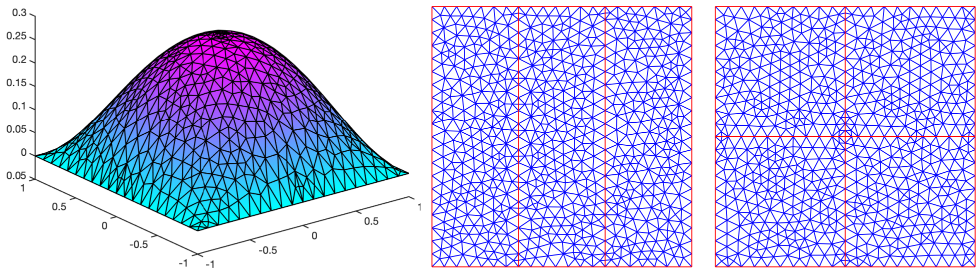

Fig. 9: Exact monodomain solution of (5.2) (left), and a 1×3 partition without a cross point of the unit square (middle), and a 2×2 partition with a cross point (right).

satisfy the homogeneous counterpart of (2.1) inside each subdomain, hence $v = 0$. Thus, by the rank-nullity which affirms that $\operatorname{rank}(T) + \dim(\ker(T)) = \dim(X_h)$, we conclude that T is a one-to-one correspondence, and in particular is onto. Now, let $(U_i^0)_{1 \le i \le N}$ be the coarse correction, and set $\widetilde{u}_i^1 := u_i^{\frac{1}{2}} + U_i^0$ the iterate after the coarse correction. The latter is chosen such that it satisfies the discrete form of the minimum jump condition in (3.3). Since T is onto, it is possible to choose U^0 in the preimage of T such that $\widetilde{u}_i^1$ for $i = 1, \dots, N$ completely cancels the jump, i.e., $q(\widetilde{u}^1) = 0$. Therefore, the $\widetilde{u}_i^1$ are continuous across the subdomains for both Dirichlet and Neumann traces. Moreover, the $\widetilde{u}_i^1$ satisfy the discrete form of (2.1) inside each subdomain. It follows that $\widetilde{u}^1$ is the discrete monodomain solution of (2.1). ◻

We test this new, optimal two-level NNM on a variable coefficient diffusion problem,

$$
\begin{aligned}
-\nabla \cdot (a(x, y)\nabla u(x, y)) &= 1, \text{ if } (x, y) \in (-1, 1) \times (-1, 1), \\
u(x, y) &= 0, \text{ if } (1 - x^2)(1 - y^2) = 0,
\end{aligned}
\tag{5.2}
$$

where $a(x, y) := 1 + x^2y^2$ is a continuously varying function along the interfaces. The exact monodomain solution is shown in Figure 9 (left). We choose two different decompositions as shown in Figure 9, one without cross points (middle), and one with a cross point (right). We mention that in the finite element setting, the discrete normal derivatives are computed as in [20]. The convergence results in Table 1 show that Algorithm 3.1 is truly optimal since it leads to convergence after one coarse correction both with and without cross point. For the 2×2 cross point case, Table 1 shows that the one level NNM iteration diverges. The reason for this divergence was identified in [5] to be the fact that NNM is not well posed in a functional setting at the continuous level in the presence of cross points. Despite these difficulties, Algorithm 3.1 still converges in one iteration; the coarse correction is still able to extract the right traces and combine them to find the exact solution.

6. Conclusion. We designed, analyzed, and tested a new coarse space correction for the Neumann-Neumann iterative method. We described the method for general second order elliptic PDEs, and analyzed it for the Poisson and the screened Poisson equation. We proved that the new coarse space is truly optimal, i.e the method converges after one coarse correction. In two and three dimensions, this optimal coarse space is however too high dimensional, and we introduced an optimized spectral approximation which can make an otherwise divergent iterative Neumann-Neumann

	1 × 3 example		2 × 2 example with cross point	
Its.	Algorithm 2.1	Algorithm 3.1	Algorithm 2.1	Algorithm 3.1
1	5.817e-01	2.741e-01	4.754e-01	9.582e-01
2	2.358e-01	8.881e-16	9.145e-01	1.382e-14
3	9.682e-02	4.440e-16	2.316e+00	4.996e-16
4	3.971e-02	4.440e-16	5.947e+00	4.440e-16
5	1.629e-02	4.718e-16	1.527e+01	7.216e-16
6	6.683e-03	4.996e-16	3.924e+01	5.551e-16
7	2.741e-03	4.163e-16	1.007e+02	3.885e-16

Table 1: Convergence of NNM and coarse corrected NNM for (5.2) using the optimal discrete coarse space defined in (5.1).

method convergent. The more we enrich our optimized coarse space, the faster the convergence becomes, and it improves both as an iterative solver and as a preconditioner. Our results in the discrete setting open also up the field for optimized coarse spaces at cross points in Neumann-Neumann methods, and other domain decomposition methods. Cross points in domain decomposition methods are currently an active field of research, especially for non-overlapping domain decomposition methods: for non-overlapping Schwarz methods, see e.g. [16, 17, 20, 28, 7, 6], and for the Neumann-Neumann method, see [4, 5]. For classical overlapping Schwarz methods, cross points are also interesting, since the partition of unity can slightly influence the convergence there [13], and for the additive Schwarz preconditioner, cross points lead to the coloring influencing the condition number estimate [32], and divergence when the additive Schwarz method is used as a stationary iteration [12, Section 3.2], an issue that can also be addressed with the coarse space, see [21].

Compared to the classical coarse space in the balancing Neumann-Neumann method which is using a constant per subdomain, our coarse space can be made richer and we showed precisely how this enhances the convergence. One then has to solve however a larger coarse problem and there is thus a trade-off, like in the recent algebraic coarse spaces GenEO, ACMS, or SHEM for Schwarz methods. An advantage of our approximate coarse space is that it is defined a priori for constant coefficient problems, and we know it at the continuous level at which the Neumann-Neumann method is defined. How to deal with approximations of the optimal coarse space in the variable coefficient case, and do enrichment in the presence of cross points will be described in future work.

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