

Chapter 1

Domain Decomposition Methods for Wave Propagation

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Abstract

We introduce a non-overlapping variant of the Schwarz algorithm for wave propagation problems with discontinuous coefficients. We derive optimal transmission conditions based on an analytic factorization of the differential operator. The optimal transmission conditions are in general non local, but our analysis reveals that the non-locality depends on the time interval under consideration. We show how to choose time windows to obtain optimal performance of the algorithm with local transmission conditions. We introduce a finite volume discretization of the optimal transmission conditions and prove stability of the obtained domain decomposition algorithm. Numerical experiments confirm the theoretical results.

1 Introduction

We propose in this paper a new approach for solving time-dependent problems by domain decomposition. Instead of discretizing first in time, we decompose the original space-time domain into subdomains and solve time dependent subproblems on each subdomain. This approach permits the use of different numerical methods in different domains, and saves communication time. We are focusing in this paper on wave propagation phenomena in the presence of discontinuous coefficients. We derive transmission conditions for overlapping and non overlapping Schwarz algorithms which lead to optimal convergence. The optimal transmission conditions are found to be corresponding to the Dirichlet to Neumann maps at the artificial interfaces. In one dimension we prove that optimal convergence can be achieved by using local transmission operators. We propose a discrete algorithm for which we prove stability and convergence either by energy estimates or using the discrete Laplace transform as in normal mode analysis. Turning to the higher-dimensional case, we prove the notion of rate of convergence to be closely related to absorbing boundary conditions. The one-dimensional analysis can be found in [5]. For other approaches see [1],[2],[3],[6].

2 A First Example

We consider the homogeneous wave equation in one dimension,

$$(1) \quad u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R}, \quad t \in [0, T],$$

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with prescribed initial conditions $u^{(0)}$ and $u^{(1)}$. The velocity c is a constant in \mathbb{R} . The domain \mathbb{R} is divided into two overlapping subdomains $\Omega_1 = (-\infty, L] \times [0, T]$ and $\Omega_2 = [0, \infty) \times [0, T]$. We introduce the "classical" Schwarz algorithm

$$(2) \quad \begin{aligned} v_{tt}^{k+1} &= c^2 v_{xx}^{k+1}, \quad x < L, & v^{k+1}(L, t) &= w^k(L, t), \quad t < T, \\ w_{tt}^{k+1} &= c^2 w_{xx}^{k+1}, \quad x > 0, & w^{k+1}(0, t) &= v^k(0, t), \quad t < T. \end{aligned}$$

At each step, these equations are supplemented with the initial conditions $u^{(0)}$ and $u^{(1)}$. Due to the finite speed of propagation, we have the following result (cf [4]) :

THEOREM 2.1. : *The algorithm converges in a finite number of iterations, when $k > \frac{Tc}{L}$.*

For example for $c = 1$, $L = 0.2$ and $T = 3$, this algorithm needs 15 iterations in order to converge. Convergence in a finite number of steps is a desirable property, but 15 iterations is rather high. The reason for the slow convergence is the Dirichlet conditions used at the artificial interfaces. These conditions lead to artificial reflections which slow the algorithm down [4]. We introduce therefore new transmission conditions which will lead to optimal convergence of the algorithm.

3 Coherent Transmission and Optimal Convergence

We consider again our first example (1) with constant velocity to explain the main idea.

3.1 The Homogeneous Case

We replace the transmission boundary conditions in (2) by the general transmission conditions

$$(3) \quad \begin{aligned} (\partial_x + \Lambda_v(\partial_t))v^{k+1}(L, t) &= (\partial_x + \Lambda_v(\partial_t))w^k(L, t), \\ (\partial_x + \Lambda_w(\partial_t))w^{k+1}(0, t) &= (\partial_x + \Lambda_w(\partial_t))v^k(0, t). \end{aligned}$$

Using a Laplace transform in time with dual variable $s = \eta + i\omega$, $\eta > 0$, it can be solved as

$$\hat{V}^k(x, s) = \hat{V}^k(0, s)e^{s(x-L)}, \quad \hat{W}^k(x, s) = \hat{W}^k(0, s)e^{-sx},$$

$$\hat{V}^{k+1}(0, s) = \frac{-\frac{s}{c} + \lambda_v(s)}{\frac{s}{c} + \lambda_v(s)} \cdot \frac{\frac{s}{c} + \lambda_w(s)}{-\frac{s}{c} + \lambda_w(s)} e^{-\frac{2s}{c}L} \hat{V}^{k-1}(0, s).$$

This defines the rate of convergence for the process

$$(4) \quad \rho(s) = \frac{-\frac{s}{c} + \lambda_v(s)}{\frac{s}{c} + \lambda_v(s)} \cdot \frac{\frac{s}{c} + \lambda_w(s)}{-\frac{s}{c} + \lambda_w(s)} e^{-\frac{2s}{c}L}.$$

We get the following results :

THEOREM 3.1. *The process converges in two iterations if and only if $\lambda_v = \frac{1}{c}\partial_t$ and $\lambda_w = -\frac{1}{c}\partial_t$. For the Dirichlet or Neumann transmission condition, the rate of convergence is equal to $e^{-\frac{2s}{c}L}$.*

Thus the optimal transmission condition is given on both sides by the transparent boundary condition. Moreover, the overlap is now irrelevant for the convergence of the algorithm and we can reduce the overlap to zero. This leads to a non-overlapping variant of the Schwarz algorithm.

3.2 The Case of Discontinuous Coefficients

We consider now the 1-D wave equation with a discontinuous velocity

$$(5) \quad \mathcal{L}(u) = \left(\frac{1}{c^2(x)} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) (u) = f(x, t)$$

with appropriate initial conditions on the domain $\mathbb{R} \times (0, T)$. The velocity $c(x)$ is a constant in each physical domain, $O_1 = \mathbb{R}^- \times (0, T)$ and $O_2 = \mathbb{R}^+ \times (0, T)$, with $c(x) = c^i$ in O_i .

The Domain Decomposition Algorithm is defined in the following way : we divide the domain \mathbb{R} into I non overlapping subdomains $\Omega_i = (a_i, a_{i+1}) \times (0, T)$, $a_j < a_i$ for $j < i$ and $a_1 = -\infty$ and $a_{I+1} = \infty$ and consider the non overlapping algorithm

$$(6) \quad \begin{aligned} \mathcal{L}(u_i^{k+1}) &= f(x, t) & x \in \Omega_i, t \in (0, T), \\ \mathcal{B}_i^-(u_i^{k+1}) &= \mathcal{B}_i^-(u_{i-1}^k) & x = a_i, t \in (0, T), \\ \mathcal{B}_i^+(u_i^{k+1}) &= \mathcal{B}_i^+(u_{i+1}^k) & x = a_{i+1}, t \in (0, T). \end{aligned}$$

We want to choose the transmission operators \mathcal{B}_i^\pm in the algorithm to obtain optimal convergence for all time $t \in [0, \infty)$. As in the homogeneous case, the optimal transmission operators are the transparent boundary operators, and they can be obtained from an exact factorization of \mathcal{L} .

THEOREM 3.2 (FACTORIZATION). *The wave operator with discontinuous coefficient defined in (5) admits the two exact factorizations*

$$(7) \quad \mathcal{L} = -(\partial_x - DtN_2(x)\partial_t)(\partial_x + DtN_2(x)\partial_t) = -(\partial_x + DtN_1(x)\partial_t)(\partial_x - DtN_1(x)\partial_t)$$

where $DtN_1(x)$ and $DtN_2(x)$ are the exact Dirichlet to Neumann operators at point x , given by

$$(8) \quad \begin{aligned} DtN_1(x_0)g(t) &= \frac{1}{c(x_0)} \left(g + 2 \sum_{k=1}^{\lfloor \frac{c_2 t}{2x_0} \rfloor} r^k g(t - 2kx_0/c_2) \right) \\ DtN_2(x_0)g(t) &= \frac{1}{c(x_0)} \left(g + 2 \sum_{k=1}^{\lfloor \frac{-c_1 t}{2x_0} \rfloor} r^k g(t - 2kx_0/c_1) \right) \end{aligned}$$

where $\lfloor x \rfloor = 0$ for $x \leq 0$. Using the factorization above, we define the transmission operators \mathcal{B}_i^\pm in the Schwarz algorithm for the wave equation by

$$(9) \quad \mathcal{B}_i^- := \partial_x - DtN_1(a_i)\partial_t, \quad \mathcal{B}_i^+ := \partial_x + DtN_2(a_{i+1})\partial_t.$$

THEOREM 3.3 (CONVERGENCE IN I STEPS GLOBALLY IN TIME). *The non-overlapping Schwarz algorithm for the wave equation (6) with transmission conditions (9) converges on all time intervals $t \in [0, T)$ in I iterations where I denotes the number of subdomains.*

But here the operators turn out to be non-local, as one can see from (8). Using the finite speed of propagation, more practical local operators can be obtained which lead to optimal performance. Since the speed of propagation is finite, not every subdomain needs to know what happens on all the others. Neighboring information suffices :

THEOREM 3.4 (CONVERGENCE IN I STEPS WITH LOCAL TRANSMISSION CONDITIONS). *For $t \in [0, T_p]$ the Schwarz algorithm (6) with local transmission operators*

$$(10) \quad \mathcal{B}_i^- := \partial_x - \frac{1}{c^-(a_i)} \partial_t, \quad \mathcal{B}_i^+ := \partial_x + \frac{1}{c^+(a_{i+1})} \partial_t$$

converges in I iterations where I denotes the number of subdomains, if $T_p < \min_{1 < i \leq I} \frac{2|a_i|}{c(a_i)}$.

It is now possible to reduce the number of iterations below I for I subdomains.

THEOREM 3.5 (CONVERGENCE IN 2 STEPS FOR $t \in [0, T_o]$). *Given $t \in [0, T_o]$ the Schwarz algorithm (6) with local transmission conditions (10) and any discontinuity aligned between subdomains converges in 2 iterations independent of the number of subdomains if*

$$T_o < \min_{1 < i < I} \frac{|a_{i+1} - a_i|}{c((a_{i+1} - a_i)/2)}.$$

4 The Discrete Algorithm

In the numerical domain Ω_i , the time and space steps are Δt_i and Δx_i . We use a vertex centered finite volume method, in order to take the boundary conditions into account. In the interior, it leads to the leap-frog scheme. The discrete transmission condition is obtained by integration of the equation on half a cell, using the boundary condition. It allows for different time scales on different domains. For instance on the left boundary of Ω_i , we get

$$\begin{aligned} \frac{1}{c^-(a_i)} \frac{U_{0,n+1}^{k+1} - U_{0,n-1}^{k+1}}{2\Delta t_i} - \frac{U_{1,n}^{k+1} - U_{0,n}^{k+1}}{\Delta x_i} + \frac{\Delta x_i}{2(c^+(a_i))^2} \frac{U_{0,n+1}^{k+1} - 2U_{0,n}^{k+1} + U_{0,n-1}^{k+1}}{\Delta t_i^2} = \\ \frac{1}{c^-(a_i)} \frac{V_{0,n+1}^k - V_{0,n-1}^k}{2\Delta t_i} - \frac{V_{0,n}^k - V_{-1,n}^k}{\Delta x_{i-1}} - \frac{\Delta x_{i-1}}{2(c^-(a_i))^2} \frac{V_{0,n+1}^k - 2V_{0,n}^k + V_{0,n-1}^k}{\Delta t_i^2} \end{aligned}$$

which is a Lax-Wendroff discretization of the local transmission conditions. Using discrete energy estimates, we have the following

THEOREM 4.1. *Suppose the velocity is continuous at the interfaces. Then the discrete algorithm is stable and convergent in a particular energy norm.*

Recall we recommend as best strategy to align the numerical interfaces with the physical interfaces. We have a partial result in the case of two subdomains, when the time step Δt is constant on the whole domain :

THEOREM 4.2. *Let $(U_{j,n}^p, V_{j,n}^p)$, be the solution to the discrete algorithm with zero data. If $(c_1 - c_2)(c_1 \frac{\Delta t}{\Delta x_1} - c_2 \frac{\Delta t}{\Delta x_2}) \geq 0$, there exists a positive constant C such that for $\eta \Delta t$ sufficiently small, one has for any p*

$$\max(\|U_{j,n}^p\|, \|V_{j,n}^p\|) \leq (1 - C\eta\Delta t)^{\lfloor \frac{p}{2} \rfloor} \max(\|U_{j,n}^0\|, \|V_{j,n}^0\|)$$

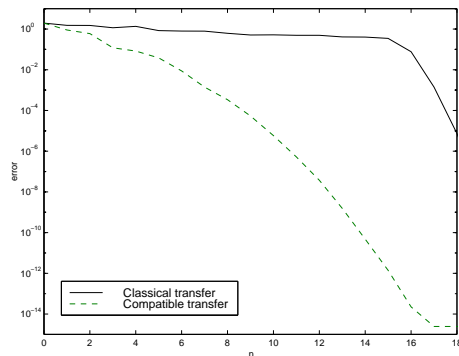
where $\|\cdot\|$ denotes the discrete l^2 -norm in time and space, with the weight $\exp -\eta\Delta t$ in time.

This result is obtained through discrete Laplace transform in time, as in normal mode analysis. Note that the GKS stability of the discrete value problem in each subdomain is violated. Numerical experiments confirm our analysis.

5 The m-dimensional case

We sketch the strategy in the homogeneous case, for two domains $\Omega_1 = (-\infty, L] \times \mathbb{R}^{m-1} \times [0, T]$ and $\Omega_2 = (0, \infty) \times \mathbb{R}^{m-1} \times [0, T]$. First of all, due to the finite speed of propagation, the overlapping algorithm corresponding to the classical approach (2) converges in a finite number of iterations. However it is not optimal. We introduce the general transmission conditions

$$\begin{aligned} (\partial x_1 + \Lambda_v(\partial t, \partial x'))v^{k+1}(L, x', t) &= (\partial x_1 + \Lambda_v(\partial t, \partial x'))w^k(L, x', t) \\ (\partial x_1 + \Lambda_w(\partial t, \partial x'))w^{k+1}(0, x', t) &= (\partial x_1 + \Lambda_w(\partial t, \partial x'))v^k(0, x', t) \end{aligned}$$


 FIG. 1. *Approximation for transverse waves*

where $x' = (x_2, \dots, x_m)$. Through a Laplace transform in time, and an $(m - 1)$ -Fourier transform in x' , we can mimic the analysis in (3) and define the rate of convergence

$$(11) \quad \rho(s) = \frac{-\sqrt{\frac{s^2}{c^2} + |\mathbf{k}|^2} + \lambda_v(s, \mathbf{k})}{\sqrt{\frac{s^2}{c^2} + |\mathbf{k}|^2} + \lambda_v(s, \mathbf{k})} \cdot \frac{\sqrt{\frac{s^2}{c^2} + |\mathbf{k}|^2} + \lambda_w(s, \mathbf{k})}{-\sqrt{\frac{s^2}{c^2} + |\mathbf{k}|^2} + \lambda_w(s, \mathbf{k})} e^{-(\frac{s^2}{c^2} + |\mathbf{k}|^2)L}$$

where \mathbf{k} is the Fourier variable corresponding to $x' = (x_2, \dots, x_m)$. As in one dimension, the algorithm converges in two iterations if and only if

$$\lambda_v(s, \mathbf{k}) = -\lambda_w(s, \mathbf{k}) = \sqrt{\frac{s^2}{c^2} + |\mathbf{k}|^2}$$

which corresponds to imposing the transparent boundary condition as transmission condition. Again, this strategy makes the overlap unnecessary. In more than one dimension, the operator Λ_v is pseudo differential in time and in the transverse variables. As in the theory of absorbing boundary conditions, we need to approximate it by a differential operator. Figure 5 shows the improvement when Λ_v is replaced by the 1-D operator ∂t .

We will present optimized transmission conditions, based on optimization of the approximately absorbing boundary conditions with respect to the natural measure, the convergence rate.

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