Low-Rank Properties in Schur Complements of Discretized Helmholtz Equations

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Abstract

We study numerically the ϵ-rank of subblocks arising in Schur complement matrices of discretized three dimensional Helmholtz problems. A small ϵ-rank is the key ingredient for $\mathcal{H}$-matrix techniques, and while Laplace-like problems have this property, the ϵ-rank for the Helmholtz case is growing with increasing wave number. We study here the growth rate in the case of a heterogeneous Helmholtz problem with a checker board wave speed distribution, and compare it to the constant wave number case.

Keywords: Heterogeneous Helmholtz Equation, Schur Complements, ϵ-Rank

1 Introduction

In contrast to Laplace-like problems, effective iterative methods for solving Helmholtz problems are rare and expensive, for an overview, see [4, 6] and references therein. Direct methods are thus attractive for such problems, and a significant effort has gone into combining reordering techniques and LU-decompositions using multifrontal methods, where additional savings are sought through compression techniques using low rank properties of subblocks arising in the factorization. While there are some theoretical results on the potential low rank property for the constant wave number case [1–3], and also comprehensive numerical experiments [5], much less is known about the case of variable wave numbers. We study here numerically the specific case of a checker board wave speed distribution.

2 Problem Setting

We study numerically the Helmholtz equation

$$\begin{aligned} \Delta u + \tfrac{(2\pi\nu)^2}{V^2} u &= \delta(\overline{r} - \overline{r}_s) f \quad \text{in } \Omega := (0, L)^3, \\ u &= 0 \qquad\qquad\quad\ \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where ν is the frequency, V is the velocity, $\overline{r}_s$ are the coordinates of the source f, and the wave number is $k := \frac{(2\pi\nu)}{V(x,y,z)}$, chosen such that we have a well posed problem with Dirichlet conditions.

We discretize the Helmholtz equation (1) using a standard seven point finite difference discretization with mesh spacing $h := \frac{1}{n}$, which leads to a sparse linear system $A\mathbf{u} = \mathbf{f}$. If we partition the system matrix into a first block A_1 corresponding to the first x-y plane of discretization points, and denote the remaining diagonal block by A_2, the linear system becomes

$$\begin{pmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{pmatrix}. \tag{2}$$

We are interested in the Schur complement matrix $S := A_1 - A_{12}A_2^{-1}A_{21} \in \mathbb{C}^{N\times N}$, $N = (n-1)^2$. We apply the singular value decomposition to the matrix subblock $S_m := S(1 : m, N-m+1 : N)$ and study the decay of its singular values σ_j as a function of h and k. We will compute for a large subblock, $m = \frac{(n-1)^2-1}{2}$, its ϵ-rank, which is defined as the smallest number r_ϵ such that $\frac{\sigma_j}{\sigma_1} < \epsilon$ for all $j > r_\epsilon$.

3 Numerical Experiments

We simulate on the cube with dimension $L = 1200m$, and frequency $\nu = 4Hz, 8Hz, 16Hz$ using the corresponding number of grid points $n = 20, 40, 80$ with velocity $V_1(x, y, z) = 2400m/s$ for 10 points per wavelength (ppw), and velocity $V_1(x, y, z) = 1200m/s$ for 5 ppw. We see in Table 1 that with 5 ppw the ϵ-rank is substantially larger than with 10 ppw. We then consider a Checker Board (ChB) type case with five fields in each direction and velocity $V_1 = 2400m/s$ in the white fields using 10 ppw, and $V_2 = \frac{V_1}{c}$ in the black fields using $10/c$ ppw, where c is a contrast parameter. We see in Table 1 that the ϵ-rank grows in the ChB case with $c = 2$ for larger ϵ more like for ν const and 10ppw, while for small ϵ the growth is more like for ν const and 5 ppw. For $c = 4$ the ϵ-rank is then much larger, while for $c = 8$ it suddenly drops, probably because the waves in the black fields are now not at all resolved any more with 1.25 ppw. We

	ϵ	1e-2	1e-4	1e-6	1e-2	1e-4	1e-6	1e-2	1e-4	1e-6	1e-2	1e-4	1e-6	1e-2	1e-4	1e-6
n	ν	V=const, ppw=10			V=const, ppw=5			V=ChB, $c=2$			V=ChB, $c=4$			V=ChB, $c=8$		
20	4	27	46	59	19	50	67	34	55	73	50	102	138	20	38	50
40	8	6	68	110	96	154	198	89	148	198	108	227	326	44	66	91
80	16	32	180	276	84	417	547	35	296	449	94	497	749	86	124	176

Table 1: ϵ-rank for a large matrix subblock S_m for a constant velocity and checker board cases with contrast c and 10 ppw in the white fields, and under-resolution in the black fields

	ϵ	1e-2	1e-4	1e-6	1e-2	1e-4	1e-6	1e-2	1e-4	1e-6	1e-2	1e-4	1e-6	1e-2	1e-4	1e-6
n	ν	V=const, ppw=10			V=const, ppw=20			V=ChB, $c=1/2$			V=ChB, $c=1/4$			V=ChB, $c=1/8$		
20	4	27	46	59	24	44	56	27	47	62	9	29	47	24	46	58
40	8	6	68	110	53	94	120	53	98	126	50	88	121	55	103	127
80	16	32	180	276	110	201	249	125	210	292	113	201	280	111	199	278

Table 2: ϵ-rank in 3d for a large matrix subblock S_m for a constant velocity and checker board cases with contrast c and 10 ppw in the white fields, and over-resolution in the back fields

next show in Table 2 the corresponding results for the over-resolved case. We see that now in all cases the ϵ-rank is growing comparably to the constant wave number case at resolution of 10 ppw in the first column (which is the same as in Table 1), which indicates that it is the lower resolution of 10 ppw in the white fields which dictates the ϵ-rank growth. The growth is $O(k^{\frac{4}{3}})$, like in the constant wave number case, see [5], where it was also noticed that increasing the resolution does not influence the ϵ-rank once the waves are well resolved.

4 Conclusion

We studied numerically how the ϵ-rank is growing in subblocks of Schur complement matrices arising from discretized heterogeneous Helmholtz problems. We found that the growth for a checker board situation is comparable to the homogeneous case, provided the resolution in all fields of the checker board is at least as good as the resolution of the homogeneous case, and there are no under-resolved fields of the checker board. The growth in 3d for a planar Schur complement is then also $O(k^{\frac{4}{3}})$, like in the constant and random wave number case studied in [5].

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