Schwarz Domain Decomposition Methods in the Course of Time

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Dirichlet Principle

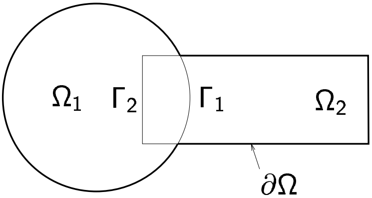
Dirichlet principle: The solution of Laplace's equation $\Delta u = 0$ on a bounded domain Ω with Dirichlet boundary conditions $u = g$ on $\partial\Omega$ is the infimum of the Dirichlet integral $\int_\Omega |\nabla v|^2$ over all functions v satisfying the boundary conditions, $v = g$ on $\partial\Omega$.

H. A. Schwarz 1869:

"Die unter dem Namen Dirichletsches Princip bekannte Schlussweise, welche in gewissem Sinne als das Fundament des von Riemann entwickelten Zweiges der Theorie der analytischen Functionen angesehen werden muss, unterliegt, wie jetzt wohl allgemein zugestanden wird, hinsichtlich der Strenge sehr begründeten Einwendungen, deren vollständige Entfernung meines Wissens den Anstrengungen der Mathematiker bisher nicht gelungen ist".

Classical Alternating Schwarz Method

Schwarz invents a method to proof that the infimum is attained: for a general domain $\Omega := \Omega_1 \cup \Omega_2$:



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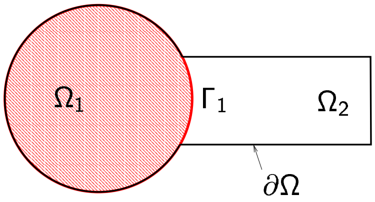

$$\begin{aligned} \Delta u_1^1 &= 0 && \text{in } \Omega_1 \\ u_1^1 &= g && \text{on } \partial\Omega \cap \overline{\Omega}_1 \\ u_1^1 &= u_2^0 && \text{on } \Gamma_1 \end{aligned}$$

solve on the disk $u_2^0 = 0$

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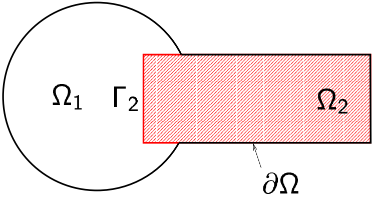

$$\begin{aligned} \Delta u_2^1 &= 0 && \text{in } \Omega_2 \\ u_2^1 &= g && \text{on } \partial\Omega \cap \overline{\Omega}_2 \\ u_2^1 &= u_1^1 && \text{on } \Gamma_2 \end{aligned}$$

solve on the rectangle

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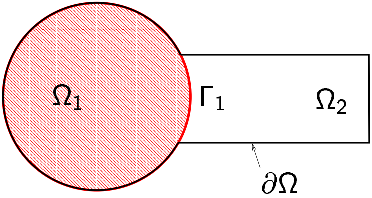

$$
\begin{aligned}
\Delta u_1^2 &= 0 && \text{in } \Omega_1\\
u_1^2 &= g && \text{on } \partial\Omega \cap \overline{\Omega}_1\\
u_1^2 &= u_2^1 && \text{on } \Gamma_1
\end{aligned}
$$

solve on the disk

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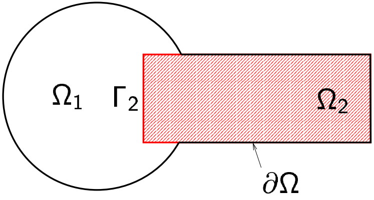

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solve on the rectangle

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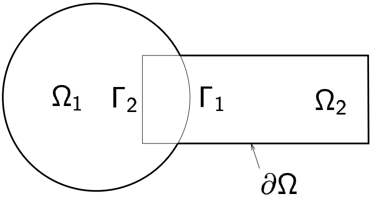

$$\begin{aligned} \Delta u_1^n &= 0 && \text{in } \Omega_1 \\ u_1^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_1 \\ u_1^n &= u_2^{n-1} && \text{on } \Gamma_1 \end{aligned} \qquad \begin{aligned} \Delta u_2^n &= 0 && \text{in } \Omega_2 \\ u_2^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_2 \\ u_2^n &= u_1^n && \text{on } \Gamma_2 \end{aligned}$$

solve on the disk solve on the rectangle

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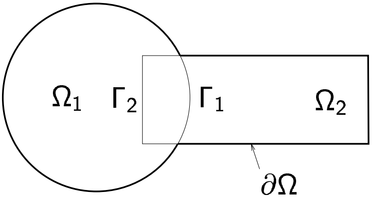

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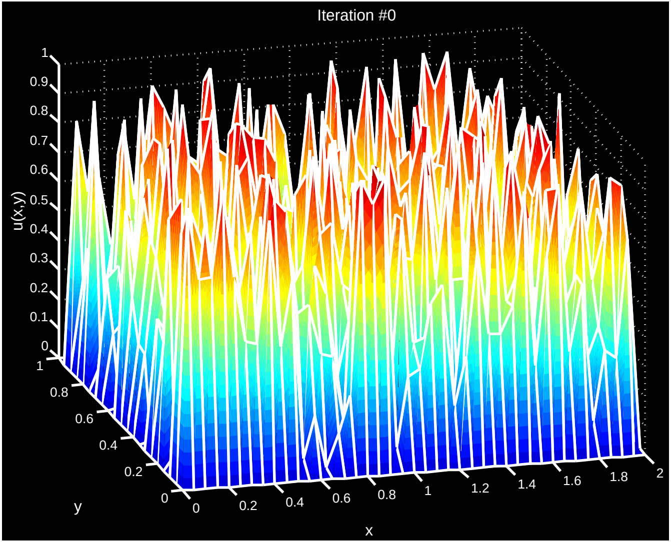
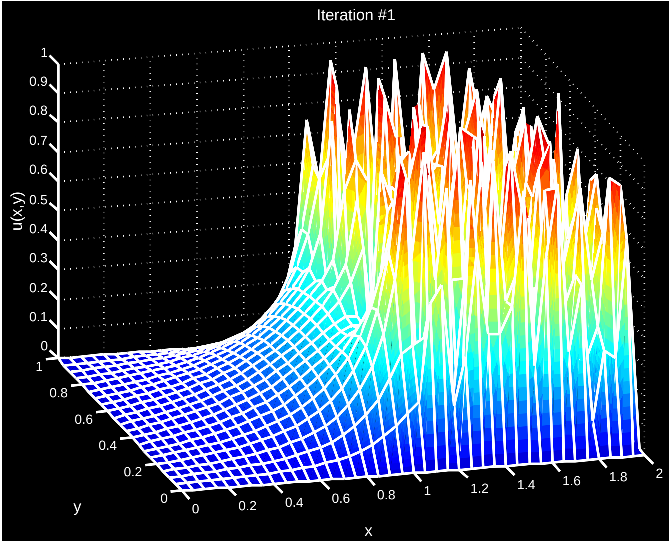
solve on the disk — solve on the rectangle

- Schwarz proved convergence in 1869 using the maximum principle.

Example how Error Decreases in Schwarz Method

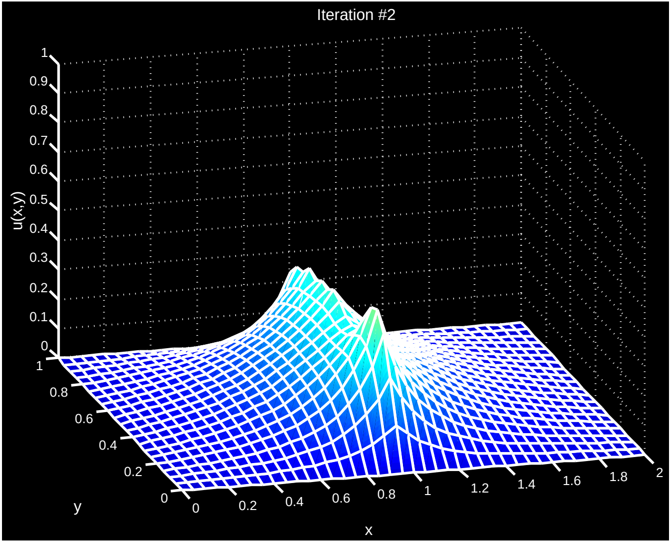


Iteration 1



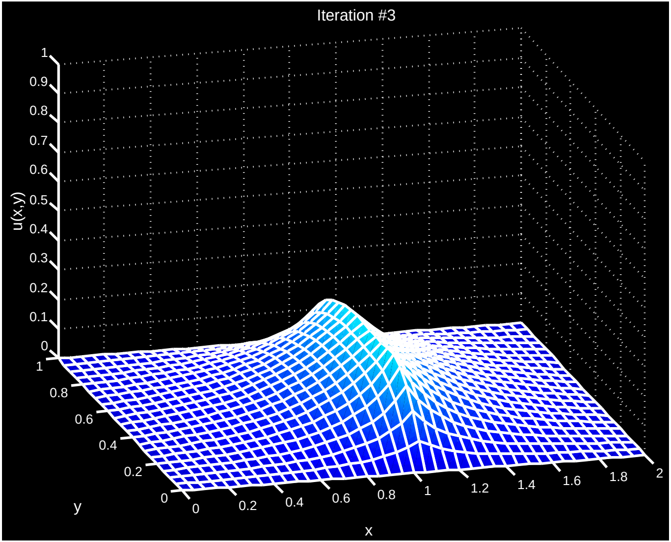
solve on the left subdomain

Iteration 2



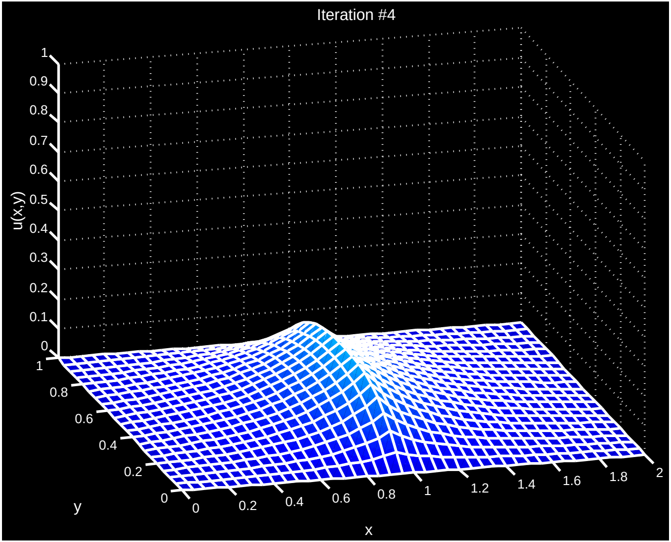
solve on the right subdomain

Iteration 3



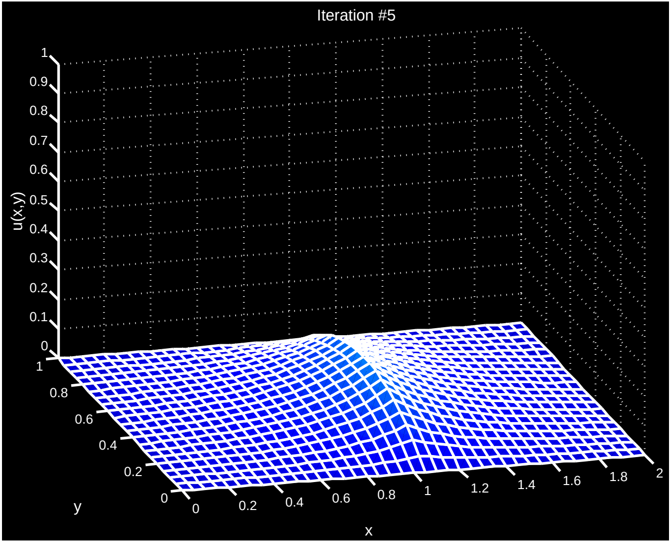
solve on the left subdomain

Iteration 4



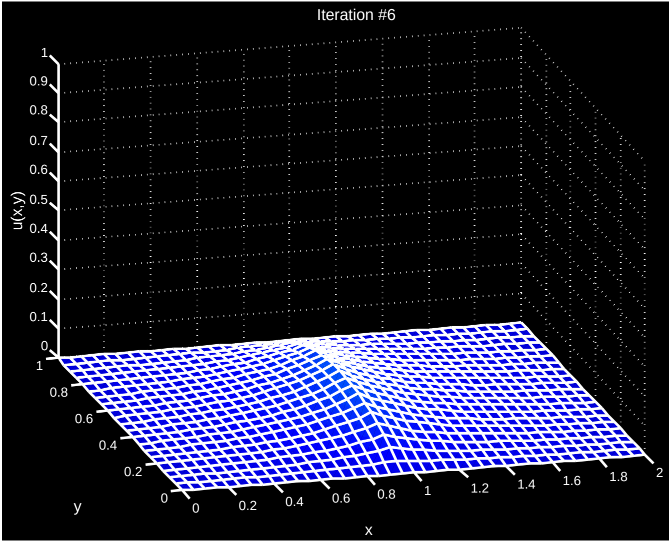
solve on the right subdomain

Iteration 5



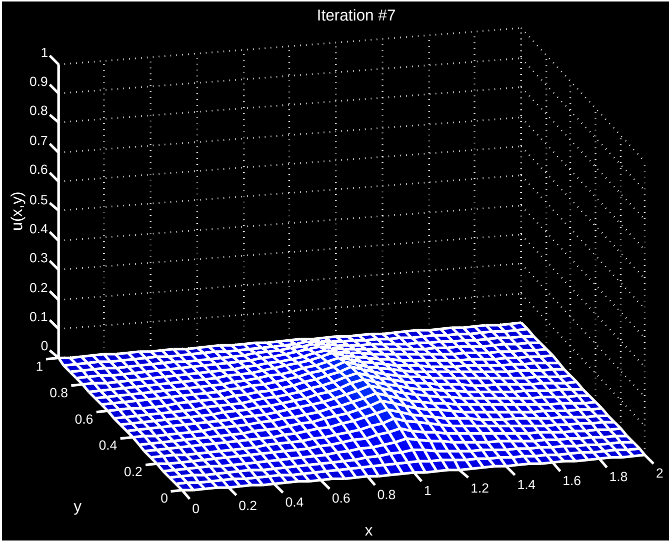
solve on the left subdomain

Iteration 6



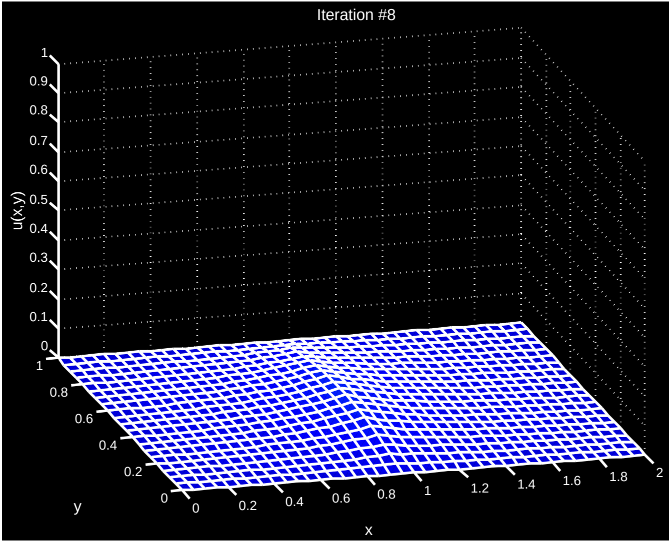
solve on the right subdomain

Iteration 7



solve on the left subdomain

Iteration 8



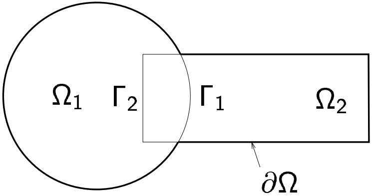
solve on the right subdomain

Classical Parallel Schwarz Method

P-L. Lions 1988:

> *The final extension we wish to consider concerns "parallel" versions of the Schwarz alternating method* $\ldots, \ldots, u_i^{n+1}$ *is solution of* $-\Delta u_i^{n+1} = f$ *in* Ω_i *and* $u_i^{n+1} = u_j^n$ *on* $\partial\Omega_i \cap \Omega_j$.

$$\mathcal{L}u = f \text{ in } \Omega$$



$$\begin{aligned} \mathcal{L}u_1^{n+1} &= f, \text{ in } \Omega_1 \\ u_1^{n+1} &= u_2^n, \text{ on } \Gamma_1 \end{aligned} \qquad \begin{aligned} \mathcal{L}u_2^{n+1} &= f, \text{ in } \Omega_2 \\ u_2^{n+1} &= u_1^n, \text{ on } \Gamma_2 \end{aligned}$$

The Multiplicative Schwarz Method (MS)

The discretizd PDE leads to the linear system

$$A\mathbf{u} = \mathbf{f}.$$

With the restriction operators

$$R_1 = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \end{bmatrix} R_2 = \begin{bmatrix} & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

and $A_j = R_j A R_j^T$ the multiplicative Schwarz method is

$$\begin{aligned} \mathbf{u}^{n+\frac{1}{2}} &= \mathbf{u}^n + R_1^T A_1^{-1} R_1(\mathbf{f} - A\mathbf{u}^n) \\ \mathbf{u}^{n+1} &= \mathbf{u}^{n+\frac{1}{2}} + R_2^T A_2^{-1} R_2(\mathbf{f} - A\mathbf{u}^{n+\frac{1}{2}}). \end{aligned}$$

Questions:

- Is MS a discretization of a continuous Schwarz method?
- How is the algebraic overlap related to the physical one?

Relation with Alternating Schwarz

If the R_j are non-overlapping, and we partition accordingly

$$A = \begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix}, \qquad \mathbf{f} = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{pmatrix},$$

we obtain from the first relation of MS, i.e.

$$\mathbf{u}^{n+\frac{1}{2}} = \mathbf{u}^n + R_1^T A_1^{-1} R_1(\mathbf{f} - A\mathbf{u}^n)$$

an interesting cancellation:

$$\begin{aligned}
R_1(\mathbf{f} - A\mathbf{u}^n) &= \mathbf{f}_1 - A_1\mathbf{u}_1^n - A_{12}\mathbf{u}_2^n \\
A_1^{-1}R_1(\mathbf{f} - A\mathbf{u}^n) &= A_1^{-1}(\mathbf{f}_1 - A_{12}\mathbf{u}_2^n) - \mathbf{u}_1^n \\
\begin{pmatrix} \mathbf{u}_1^{n+\frac{1}{2}} \\ \mathbf{u}_2^{n+\frac{1}{2}} \end{pmatrix} &= \begin{pmatrix} \mathbf{u}_1^n \\ \mathbf{u}_2^n \end{pmatrix} + \begin{pmatrix} A_1^{-1}(\mathbf{f}_1 - A_{12}\mathbf{u}_2^n) - \mathbf{u}_1^n \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} A_1^{-1}(\mathbf{f}_1 - A_{12}\mathbf{u}_2^n) \\ \mathbf{u}_2^n \end{pmatrix}
\end{aligned}$$

Relation with Alternating Schwarz

Similarly, from the second relation of MS, i.e.

$$\mathbf{u}^{n+1} = \mathbf{u}^{n+\frac{1}{2}} + R_2^T A_2^{-1} R_2(\mathbf{f} - A\mathbf{u}^{n+\frac{1}{2}})$$

we obtain

$$\begin{pmatrix} \mathbf{u}_1^{n+1} \\ \mathbf{u}_2^{n+1} \end{pmatrix} = \begin{pmatrix} A_1^{-1}(\mathbf{f}_1 - A_{12}\mathbf{u}_2^n) \\ A_2^{-1}(\mathbf{f}_2 - A_{21}\mathbf{u}_1^{n+1}) \end{pmatrix},$$

which can be rewritten in the equivalent form

$$A_1\mathbf{u}_1^{n+1} = \mathbf{f}_1 - A_{12}\mathbf{u}_2^n, \qquad A_2\mathbf{u}_2^{n+1} = \mathbf{f}_2 - A_{21}\mathbf{u}_1^{n+1}$$

and is therefore a discretization of the alternating Schwarz method from 1869,

$$\begin{aligned} \mathcal{L}u_1^{n+1} &= f, \text{ in } \Omega_1 \\ u_1^{n+1} &= u_2^n, \text{ on } \Gamma_1 \end{aligned} \qquad \begin{aligned} \mathcal{L}u_2^{n+1} &= f, \text{ in } \Omega_2 \\ u_2^{n+1} &= u_1^{n+1}, \text{ on } \Gamma_2 \end{aligned}$$

MS is also a block Gauss Seidel method

MS is also equivalent to a block Gauss Seidel method, since

$$A_1\mathbf{u}_1^{n+1} = \mathbf{f}_1 - A_{12}\mathbf{u}_2^n, \quad A_2\mathbf{u}_2^{n+1} = \mathbf{f}_2 - A_{21}\mathbf{u}_1^{n+1}$$

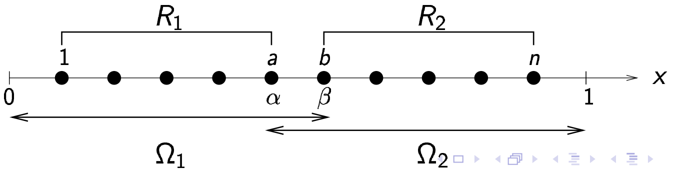
leads in matrix form to the iteration

$$\begin{bmatrix} A_1 & 0 \\ A_{21} & A_2 \end{bmatrix} \begin{pmatrix} \mathbf{u}_1^{n+1} \\ \mathbf{u}_2^{n+1} \end{pmatrix} = \begin{bmatrix} 0 & -A_{12} \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \mathbf{u}_1^n \\ \mathbf{u}_2^n \end{pmatrix} + \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{pmatrix}$$

So why the complicated R_j notation ?

- With R_j, one can also use overlapping blocks.
- With R_j, there is a global approximate solution $\mathbf{u}^n$.

Note that even the algebraically non-overlapping case above implies overlap at the PDE level:



The Additive Schwarz Method (AS)

M. Drjya and O. Widlund 1989:

> *The basic idea behind the additive form of the algorithm is to work with the simplest possible polynomial in the projections. Therefore the equation* $(P_1 + P_2 + \ldots + P_N)u_h = g_h'$ *is solved by an iterative method.*

Using the same notation as before, the preconditioned system is

$$(R_1^T A_1^{-1} R_1 + R_2^T A_2^{-1} R_2)A\mathbf{u} = (R_1^T A_1^{-1} R_1 + R_2^T A_2^{-1} R_2)\mathbf{f}$$

Writing this as a stationary iterative method yields

$$\mathbf{u}^n = \mathbf{u}^{n-1} + (R_1^T A_1^{-1} R_1 + R_2^T A_2^{-1} R_2)(\mathbf{f} - A\mathbf{u}^{n-1})$$

Question: Is AS equivalent to a discretization of Lions parallel Schwarz method ?

Algebraically non-overlapping case

If the R_j are non-overlapping, we obtain now

$$\begin{pmatrix} \mathbf{u}_1^{n+1} \\ \mathbf{u}_2^{n+1} \end{pmatrix} = \begin{pmatrix} A_1^{-1}(\mathbf{f}_1 - A_{12}\mathbf{u}_2^n) \\ A_2^{-1}(\mathbf{f}_2 - A_{21}\mathbf{u}_1^n) \end{pmatrix},$$

which can be rewritten in the equivalent form

$$A_1\mathbf{u}_1^{n+1} = \mathbf{f}_1 - A_{12}\mathbf{u}_2^n, \qquad A_2\mathbf{u}_2^{n+1} = \mathbf{f}_2 - A_{21}\mathbf{u}_1^n.$$

This a discretization of Lions parallel Schwarz method from 1988,

$$\begin{array}{rcl} \mathcal{L}u_1^{n+1} & = & f, \text{ in } \Omega_1 \\ u_1^{n+1} & = & u_2^n, \text{ on } \Gamma_1 \end{array} \qquad \begin{array}{rcl} \mathcal{L}u_2^{n+1} & = & f, \text{ in } \Omega_2 \\ u_2^{n+1} & = & u_1^n, \text{ on } \Gamma_2 \end{array}$$

In the algebraically non-overlapping case, AS is also equivalent to a block Jacobi method,

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{pmatrix} \mathbf{u}_1^{n+1} \\ \mathbf{u}_2^{n+1} \end{pmatrix} = \begin{bmatrix} 0 & -A_{12} \\ -A_{21} & 0 \end{bmatrix} \begin{pmatrix} \mathbf{u}_1^n \\ \mathbf{u}_2^n \end{pmatrix} + \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{pmatrix}$$

What happens if the R_j overlap ?

If the R_j overlap, the cancellation is more complicated:

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \begin{pmatrix} \boxed{A_1^{-1}(\mathbf{f}_1 - A_{12}\mathbf{u}_2^n) - \mathbf{u}_1^n} \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \boxed{A_2^{-1}(\mathbf{f}_2 - A_{21}\mathbf{u}_1^n) - \mathbf{u}_2^n} \end{pmatrix}$$

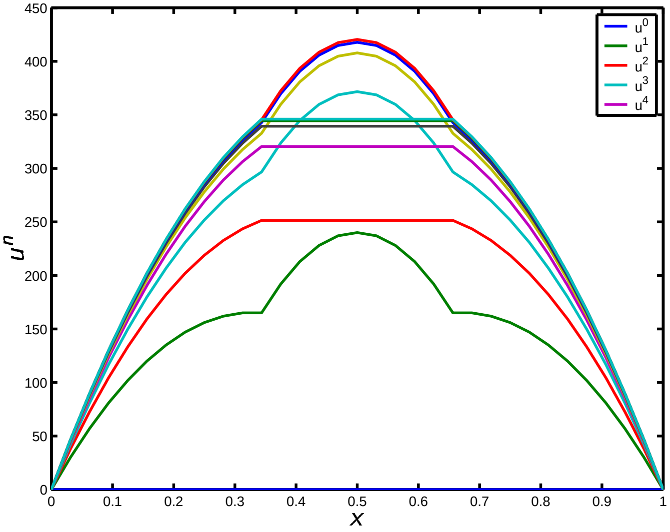
In the overlap, the current iterate is subtracted twice, and a new approximation from the left and right solve is added.

Remarks:

- Method does not converge in the overlap: the spectral radius of the AS iteration operator equals 1 for two subdomains.
- The method converges outside of the overlap for two subdomains.
- For more than two subdomains with cross points the method diverges everywhere.

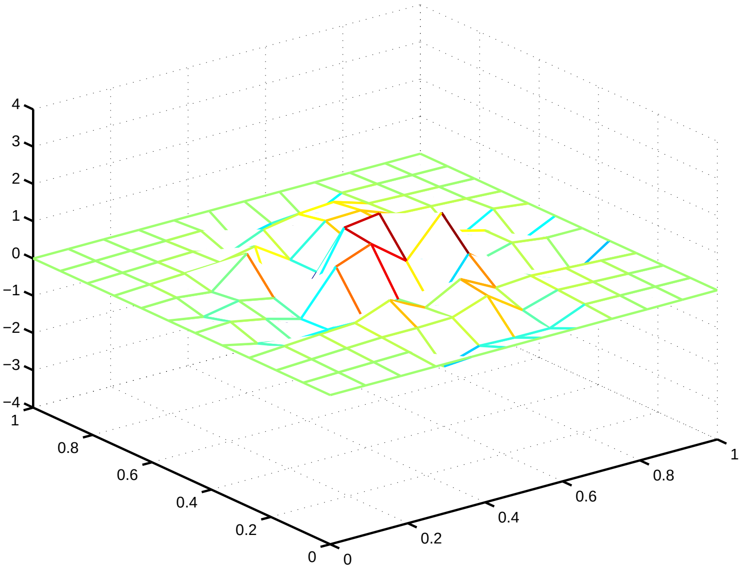
AS is thus not equivalent to a discretization of Lions parallel Schwarz method for more than minimal physical overlap.

An Example for 2 Subdomains: $-\frac{\partial^2 u}{\partial x^2}=1$

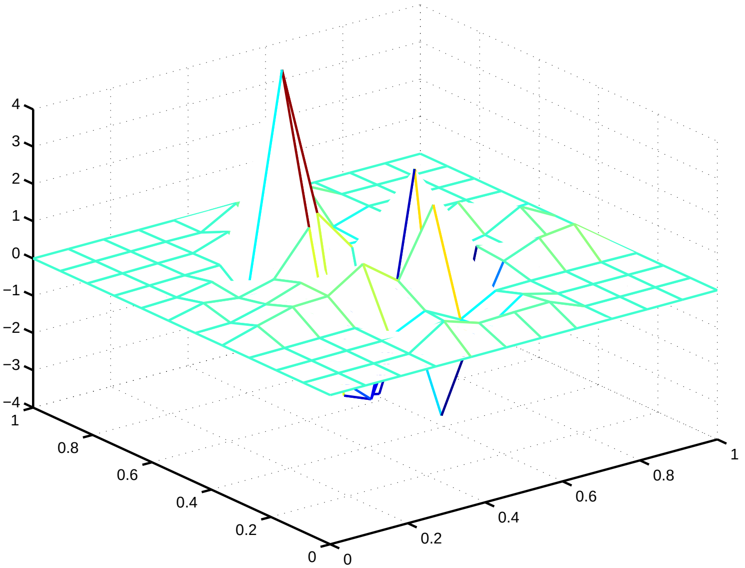


Method does not converge in the overlap
Method converges outside the overlap.

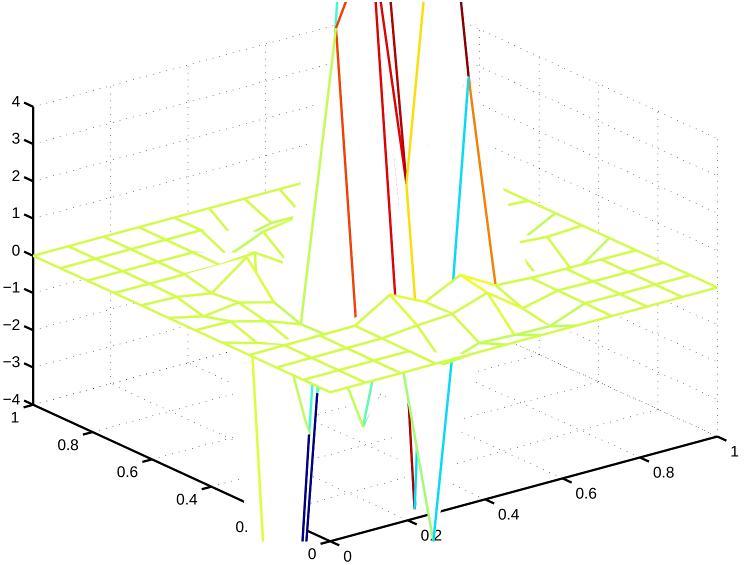
An Example for four Subdomains: initial error



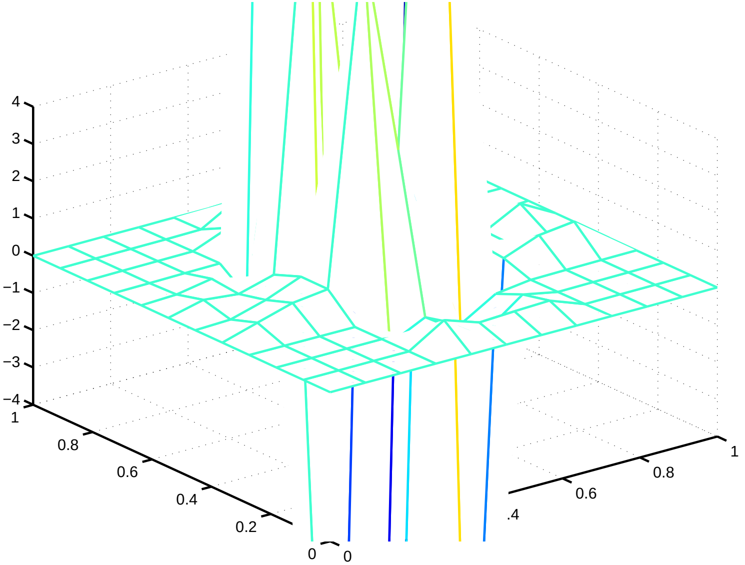
Error at iteration 2



Error at iteration 3



Error at iteration 4: method diverges



Krylov Acceleration

Krylov acceleration of the iterative method

$$\mathbf{u}^n = \mathbf{u}^{n-1} + M_{AS}^{-1}(\mathbf{f} - A\mathbf{u}^{n-1})$$

$\Leftrightarrow$ solving with a Krylov method the preconditioned system

$$M_{AS}^{-1}A\mathbf{u} = M_{AS}^{-1}\mathbf{f}.$$

Theorem (M. Drjya and O. Widlund 1989:)

The condition number of the additive Schwarz preconditioned system satisfies

$$\kappa(M_{AS}^{-1}A) \leq C\left(1 + \frac{H}{\delta}\right).$$

Here δ is the overlap and H is the characteristic coarse mesh size of a coarse grid correction

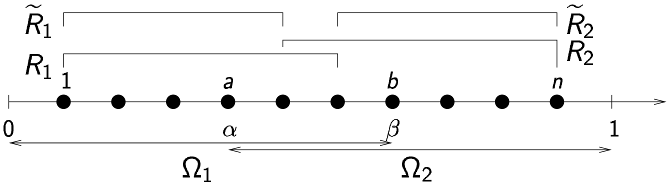
$$M_{AS}^{-1} := \sum_{j=1}^{n} R_j^T A_j^{-1} R_j + R_0^T A_0^{-1} R_0$$

Restricted Additive Schwarz (RAS)

X. Cai and M. Sarkis 1998:

> *While working on an AS/GMRES algorithm in an Euler simulation, we removed part of the communication routine and surprisingly the "then AS" method converged faster in both terms of iteration counts and CPU time.*

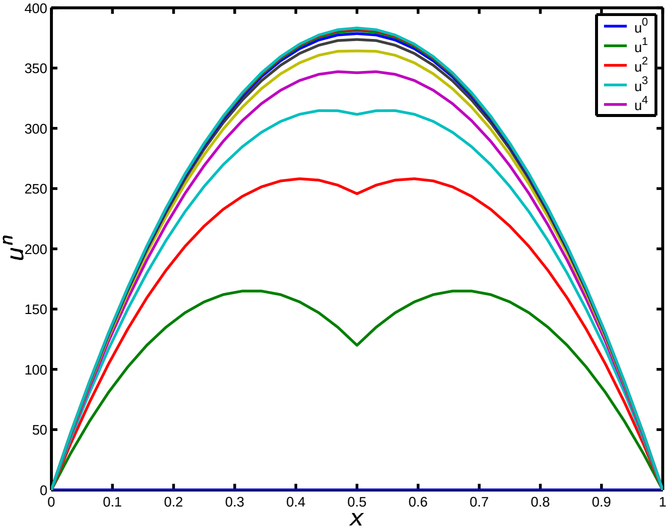
$$u^{n+1} = u^n + (\widetilde{R}_1^T A_1^{-1} R_1 + \widetilde{R}_2^T A_2^{-1} R_2)(f - Au^n)$$



Remarks:

- RAS is equivalent to a discretization of Lions parallel Schwarz method (Efstathiou, G. 2003, G. 2008)
- the preconditioner is **non symmetric**, even if A_j is symmetric

An Example for 2 Subdomains



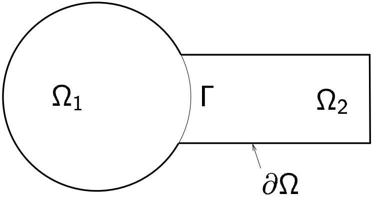
RAS corrects problem of AS in the overlap $\Longrightarrow$ convergence

Problems of classical Schwarz: Overlap Necessary

P-L. Lions 1990:

> *However, the Schwarz method requires that the subdomains overlap, and this may be a severe restriction - without speaking of the obvious or intuitive waste of efforts in the region shared by the subdomains.*

$$\mathcal{L}u = f \text{ in } \Omega$$

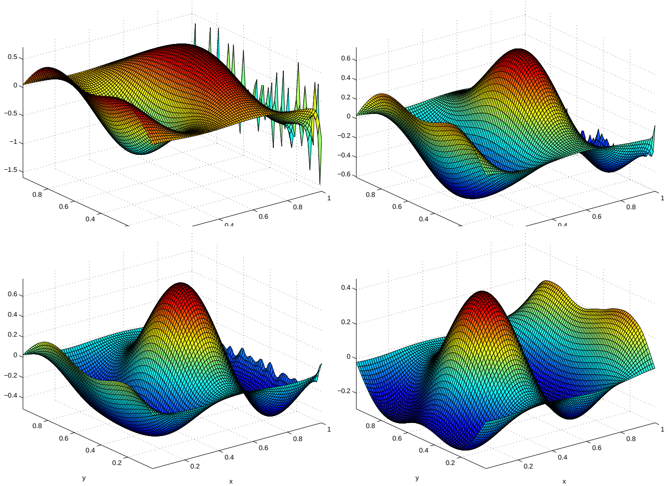


$$\begin{aligned} \mathcal{L}u_1^n &= f \quad \text{in } \Omega_1 \\ (\partial_{n_1}+p_1)u_1^n &= (\partial_{n_1}+p_1)u_2^{n-1} \text{ on } \Gamma \end{aligned} \qquad \begin{aligned} \mathcal{L}u_2^n &= f \quad \text{in } \Omega_2 \\ (\partial_{n_2}+p_2)u_2^n &= (\partial_{n_2}+p_2)u_1^{n} \text{ on } \Gamma \end{aligned}$$

P-L. Lions 1990:

> *First of all, it is possible to replace the constants in the Robin conditions by two proportional functions on the interface, or even by local or nonlocal operators.*

Other Problem: Lack of Convergence



B. Després 1990:

> *L'objectif de ce travail est, après construction d'une méthode de décomposition de domaine adaptée au problème de Helmholtz, d'en démontrer la convergence.*

Further Problem: Convergence Speed

T. Hagstrom, R. P. Tewarson and A. Jazcilevich 1988: Numerical experiments on a domain decomposition algorithm for nonlinear elliptic boundary value problems

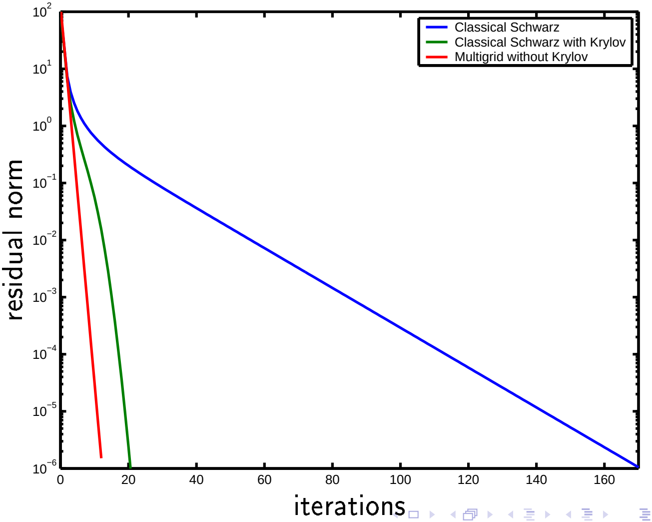
> *In general, [the coefficients in the Robin transmission conditions] may be operators in an appropriate space of function on the boundary. Indeed, we advocate the use of nonlocal conditions.*

W.-P. Tang 1992: Generalized Schwarz Splittings

> *In this paper, a new coupling between the overlap[ping] subregions is identified. If a successful coupling is chosen, a fast convergence of the alternating process can be achieved without a large overlap.*

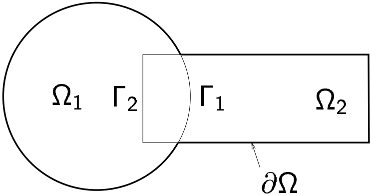
Comparison of Classical Schwarz with Multigrid

Comparison of MS with two subdomains as an iterative solver and a preconditioner for a Krylov method, with a standard multigrid solver:



Continuous Optimized Schwarz Methods

$\mathcal{L}u = f$ in Ω



Instead of the classical alternating Schwarz method

$$\begin{aligned} \mathcal{L}u_1^n &= f, \text{ in } \Omega_1 \\ u_1^n &= u_2^{n-1}, \text{ on } \Gamma_1 \end{aligned} \qquad \begin{aligned} \mathcal{L}u_2^n &= f, \text{ in } \Omega_2 \\ u_2^n &= u_1^n, \text{ on } \Gamma_2 \end{aligned}$$

one uses transmission conditions adapted to the PDE,

$$\mathcal{B}_1 u_1^n = \mathcal{B}_1 u_2^{n-1}, \text{ on } \Gamma_1 \quad \mathcal{B}_2 u_2^n = \mathcal{B}_2 u_1^n, \text{ on } \Gamma_2$$

Remarks:

- optimal choice for $\mathcal{B}_j$ is $\partial_{n_j} + DtN_j$
- good approximation is $\mathcal{B}_j = \partial_{n_j} + p_j + r_j\partial_\tau + q_j\partial_{\tau\tau}$
- method can converge even without physical overlap

How to choose the parameters

Contraction factor using Fourier analysis:

$$\rho(z,s) = \left(\frac{s(z) - f_{PDE}(z)}{s(z) + f_{PDE}(z)}\right)^2 e^{-Lf_{PDE}(z)}$$

- z related to the Fourier parameters
- s polynomial with coefficients to be optimized.
- f_{PDE} symbol of the DtN of the PDE to be solved.

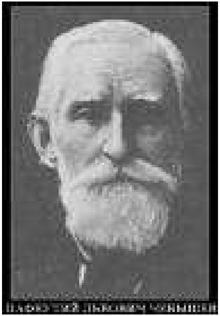
For a fast algorithm, we need to minimize ρ, i.e.

$$\inf_{s\in\mathbb{P}_n} \sup_{z\in K} |\rho(z,s)|$$

- $\mathbb{P}_n$ set of complex polynomials of degree $\leq n$
- K is a bounded or unbounded set in the complex plane

Best Approximation Problems

Chebyshev (1854): Théorie des mécanismes connus sous le nom de parallélogrammes.



$$\min_{p \in \mathbb{P}_n} \max_{x \in K} |f(x) - p(x)|$$

. . . la différence $f(x) - p$ jouira, comme on le sait, de cette propriété: Parmi les valeurs les plus grandes et les plus petites de la différence $f(x) - p$ entre les limites, on trouve au moins $n + 2$ fois la même valeur numérique.

Theorem (Bennequin, G, Halpern 2005)

If $L = 0$ and K is compact, then for every $n \geq 0$, there exists a unique solution s_n^*, and there exist at least $n + 2$ points $z_1, \cdots, z_{n+2}$ in K such that

$$\left| \frac{s_n^*(z_i) - f(z_i)}{s_n^*(z_i) + f(z_i)} \right| = \left\| \frac{s_n^* - f}{s_n^* + f} \right\|_\infty$$

Case $L > 0$

Without assuming that K is compact, one can show (Bennequin, G, Halpern 2006):

Theorem (Existence)

Let K be a closed set in $\mathbb{C}$, containing at least $n+2$ points. Let f satisfy $\Re f(z) > 0$ and

$$\Re f(z) \longrightarrow +\infty \text{ as } z \longrightarrow \infty \text{ in } K.$$

Then for L small enough, there exists a solution.

Theorem (Equioscillation)

Under the same assumptions, if s_n^* is a solution for $L > 0$, then there exist at least $n+2$ points $z_1, \cdots, z_{n+2}$ in K such that

$$\left| \frac{s_n^*(z_i) - f(z_i)}{s_n^*(z_i) + f(z_i)} e^{-Lf(z_i)} \right| = \left\| \frac{s_n^* - f}{s_n^* + f} e^{-Lf} \right\|_\infty = \delta_n(L)$$

Uniqueness, Local Minima and Symmetry

Theorem (Uniqueness)

With the same assumptions, and if K is compact, and L satisfies

$$\delta_n(L)e^{L\sup_{z\in K}\Re f(z)} < 1,$$

where $\delta_n(L)$ is the minimum, then the solution is unique.

Theorem (Local Minima)

If K is compact, and L is small, then if s_n^* is a strict local minimum, then it is the global minimum.

Theorem (Symmetry $\Longrightarrow$ real coefficients)

If K is compact and symmetric with respect to the x-axis, and $f(\bar{z}) = \overline{f(z)}$ in K, then for L small, s_n^* has real coefficients.

Optimized Parameters for a Model Problem

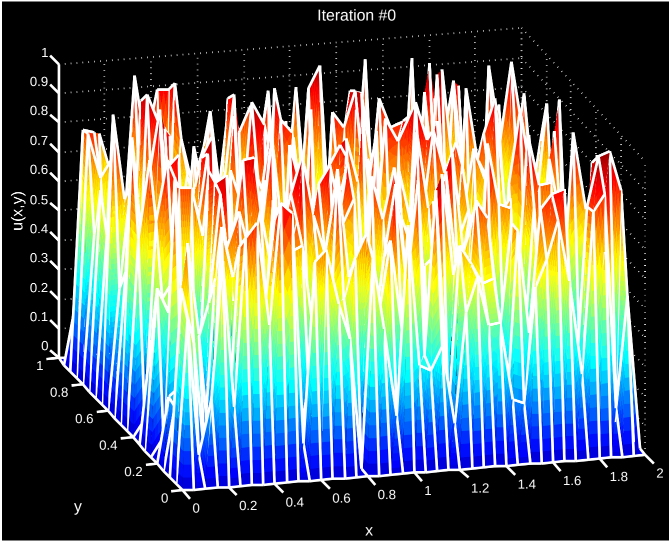
For the self adjoint coercive problem

$$\mathcal{L}u = (\eta - \Delta)u = f$$

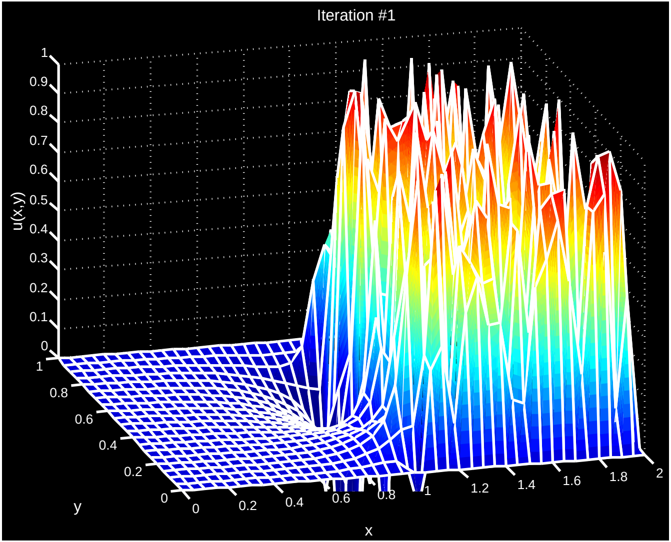
the asymptotically optimal parameters are (G 2006)

	p	q
OO0	$\frac{\sqrt{\pi}(k_{\min}^2+\eta)^{1/4}}{h^{1/2}}$	0
OO0(Ch)	$\frac{(k_{\min}^2+\eta)^{1/3}}{2^{1/3}(Ch)^{1/3}}$	0
OO2	$\frac{\pi^{1/4}(k_{\min}^2+\eta)^{3/8}}{2^{1/2}h^{1/4}}$	$\frac{h^{3/4}}{2^{1/2}\pi^{3/4}(k_{\min}^2+\eta)^{1/8}}$
OO2(Ch)	$\frac{(k_{\min}^2+\eta)^{2/5}}{2^{3/5}(Ch)^{1/5}}$	$\frac{(Ch)^{3/5}}{2^{1/5}(k_{\min}^2+\eta)^{1/5}}$
TO0	$\sqrt{\eta}$	0
TO2	$\sqrt{\eta}$	$\frac{1}{2\sqrt{\eta}}$

Example of Error Decrease in Optimized Schwarz

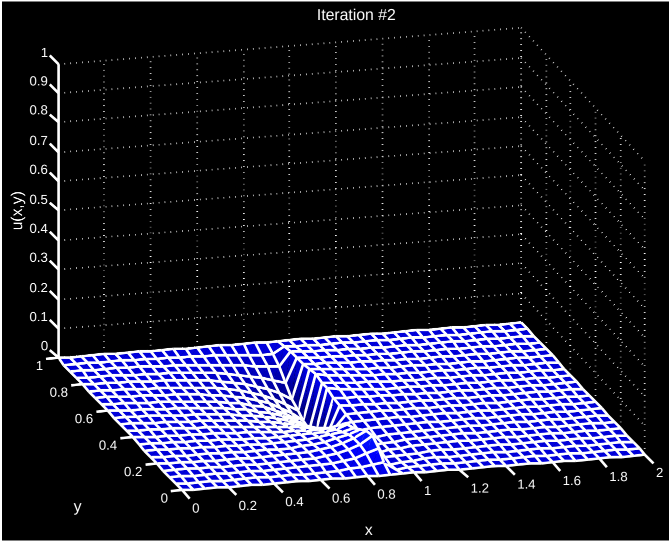


Iteration 1



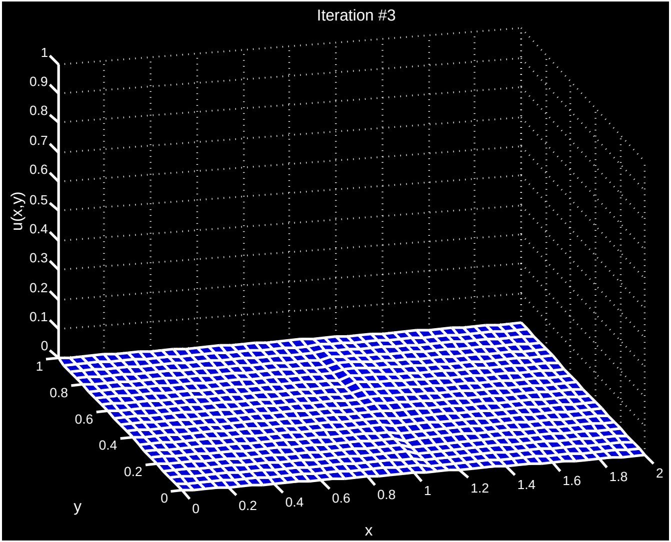
solve on the left subdomain

Iteration 2



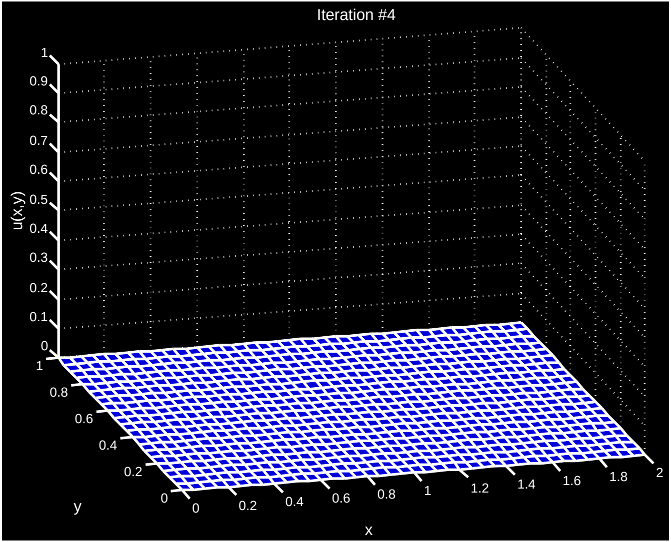
solve on the right subdomain

Iteration 3



solve on the left subdomain

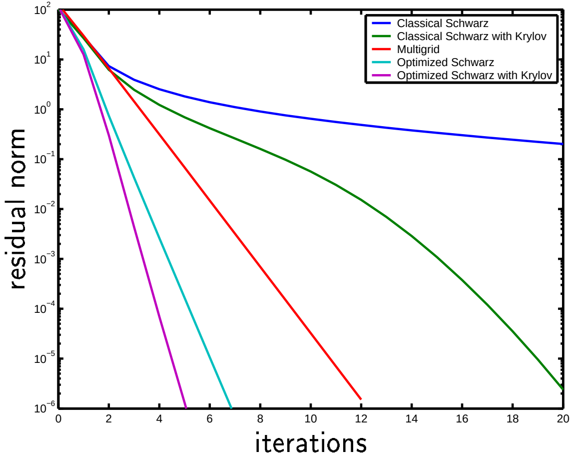
Iteration 4



solve on the right subdomain

Comparison of Optimized Schwarz with Multigrid

Comparison of MS as an iterative solver, as a preconditioner, multigrid, and an optimized Schwarz methods used iteratively and as a preconditioner:



Discrete Optimized Schwarz Methods

How does one have to change the RAS

$$M_{RAS}^{-1} = (\tilde{R}_1^T A_1^{-1} R_1 + \tilde{R}_2^T A_2^{-1} R_2)$$

and the MS preconditioner

$$M_{MS}^{-1} = \left[I - \prod_{j=1}^{J} \left(I - R_j^T A_j^{-1} R_j A \right) \right] A^{-1},$$

to obtain an optimized method ?

Simply replace A_j by a slightly modified $\tilde{A}_j$!

(St-Cyr, G and Thomas, 2007)

An Example

$$\mathcal{L}u = (\eta - \Delta)u = f, \quad \text{in } (0,1)^2$$

Finite difference/volume discretization leads to

$$A\mathbf{u} = \mathbf{f}$$

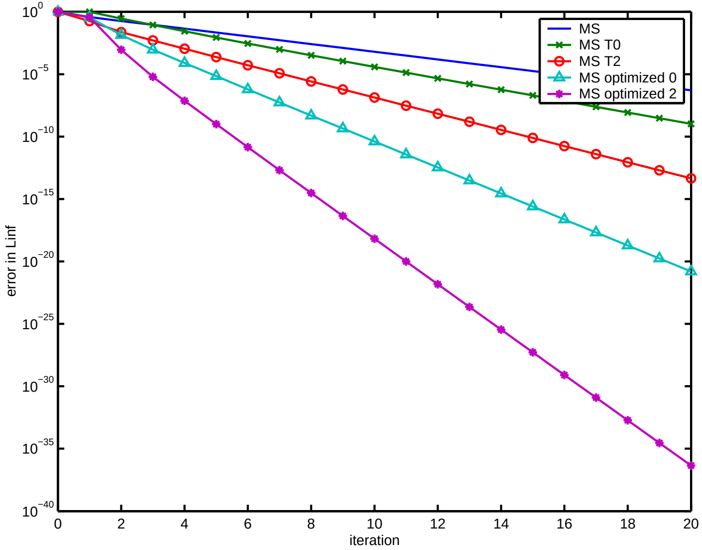
$$A = \frac{1}{h^2}\begin{bmatrix} T_\eta & -I & \\ -I & T_\eta & \ddots \\ & \ddots & \ddots \end{bmatrix}, \quad T_\eta = \begin{bmatrix} \eta h^2 + 4 & -1 & \\ -1 & \eta h^2 + 4 & \ddots \\ & \ddots & \ddots \end{bmatrix}$$

The classical subdomain matrices are $A_j = R_j A R_j^T$.
The optimized $\tilde{A}_j$ are obtained from A_j by simply replacing the interface diagonal block T_η by

$$\tilde{T} = \frac{1}{2}T_\eta + phI + \frac{q}{h}(T_0 - 2I), \quad T_0 = T_\eta|_{\eta=0},$$

where p and q are solutions of the associated min-max problem.

Result for the Example



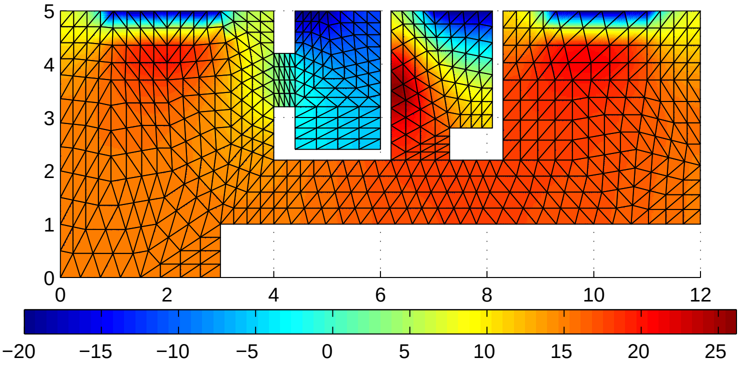
Results for other PDEs

Similar analysis and asumptotic formulas are also available for:

- The steady advection-reaction-diffusion equations (Japhet 1998, Dubois and G. 2007)
- The indefinite Helmholtz equation (Chevalier 1998, G., Magoules, Nataf 2001, G. Halpern 2006)
- The unsteady heat equation (G. Halpern 2003, Binh 2009)
- The second order wave equation (G., Halpern, Nataf 2003, G. Halpern 2006)
- The shallow water equations (Martin 2005, 2006)
- The Cauchy-Riemann equations (Dolean, G. 2007)
- The unsteady advection reaction diffusion equation (Bennequin, G. Halpern 2008)
- The Maxwell's equations (Dolean, G., Gerardo-Giorda 2008)

Optimized Schwarz Application

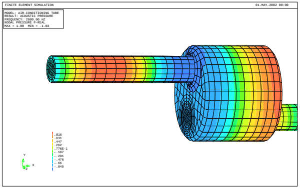
Result of a non-overlapping optimized Schwarz method with Robin transmission conditions:

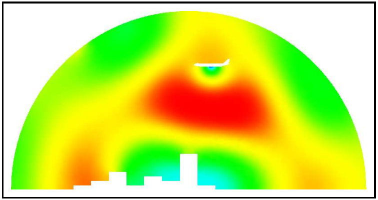


With the optimal parameter p^* from the two subdomain theory, the convergence factor ratio is in the iterative case $32/25 = 1.28 \approx 2^{1/3} = 1.26$, as predicted by the two subdomain theory.

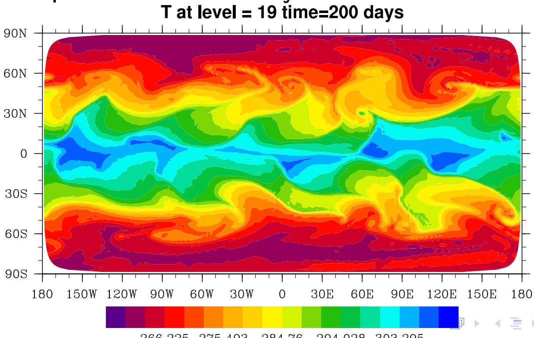
Large Scale Optimized Schwarz Computations

- Exhaust system, and airplane in approach over a city:



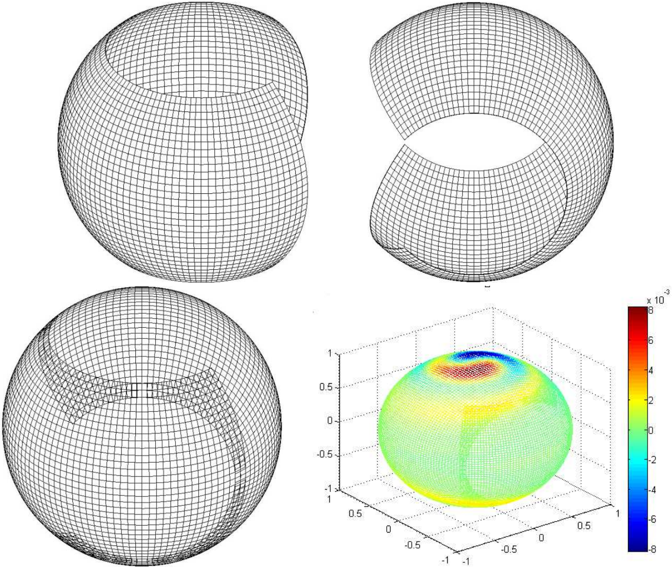


- Held Suarez test, temperature field, at the surface of the planet after 200 days of simulation.



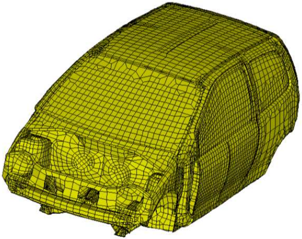
Global Weather Simulation: Cyclogenesis Test

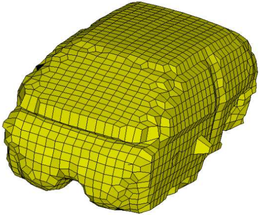
On the Yin-Yang grid (with Côté and Qaddouri 2006)

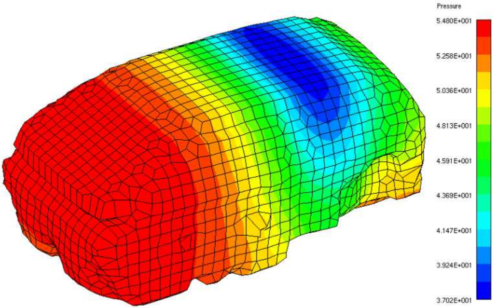


Large Scale Optimized Schwarz Computations

- Twingo, noise simulation in the passenger compartment:

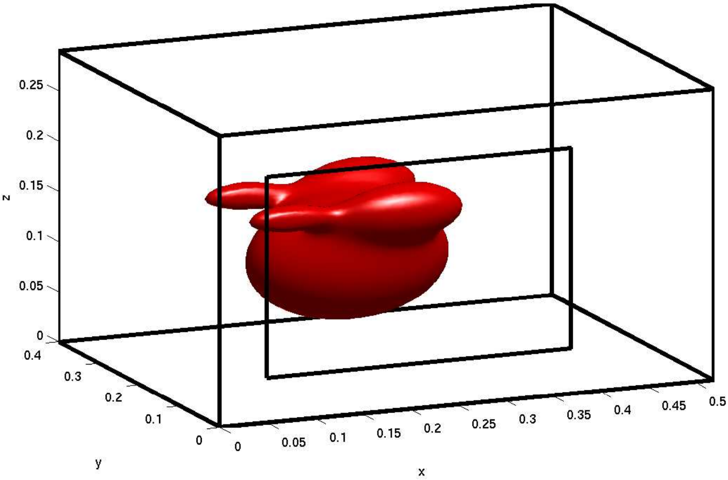






The Chicken Problem

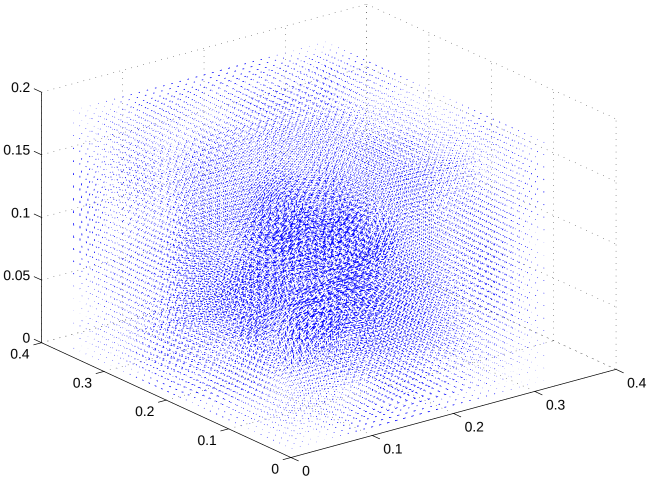
Heating a chicken in our Whirlpool Talent Combi 4 microwave:



Joint work with Dolean and Gerardo-Giorda (2009)

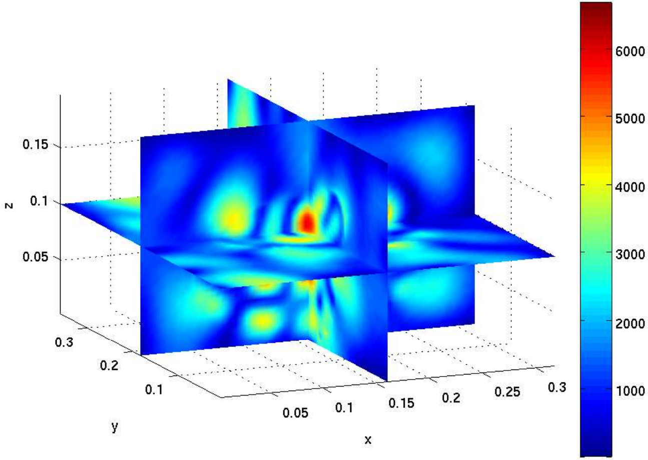
The Chicken Problem

The electric field computed with an 8 subdomain decomposition:



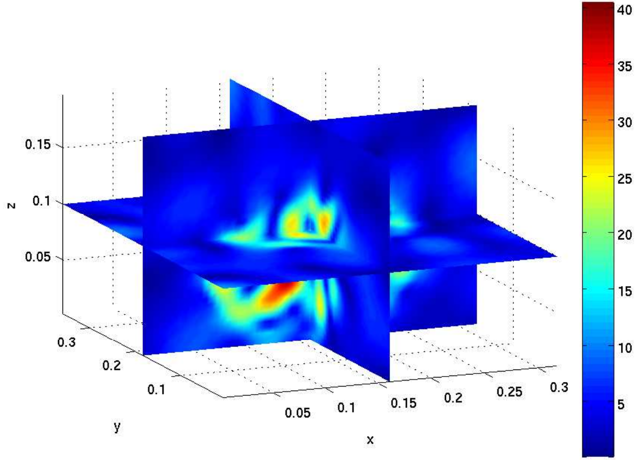
The Chicken Problem

The electric field intensity in the chicken



The Chicken Problem

The magnetic field intensity in the chicken:



Conclusions

- Discrete Schwarz methods are in most cases discretizations of continuous Schwarz methods (exception: AS with overlap!)
- Optimized Schwarz Methods use transmission conditions adapted to the underlying PDE, which can greatly improve their convergence rate
- Replacing classical subdomain matrices A_j by optimized ones, leads to optimized MS, RAS and AS (on an augmented system)

Important current problems (2009):

- General convergence proof for overlapping optimized Schwarz methods (Kimn 2006, Loisel and Szyld 2009)
- Coarse grid corrections for optimized Schwarz (G. and Dubois 2009)
- Algebraically optimized $\tilde{A}_j$ (G. and St-Cyr 2009, G. and Szyld 2009)