A New Coarse Space for a Space-Time Schwarz Waveform Relaxation Method

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1 Introduction and Model Problem

Coarse spaces are in general needed to achieve scalability in domain decomposition methods, see [16] and references therein. There are however exceptions, where one level domain decomposition methods are scalable, which can be due to geometry and/or the operator, see [2] and references therein. In particular for space-time problems this can happen when solving parabolic problems on short time intervals, see [11] for a continuous analysis, [1] for Additive Schwarz applied to each time step, and [4] for hyperbolic problems.

We are interested here in space-time parallel solvers for parabolic problems over longer time intervals, where a coarse correction is needed for scalability. While for elliptic problems there are new coarse spaces constructed by improving directly general condition number estimates, like GenEO [15] and GDSW [12], there are so far no such estimates for evolution problems. We thus base our new coarse space construction for space-time problems on approximating an optimal coarse space, optimal in the sense that the resulting method converges after one coarse correction, see [8, 9] and references therein for elliptic problems.

With the invention of the parareal algorithm [13], research activity increased again tremendously to develop space-time parallel solvers, see the

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review [3] and references therein. While the parareal algorithm can be combined with Schwarz waveform relaxation [10] to obtain a general space-time parallel solver [14, 5], whose convergence was analyzed in [6], we design here a new space-time two level Schwarz waveform relaxation method for evolution problems. For simplicity, we consider the one dimensional heat equation

$$\mathcal{L}u := \partial_t u - \partial_{xx} u = f, \quad \text{in } \Omega \times (0,T), \tag{1}$$

where $\Omega = (a,b)$, $a < b$, with initial condition $u(x,0) = u_0(x)$, $x \in \Omega$, and boundary conditions $u(a,t) = g_1(t)$ and $u(b,t) = g_2(t)$, $t \in [0,T]$.

2 New Two Level Schwarz Waveform Relaxation

We divide the spatial domain (a,b) into I overlapping subdomains $\Omega_i := (a_i, b_i)$, $i = 1,2,\ldots,I$, with $a_1 := a$, $b_I := b$, and decompose the time interval $(0,T)$ into N time subintervals, $0 =: T_0 \leq \cdots \leq T_n := n\Delta T \leq \cdots \leq T_N := T$, $\Delta T := T/N$. This defines the space-time subdomains $\Omega_{i,n} := \Omega_i \times (T_n, T_{n+1})$, $i = 1,2,\ldots,I$, $n = 0,\ldots,N-1$. In [5, 6], the initial conditions in the space-time subdomains were updated using a parareal mechanism, while the boundary conditions were updated using Schwarz waveform relaxation techniques. In contrast, our new two level space-time Schwarz waveform relaxation algorithm consists of iterating two steps: a solve on each space-time subdomain, and a new coarse grid correction. The solver on each space-time subdomain $\Omega_{i,n}$ solves for given initial value $u_{i,n,0}$ and boundary value $\mathcal{B}_{i,n}\bar{u}$

$$\begin{aligned} \mathcal{L}u_{i,n} &= f, && \text{in } \Omega_{i,n}, \\ u_{i,n}(x,T_n) &= u_{i,n,0}, && x \in \Omega_i, \\ \mathcal{B}_{i,n}u_{i,n} &= \mathcal{B}_{i,n}\bar{u}, && \text{on } \partial\Omega_i \times (T_n, T_{n+1}). \end{aligned} \tag{2}$$

Here the operators $\mathcal{B}_{i,n}$ are transmission operators, which can be of Dirichlet, Robin or higher order type. We discretize (1) by a centered finite difference scheme in space and backward Euler in time, to get the linear space-time system $L^h\mathbf{u} = \mathbf{f}$. We denote by Ω^h, $\Omega^h_{i,n}$ the discretized spaces corresponding to Ω and $\Omega_{i,n}$, $i = 1,2,\ldots,I, n = 0,1,\ldots,N-1$. Also denoting by $\Gamma_{ij,n} := \partial\Omega_{i,n} \cap \Omega_{j,n}$ the interfaces, and $\Gamma_{i,n}$ the initial line for the space-time subdomain $\Omega_{i,n}$, $\Gamma^h_{ij,n}$ and $\Gamma^h_{i,n}$ are the corresponding discretized spaces. Furthermore, we let $N_{\Gamma^h_{ij,n}}$ and $N_{\Gamma^h_{i,n}}$ be the number of degrees of freedom (DOFs) on the interface $\Gamma^h_{ij,n}$ and the initial line $\Gamma^h_{i,n}$ for the space-time subdomain $\Omega^h_{i,n}$.

Then for any initial guess of the initial values $\mathbf{u}^0_{i,n,0}$ on the initial line $\Gamma^h_{i,n}$ and the interface values $\mathcal{B}^h_{i,n}\mathbf{u}^0$ for the space-time subdomain $\Omega^h_{i,n}$, our new two level Schwarz waveform relaxation method computes iteratively for $k = 0,1,\ldots$, and for all subdomain indices $i = 1,2,\ldots,I$, $n = 0,1,\ldots,N-1$:

Step I. Solve the subdomain problems on each space-time subdomain $\Omega^h_{i,n}$,

$$\begin{aligned} L^h \mathbf{u}_{i,n}^{k+1/2} &= \mathbf{f}, && \text{in } \Omega^h_{i,n}, \\ \mathbf{u}_{i,n}(x,T_n)^{k+1/2} &= \mathbf{u}^0_{i,n,0}, && x \in \Gamma^h_{i,n}, \\ \mathcal{B}_{i,n}\mathbf{u}_{i,n}^{k+1/2} &= \mathcal{B}_{i,n}\mathbf{u}^k, && \text{on } \Gamma^h_{ij,n}. \end{aligned} \tag{3}$$

Step II. Denoting by $\bar{\mathbf{u}}^{k+1/2}$ a composed approximate solution from the subdomain solutions $\mathbf{u}_{i,n}^{k+1/2}$ using a partition of unity, the coarse correction step reads

$$\mathbf{u}^{k+1} = \bar{\mathbf{u}}^{k+1/2} + R_c^T L_c^{-1} R_c(\mathbf{f} - L^h \bar{\mathbf{u}}^{k+1/2}), \tag{4}$$

where R_c is a restriction matrix to a coarse space, and $L_c := R_c L^h R_c^T$. Finally we set $\mathbf{u}^{k+1}_{i,n,0} = \mathbf{u}^{k+1}$ on the initial lines $\Gamma^h_{i,n}$.

Definition 1 (Complete coarse space) A complete coarse space for the two level space-time Schwarz waveform relaxation method (3)-(4) for the model problem (1) is given by R_c such that (3)-(4) converges after one iteration for an arbitrary initial guess $\mathbf{u}^0_{i,n,0}$ and $\mathcal{B}_{i,n}\mathbf{u}^0$, i.e. the method becomes a direct solver.

To give an example of such a complete coarse space for the two level space-time Schwarz waveform relaxation method (3)-(4) for the model problem (1), we define $\boldsymbol{\phi}^l_{ij,n,\mathrm{cs}}$ for each DOF $l = 1, \ldots, N_{\Gamma^h_{i,n}}$ on the interface $\Gamma^h_{ij,n}$ to be the extension

$$\begin{aligned} L^h \boldsymbol{\phi}^l_{ij,n,\mathrm{cs}} &= 0 && \text{in } \Omega^h_{i,n}, \\ \boldsymbol{\phi}^l_{ij,n,\mathrm{cs}} &= 1 && \text{at DOF } l \text{ of } \Gamma^h_{ij,n}, \\ \boldsymbol{\phi}^l_{ij,n,\mathrm{cs}} &= 0 && \text{on } \Gamma^h_{i,n} \text{ and the rest of } \Gamma^h_{ij,n} \text{ and } \Omega^h. \end{aligned} \tag{5}$$

Similarly, we define $\boldsymbol{\phi}^l_{i,n,\mathrm{cs}}$ for each DOF $l = 1, \ldots, N_{\Gamma^h_{i,n}}$ on $\Gamma^h_{i,n}$ to be the extension

$$\begin{aligned} L^h \boldsymbol{\phi}^l_{i,n,\mathrm{cs}} &= 0 && \text{in } \Omega^h_{i,n}, \\ \boldsymbol{\phi}^l_{i,n,\mathrm{cs}} &= 1 && \text{at DOF } l \text{ of } \Gamma^h_{i,n}, \\ \boldsymbol{\phi}^l_{i,n,\mathrm{cs}} &= 0 && \text{on } \Gamma^h_{ij,n} \text{ and the rest of } \Gamma^h_{i,n} \text{ and } \Omega^h. \end{aligned} \tag{6}$$

We then define our complete coarse space by

$$V_{0,\mathrm{cs}} := \mathrm{span}\{\{\boldsymbol{\phi}^l_{ij,n,\mathrm{cs}}\}_{l=1}^{N_{\Gamma^h_{ij,n}}}\}_{i=1,n=1}^{i=I,n=N-1} \cup \{\{\boldsymbol{\phi}^l_{i,n,\mathrm{cs}}\}_{l=1}^{N_{\Gamma^h_{i,n}}}\}_{i=1,n=1}^{i=I,n=N-1}. \tag{7}$$

Theorem 1 *A complete coarse space for the two level space-time Schwarz waveform relaxation method* (3)-(4) *for the model problem* (1) *is given by* R_c *containing in its columns the vectors of* $V_{0,cs}$ *from* (7).

Proof The proof is technical [7], for an illustration see Section 3. □

The dimension of the complete coarse space (7) corresponds only to the size of the interfaces and initial lines, but can still become prohibitively large, when the size of the problem increases, and we need to consider approximations of (7), which we call optimized coarse spaces, formed by extensions of linear and spectral functions along the interfaces $\Gamma^h_{ij,n}$ and initial lines $\Gamma^h_{i,n}$. The linear functions on the interfaces are $\boldsymbol{\psi}^{-1}_{ij}$ with $\boldsymbol{\psi}^{-1}_{ij}(T_n) = 0$, $\boldsymbol{\psi}^{-1}_{ij}(T_{n+1}) = 1$, and $\boldsymbol{\psi}^0_{ij}$ with $\boldsymbol{\psi}^0_{ij}(T_n) = 1$, $\boldsymbol{\psi}^0_{ij}(T_{n+1}) = 0$, and the spectral functions are $\boldsymbol{\psi}^l_{ij} = \sin(\frac{l\pi(t-T_n)}{T_{n+1}-T_n}), t \in [T_n, T_{n+1}]$. Let $\boldsymbol{\phi}^l_{ij,n,\text{app}}$ be defined by the extension

$$\begin{aligned} L^h\boldsymbol{\phi}^l_{ij,n,\text{app}} &= 0 && \text{in } \Omega^h_{i,n}, \\ \boldsymbol{\phi}^l_{ij,n,\text{app}} &= \boldsymbol{\psi}^l_{ij} && \text{on } \Gamma^h_{ij,n},\ l = -1, 0, 1, \ldots, \ell_t, \\ \boldsymbol{\phi}^l_{ij,n,\text{app}} &= 0 && \text{on } \Gamma^h_{i,n} \text{ and the rest of } \Gamma^h_{ij,n} \text{ and } \Omega^h. \end{aligned} \tag{8}$$

Similarly the linear functions along the initial lines $\Gamma^h_{i,n}$ are $\boldsymbol{\psi}^{-1}_i$ with $\boldsymbol{\psi}^{-1}_i(a_i) = 0$, $\boldsymbol{\psi}^{-1}_i(b_i) = 1$, and $\boldsymbol{\psi}^0_i$ with $\boldsymbol{\psi}^0_i(a_i) = 1$, $\boldsymbol{\psi}^0_i(b_i) = 0$, and the spectral functions are $\boldsymbol{\psi}^l_i = \sin(\frac{l\pi(x-a_i)}{b_i-a_i}), x \in [a_i, b_i]$. Let $\boldsymbol{\phi}^l_{i,n,\text{app}}$ be defined by the extension

$$\begin{aligned} L^h\boldsymbol{\phi}^l_{i,n,\text{app}} &= 0 && \text{in } \Omega^h_{i,n}, \\ \boldsymbol{\phi}^l_{i,n,\text{app}} &= \boldsymbol{\psi}^l_i && \text{on } \Gamma^h_{i,n},\ l = -1, 0, 1, \ldots, \ell_x, \\ \boldsymbol{\phi}^l_{i,n,\text{app}} &= 0 && \text{on } \Gamma^h_{ij,n} \text{ and the rest of } \Omega^h. \end{aligned} \tag{9}$$

Our optimized coarse space is then given by

$$V_{0,\text{cs-l}} := \text{span}\{\{\boldsymbol{\phi}^l_{ij,n,\text{app}}\}^{\ell_t}_{l=-1}\}^{i=I,n=N-1}_{i=1,n=1} \cup \{\{\boldsymbol{\phi}^l_{i,n,\text{app}}\}^{\ell_x}_{l=-1}\}^{i=I,n=N-1}_{i=1,n=1}. \tag{10}$$

3 Numerical Experiments

We solve the model problem on $\Omega \times (0, T) := (0, 1) \times (0, 1)$ with the source term $f \equiv 0$, zero boundary conditions, and the initial value $u_0 = \exp(-3(0.5-x)^2)$, discretized by centered finite differences in space using an overlap $4h$ with $h = 1/40$ being the mesh parameter and backward Euler in time with time step $\Delta t = 1/40$. The initial guesses along the interfaces and the initial lines of the space-time subdomains are all random. We first decompose the domain Ω into two overlapping subdomains and the time interval $(0, T)$ also into two time subintervals. Figure 1 shows two examples of basis functions from the complete coarse space: one coming from the interface (left) and one from the initial line (right). In Figure 2 we show on the left the residual after the first **Step I** of the new space-time Schwarz waveform relaxation algorithm, which shows that the residual is only non-zero along the interfaces

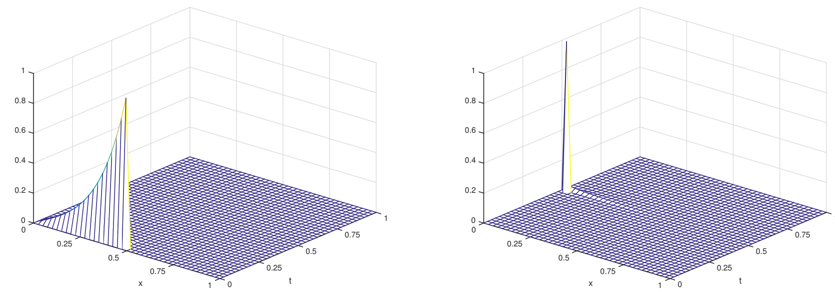

Fig. 1 First basis function of the complete coarse space from the interface $\Gamma_{12,1}^{h}$ of $\Omega_{1,1}^{h}$ (left) and from the initial line of $\Omega_{1,2}^{h}$ (right).

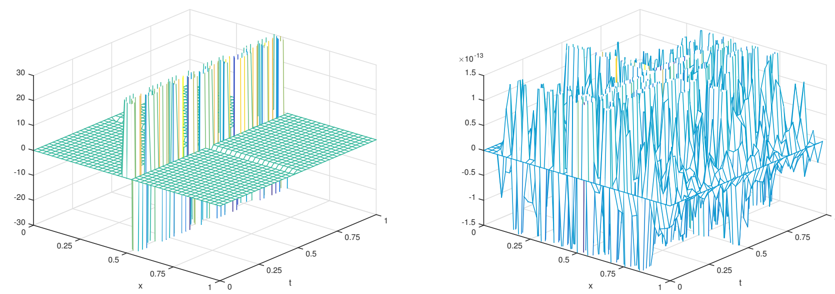

Fig. 2 Residual after the first execution of **Step I** of our new space-time Schwarz waveform relaxation algorithm (left) and after the following coarse correction **Step II** with the complete coarse space (right, note the different scale!).

and the initial line of the space-time subdomains. On the right we show the effect of the following coarse correction **Step II** using the complete coarse space, which reduces the residual to machine precision: the method becomes a direct solver.

We next show basis functions of our optimized coarse space: in Figure 3 basis functions from the interface of the space-time subdomains, and in Figure 4 basis functions from the initial line of the space-time subdomain. We show in Figure 5 the influence on the convergence of the optimized coarse space for both 2 spatial subdomains with 2 time subintervals (left) and 4 spatial subdomains with 4 time subintervals (right). The size of the coarse problem is $12, 18, 24, 30$ corresponding to $\ell = 0, 1, 2, 3$ for the first case, and $72, 108, 144, 180$ for the second case. Here the size of the fine problem is 1560 for the all-at-once discretization. We see that the coarse space indeed makes the new two level space-time Schwarz waveform relaxation method scalable, and increasing the number of spectral functions ℓ in the enrichment improves convergence.

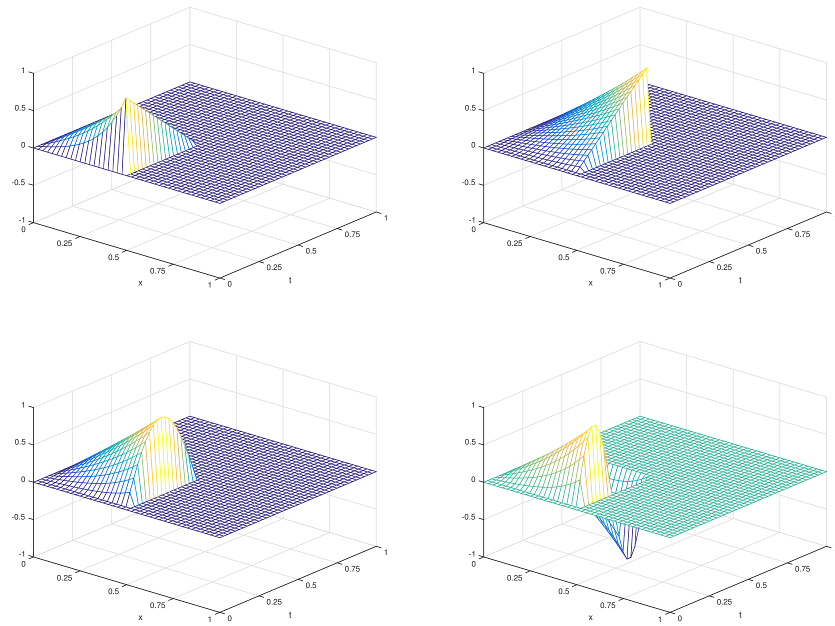

Fig. 3 First two linear basis functions extended to $\Omega^h_{1,1}$ from the interface (top), and first two spectral basis functions for the same subdomain (bottom).

4 Conclusions

We presented a new two level parallel space-time Schwarz waveform relaxation method. The method alternates between solving subproblems in space-time subdomains in parallel, and a new coarse correction which is a spectral approximation of a complete coarse space in space-time. We tested both the complete coarse space and its spectral approximation for a heat equation model problem, but the algorithm definition is valid for much more general equations and also higher dimensions.

Acknowledgements This work was supported by the Natural Science Foundation of China (NSFC) under grant 11801449, 11871393, the International Science and Technology Cooperation Program of Shaanxi Key Research & Development Plan under grant 2019KWZ-08, the Natural Science Basic Research Plan in Shaanxi Province of China (Program No. 2019JQ-617), China Postdoctoral Science Foundation under grant 2020M673465, and the Fundamental Research Funds for the Central Universities under grant G2018KY0306.

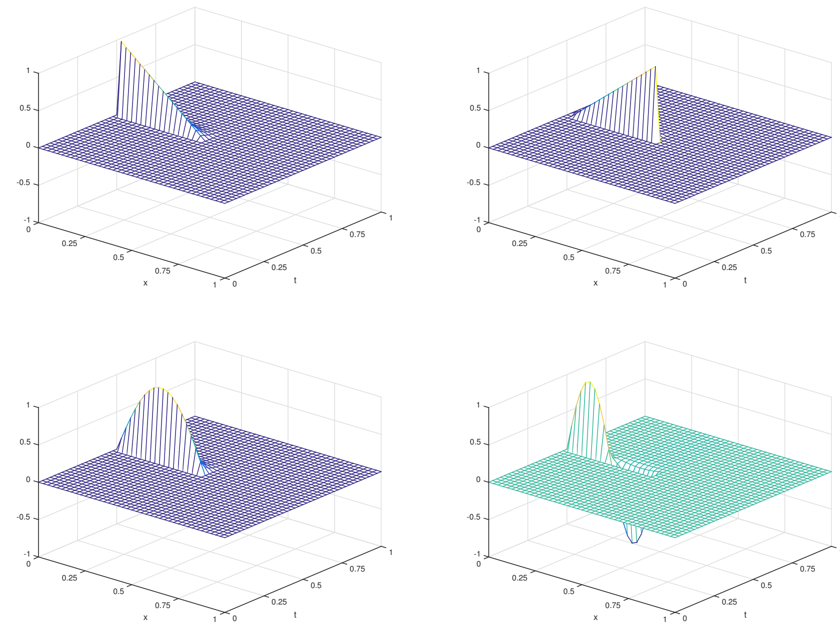

Fig. 4 First two linear basis functions extended to $\Omega^h_{1,2}$ from the initial line (top), and first two spectral basis functions for the same subdomain (bottom).

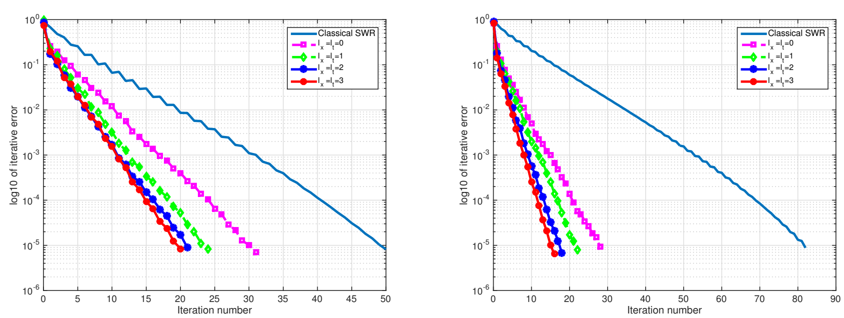

Fig. 5 Comparison of the numerically measured convergence rates using the new optimized coarse space with ℓ spectral function enrichment for 2 spatial subdomains with 2 time subintervals (left) and 4 spatial subdomains with 4 time subintervals (right).

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