AILU for Helmholtz problems:
A new Preconditioner Based on an Analytic Factorization.

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Abstract

We investigate a new type of preconditioner which is based on the analytic factorization of the operator into two parabolic factors. Approximate analytic factorizations lead to new block ILU preconditioners. We analyze the preconditioner at the continuous level where it is possible to optimize its performance. Numerical experiments illustrate the effectiveness of the new approach.

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Résumé

Nous étudions un nouveau type de préconditioneurs qui est basé sur la factorisation analytique de l'opérateur en deux opérateurs paraboliques. Ainsi des factorisations analytiques approchées conduisent à de nouveaux préconditioneurs ILU par bloc. Le préconditioneur est analysé au niveau continu où il est possible d'optimiser ses performances. Des résultats numériques illustrent l'efficacité de cette nouvelle approche.

Version française abrégée

Nous considérons l'opérateur d'Helmholtz $\mathcal{L} = -\omega^2 - \Delta$ en dimension deux et trois. Nous voulons résoudre l'équation

$$\mathcal{L}(u) = f \tag{1}$$

dans un domaine Ω, avec des conditions aux bords. La discrétisation de ce problème par une méthode d'éléments finis ou de différences finies sur une grille structurée conduit à un système linéaire de grande taille

$$K\boldsymbol{u} = \boldsymbol{f} \tag{2}$$

où l'opérateur discret K a une structure bloc

$$K = \begin{bmatrix} D_1 & L_{1,2} & & \\ L_{2,1} & D_2 & \ddots & \\ & \ddots & \ddots & L_{n-1,n} \\ & & L_{n,n-1} & D_n \end{bmatrix}. \tag{3}$$

Les blocs diagonaux D_i représentent la discrétisation selon y (et z en dimension trois) de l'opérateur $\mathcal{L}$ ainsi que la partie diagonale de la discrétisation selon x de l'opérateur, le reste étant contenu dans $L_{i,j}$. La matrice K étant creuse, il serait intéressant de le résoudre par une méthode itérative et donc de le préconditioner pour avoir une méthode efficace.

Les méthodes de décomposition de domaines peuvent être considérées comme des préconditioneurs, voir par exemple [Des91], [CN98], [dLBFM$^+$98] or [CCEW98]. Nous ne considérons pas cette approche mais plutôt les préconditioneurs basés sur la matrice: les inverses approchés et les factorisations incomplètes [BT98]. Ces deux techniques ne prennent pas en compte la forme spécifique de l'opérateur dont la matrice est issue. Nous allons obtenir un préconditioneur de type ILU de manière analytique à partir de l'opérateur aux dérivées partielles avant discrétisation. De plus, l'analyse continue permet l'optimisation du préconditioneur pour une équation aux dérivées partielles donnée.

Bien que la factorisation parabolique d'opérateurs soit classique en acoustique pour l'approximation de l'équation des ondes (voir [Cla76]), elle n'est pas utilisée comme préconditioneur. Une première utilisation de cette notion pour la résolution d'équations a été proposée dans [Nat90] et étendue dans [NLS93]. L'idée a aussi été utilisée par Giladi et Keller, à partir d'une analyse asymptotique dans [GK97]. Les principales difficultés de ces approches sont leur faible qualité en tant que préconditioneurs ce qui limite leur utilisation. Dans ces derniers travaux, l'opérateur factorisé est non symétrique et les factorisations approchées étaient valables pour une petite diffusion ce qui simplifie ce type d'approximation. L'opérateur considéré ici est symétrique mais non défini positif. Nous nous servons d'une analogie entre la factorisation analytique et la factorisation ILU par bloc. Notre approche est liée à des travaux antérieurs au niveau discret de Wittum dans [Wit91, Wit92] généralisés plus tard par Wagner dans [Wag97, Wag97].

Dans le § 2, nous considérons la factorisation de l'opérateur continu ainsi que de l'opérateur semi-discrétisé $-\omega^2 - D_x^+ D_x^- - \partial_{yy}$ sous la forme $-\frac{1}{h^2\tau}(D_x^- + \lambda_1)(D_x^+ - \lambda_2)$ où τ et λ_i sont des opérateurs agissant dans la direction y et qui sont donnés par leur symbole de Fourier:

$$\tau = \frac{1}{h^2} + \frac{-\omega^2 + k^2}{2} + \frac{1}{2h}\sqrt{(-\omega^2 + k^2)^2 h^2 + 4(-\omega^2 + k^2)}, \tag{4}$$

$$\lambda_1 = \lambda_2 = \tau h - \frac{1}{h} = h\frac{-\omega^2 + k^2}{2} + \frac{1}{2}\sqrt{(-\omega^2 + k^2)^2 h^2 + 4(-\omega^2 + k^2)}.$$

Le préconditioneur AILU (ILU analytique) est introduit au § 3. L'opérateur τ est approché par une approximation locale, soit en Fourier par un polynôme d'ordre 2 en k. On obtient alors une factorisation approchée

$$\mathcal{L}_{app} = -\frac{1}{h^2 \tau_{app}}(D_x^- + \lambda_1^{app})(D_x^+ - \lambda_2^{app})$$

où $\tau_{app} = \frac{1}{h^2} + \frac{-\omega^2+k^2}{2} + \frac{1}{2h}(p + qk^2)$ et $\lambda_i^{app} = \tau_{app} h - \frac{1}{h}$. Les coefficients $p, q \in C$ sont choisis de façon à ce que $\mathcal{L}_{app}^{-1}\mathcal{L}$ soit le plus proche possible de l'identité sauf pour quelques valeurs propres qui seront pris en charge par la méthode de Krylov. Des résultats numériques sont donnés dans le § 4.

English version

1 Introduction

Given the Helmholtz operator $\mathcal{L} = -\omega^2 - \Delta$ acting on $u : \mathbf{R}^n \longrightarrow \mathbf{R}$, $n = 2, 3$ we are interested to solve the elliptic partial differential equation

$$\mathcal{L}(u) = f \tag{5}$$

in a given domain $\Omega \subset \mathbf{R}^n$ with appropriate boundary conditions. Discretizing the elliptic operator with a finite element or finite difference method on a structured grid, we obtain a large system of linear equations

$$K\boldsymbol{u} = \boldsymbol{f} \tag{6}$$

where the discrete elliptic operator K has the block structure

$$K = \begin{bmatrix} D_1 & L_{1,2} & & \\ L_{2,1} & D_2 & \ddots & \\ & \ddots & \ddots & L_{n-1,n} \\ & & L_{n,n-1} & D_n \end{bmatrix}. \tag{7}$$

The diagonal blocks D_i represent in our notation the discretization of the y part of the elliptic operator $\mathcal{L}$ and include on the diagonal a part of the discretization of the x part of the elliptic operator, the rest being contained in $L_{i,j}$. Since K is sparse, it is interesting to solve (6) by an iterative method and it is necessary to precondition the system to obtain the solution efficiently.

Domain decomposition methods can be used as preconditioners, see [Des91], [CN98], [dLBFM$^+$98] or [CCEW98]. This approach is not considered here. Instead, we focus on the the matrix based preconditioners called incomplete factorizations [BT98]. These techniques do not relate to the underlying differential operator in general. To link the factorization to the underlying operator, we derive an ILU preconditioner from the analytic factorization of the differential operator itself before it is discretized. Such a preconditioner is approximating well the continuous operator. In addition the continuous analysis allows us to optimize the preconditioner for the given elliptic PDE.

Although the parabolic factorization of elliptic operators has been a topic of interest for a while [Cla76, SB93] the first use of this approach for iterative solvers was proposed by Nataf in [Nat90] and extended in [NLS93]. The idea was also picked up by Giladi and Keller, motivated by an asymptotic analysis in [GK97]. The main difficulties remaining in this approach are the low quality of the approximate factorization and thus the limited applicability. In all the previous work the factored operator was non symmetric and the approximate factorization was only considered for small diffusion coefficients which simplifies this type of approximation. We consider here symmetric operators and using a link between the analytic factorization and the exact block LU decomposition we obtain approximate factorizations of high quality. Our approach is related to earlier work at the discrete level by Wittum in [Wit91, Wit92] extended later by Wagner [Wag97, Wag97].

2 Analytic Parabolic Factorization

Given an elliptic operator $\mathcal{L}(u)$ we write the operator as a product of two parabolic operators,

$$\mathcal{L}(u) = -(\partial_x + \Lambda_1)(\partial_x - \Lambda_2)(u) \tag{8}$$

where Λ_1 and Λ_2 are positive operators up to a compact operator. The first factor represents a parabolic operator acting in the positive x direction and the second one a parabolic operator acting in the negative x direction.

In the sequel we restrict ourselves for the analysis to the case of $\mathcal{L} = (-\omega^2 - \Delta)$, where Δ denotes the Laplacian in two dimensions and $\omega \geq 0$.

Our results are based on Fourier analysis. We take a Fourier transform of $\mathcal{L} = (-\omega^2 - \Delta)$ in y to obtain

$$\mathcal{F}_y(-\omega^2 - \Delta) = -\partial_{xx} + k^2 - \omega^2 = -(\partial_x + \sqrt{k^2 - \omega^2})(\partial_x - \sqrt{k^2 - \omega^2}) \tag{9}$$

and thus we have the continuous parabolic factorization

$$(-\omega^2 - \Delta) = -(\partial_x + \Lambda_1)(\partial_x - \Lambda_2) \tag{10}$$

where $\Lambda_1 = \Lambda_2 = \mathcal{F}_y^{-1}(\sqrt{k^2 - \omega^2})$. Note that the Λ_i are non local operators in y because of the square root.

To relate this parabolic factorization to the exact block LU decomposition of the discrete matrix operator, we discretize the x direction of $(-\omega^2 - \Delta)$ and compute the analytic factorization (9) for the semi discrete operator $(-\omega^2 - \Delta_h)$. We have

$$\Delta_h = D_x^- D_x^+ + \partial_{yy}$$

where $D_x^+(x) := (x_{i+1} - x_i)/h$ and $D_x^-(x) := (x_i - x_{i-1})/h$ represent the discrete derivatives on a given mesh. Taking a Fourier transform in y of $-\omega^2 - \Delta_h$ as before we obtain the factored form

$$\mathcal{F}_y(-\omega^2 - \Delta_h) = -\frac{1}{h^2\tau}(D_x^- + \lambda_1)(D_x^+ - \lambda_2) = -\frac{1}{h^2\tau}(D_x^- D_x^+ - \lambda_2 D_x^- + \lambda_1 D_x^+ - \lambda_1\lambda_2),$$

with the unknowns λ_1, λ_2 and the additional parameter τ introduced because of the discretization. Using $D_x^+ - D_x^- = hD_x^- D_x^+$ to replace the term with D_x^- we find for

$$\tau = \frac{1}{h^2} + \frac{-\omega^2 + k^2}{2} + \frac{1}{2h}\sqrt{(-\omega^2 + k^2)^2 h^2 + 4(-\omega^2 + k^2)}, \tag{11}$$

where we chose the positive root, since we defined λ_1 and λ_2 to be positive operators for $|k| > \omega$. Similarly, we find

$$\lambda_1 = \lambda_2 = \tau h - \frac{1}{h} = h\frac{-\omega^2 + k^2}{2} + \frac{1}{2}\sqrt{(-\omega^2 + k^2)^2 h^2 + 4(-\omega^2 + k^2)}$$

which are positive for $|k| > \omega$. The semi discrete analytic parabolic factorization is thus given by

$$\mathcal{F}_y(-\omega^2 - \Delta_h) = -\left(D_x^- + (\tau h - \frac{1}{h})\right)\frac{1}{h^2\tau}\left(D_x^+ - (\tau h - \frac{1}{h})\right). \tag{12}$$

Note that as we take the limit for $h \longrightarrow 0$ in (12) we recover again the continuous parabolic factorization (9) since the middle term disappears in the limit. For discrete problems it is however important to include the middle factor, which was not the case in previous work on continuous parabolic factorizations. One can show that (12) corresponds to the exact block-LU decomposition of the fully discrete matrix operator (7), see [GN99].

3 The AILU Preconditioner

One could use directly the parabolic factorization given in (12) to solve the original problem (6). Instead of solving the linear system, one would have to solve two lower dimensional parabolic problems, one in the positive and one in the negative x direction, corresponding to a forward and a backward solve of the exact block LU decomposition. This is however not advisable since the parabolic factorization contains nonlocal operators in y. We therefore approximate the parabolic factorization by local operators and use the factorization as a preconditioner corresponding to a new type of ILU preconditioner we call AILU (Analytic ILU). We replace the nonlocal operator τ in (12) by a local approximation of the form

$$\tau_{app} = \frac{1}{h^2} + \frac{-\omega^2 + k^2}{2} + \frac{1}{2h}(p + qk^2), \quad p, q \in \mathbf{C},\ \Re(q) > 0,$$

which leads to a classical linear second order parabolic problem. Since we have the analytic parabolic factorization, we can use the parameters p and q to optimize the performance of the AILU preconditioner. We insert the approximation τ_{app} into the factorization (12) and obtain the operator resulting from the approximate factorization of $-\omega^2 + k^2 - D_x^+ D_x^-$ in the form

$$\mathcal{L}_{app} = -D^- D^+ + \tau_{app} + \frac{1}{\tau_{app} h^4} - \frac{2}{h^2}.$$

The complex numbers p and q are to be chosen so that $\mathcal{L}_{app}^{-1}\mathcal{L}$ is as close as possible to the identity except for a few frequencies which will be taken into account by the Krylov method. We find after some calculation that we have to minimize

$$\rho(k) := \left| 1 - \frac{2(-\omega^2 + k^2)(2 - \omega^2 h^2 + ph + h(h+q)k^2)}{(p - \omega^2 h + (q+h)k^2)^2} \right|.$$

In a numerical setting, the frequency parameter k can not vary arbitrarily. For Dirichlet boundary conditions, it is bounded from below by the size of the domain, $k > \frac{\pi}{L}$ where L denotes the size of the domain in the y direction. From above, k is bounded by the mesh size h, $k < \frac{\pi}{h}$. We chose p such that $\rho(k)$ is equal to zero for $k = \pi/L$. The other parameter q is chosen so that the symbol of $\rho(k)$ is as close as possible to zero except for the frequency $|k| = \omega$ where $\rho = 1$ for any parameter q.

4 Numerical Experiments

We first consider a two dimensional open cavity problem,

$$-\omega^2 u - \partial_{xx} u - \partial_{yy} u = \delta(x - \frac{1}{2})\delta(y - \frac{1}{2}), \quad 0 < x, y < 1$$

with homogeneous Dirichlet boundary conditions on the top, bottom and right boundaries. On the left boundary ($x = 0$), we impose an absorbing boundary condition $(-\partial_x + i\omega)(u) = 0$. The right hand side corresponds to a point source in the center of the cavity. We start QMR with the initial guess $u^{(0)} = 0$ and we compare the new preconditioner with the unpreconditioned QMR algorithm, and the preconditioners ILU('0') and ILU(1e-2) using the QMR algorithm and a tolerance of 1e-6. Table 1 shows the results obtained from numerical experiments for different values of ω with 10 points per wavelength in each spatial direction. The new AILU

ω	Iteration count				Solution process only in Mflops				Precond. cost in Mflops		
	QMR	ILU('0')	ILU(1e-2)	AILU	QMR	ILU('0')	ILU(1e-2)	AILU	ILU('0')	ILU(1e-2)	AILU
5	197	60	22	23	120.1	60.4	28.3	28.3	0.2	5.1	0.3
10	737	370	80	36	1858.2	1489.3	421.4	176.2	0.8	38.6	0.9
15	1775	> 2000	220	43	10185.2	> 18133.2	2615.1	475.9	1.9	127.5	2.0
20	> 2000	—	> 2000	64	> 20335.1	—	> 42320.1	1260.2	3.4	298.2	3.6
30	—	—	—	90	—	—	—	3984.1	7.8	996.1	8.1
50	—	—	—	285	—	—	—	24000.4	–	–	22.5

Table 1: Two-dimensional open cavity test with 10 points per wavelength. Comparison of iteration count, flop count in mega flops for the solution process and computing the preconditioner separately.

Iteration count			Solution process only in Mflops			Precond. cost in Mflops	
QMR	ILU(1e-2)	AILU	QMR	ILU(1e-2)	AILU	ILU(1e-2)	AILU
285	58	27	18148.3	7588.2	8866.2	23845.6	1288.9

Table 2: Three-dimensional tube test, $\omega = 6$ and $h = 1/60$. Comparison of iteration count and flop count for the solution process only and computing the preconditioner separately in mega flops. For this problem, we were not able to compute ILU('0') in Matlab.

preconditioner shows an excellent reduction of the iteration count and permits the solution of the given problem in significantly less flops than any of the other methods tested. For big problems it was the only successful solver. The construction of AILU however is as cheap as the construction of ILU('0').

As a next example we consider a three dimensional model problem,

$$-\omega^2 u - \partial_{xx} u - \partial_{yy} u - \partial_{zz} u = \delta(x - \frac{1}{2})\delta(y - \frac{1}{2})\delta(z - \frac{1}{2}), \quad 0 < x, y, z < 1$$

with homogeneous Dirichlet boundary conditions on all except on the the left and right boundaries where we set $(\partial_{\boldsymbol{n}} + i\omega)(u) = 0$. We compare the new preconditioner with unpreconditioned QMR and ILU(1e-2) using a tolerance of 1e-6. We were not able to compute ILU('0') for this problem size in Matlab. Table 2 shows the iteration and the flop counts. Again AILU shows an excellent reduction in the flop count and the total cost of the computation (i.e. including the construction of the preconditioner) is in favor of AILU even though we have chosen here to solve the parabolic problems within AILU exactly by factorization to conform with the analysis. For big problems it will be better to use an iterative solver within AILU as well. In addition the parabolic problems need not to be solved very accurately, since they describe themselves only an approximation. This will lead to a significant reduction in the cost of the preconditioner both in terms of flop counts and memory for three-dimensional problems. This issue is under current investigation.

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