Dimensional reduction by Fourier analysis of a Stokes-Darcy fracture model

Martin J. Gander[1], Julian Hennicker[2], Roland Masson[3], and Tommaso Vanzan[4]

[1] Université de Genève, Switzerland, martin.gander@unige.ch,
[2] julian.hennicker@gmail.com,
[3] Université Côte d'Azur, Inria, CNRS, Laboratoire J.A. Dieudonné, team Coffee, France, roland.masson@univ-cotedazur.fr,
[4] CSQI Chair, Ecole Polytecnique Fédérale de Lausanne, Switzerland, tommaso.vanzan@epfl.ch

Abstract. We consider a Stokes flow along a thin fracture coupled to a Darcy flow in the surrounding matrix domain. In order to derive a dimensionally reduced model representing the fracture as an interface coupled to the surrounding matrix, we extend the methodology based on Fourier analysis developed in [1] for a Darcy-Darcy coupling. We show that this approach not only allows us to derive error estimates between the solutions of the full and mixed-dimensional models, but also leads to a model correction term compared with what is obtained from the classical reduction technique based on integration along the fracture width combined with profile closure assumptions [3,2].

Keywords: Dimensionally reduced fracture model, mixed-dimensional model, Discrete Fracture Matrix model, Stokes-Darcy coupling, Fourier analysis

1 Stokes-Darcy fracture model

Let us consider the matrix domains $\Omega_1 = (-L_1, -\delta) \times \mathbb{R}$, $\Omega_2 = (\delta, L_2) \times \mathbb{R}$ and the fracture domain $\Omega_f = (-\delta, \delta) \times \mathbb{R}$ as illustrated in Figure 1. We consider the following Darcy (in the matrix) Stokes (in the fracture) coupled model:

$$
\begin{aligned}
-\mu \Delta \mathbf{u} + \nabla p &= 0 && \text{on } \Omega_f,\\
\operatorname{div} \mathbf{u} &= 0 && \text{on } \Omega_f,\\
\operatorname{div}(\mathbf{u}_i) &= f_i && \text{on } \Omega_i, i = 1,2,\\
\mathbf{u}_i &= -\mathbb{K}_i \nabla p_i && \text{on } \Omega_i, i = 1,2,
\end{aligned}
$$

combined with the following coupling conditions on $\Gamma_1 = \{-\delta\} \times \mathbb{R}$ and $\Gamma_2 = \{\delta\} \times \mathbb{R}$:

$$\mathbf{u}_i \cdot \mathbf{n}_i = \mathbf{u} \cdot \mathbf{n}_i \qquad \text{on } \Gamma_i, i = 1,2, \tag{1}$$
eq_fluxcont

$$p_i = p - \mu (\nabla \mathbf{u}\, \mathbf{n}_i) \cdot \mathbf{n}_i \qquad \text{on } \Gamma_i, i = 1,2, \tag{2}$$
eq_pjump

$$\mu (\nabla \mathbf{u}\, \mathbf{n}_i) \cdot \boldsymbol{\tau} = \alpha \mathbf{u} \cdot \boldsymbol{\tau} \qquad \text{on } \Gamma_i, i = 1,2, \tag{3}$$
eq_BJ

where $\mathbf{n}_i$ is the unit normal vector on Γ_i, oriented outward of Ω_i, $\boldsymbol{\tau}$ is the unit vector tangent to the interfaces oriented in the positive y direction, $\mu > 0$ is the fluid kinematic

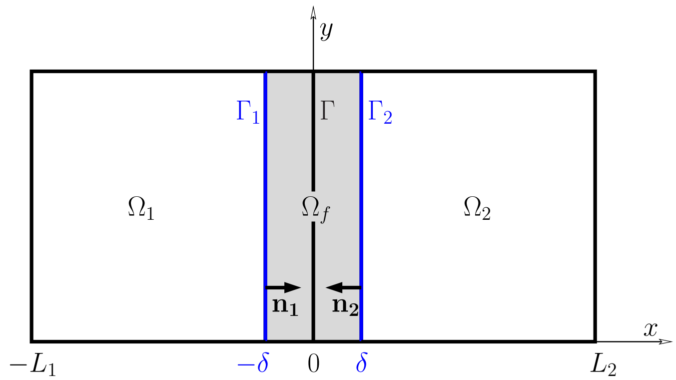

Fig. 1. Model problem geometry, with $\Omega_1 = (-L_1, -\delta) \times \Gamma$, $\Omega_2 = (\delta, L_2) \times \Gamma$, $\Gamma_1 = \{-\delta\} \times \Gamma$, $\Gamma_2 = \{\delta\} \times \Gamma$, and $\Omega_f = (-\delta, \delta) \times \Gamma$. The unit normals on Γ_j pointing outside of Ω_j are denoted by $\mathbf{n}_j$, $j = 1,2$. Note that the Fourier analysis below will be carried out on unbounded domains by setting $\Gamma = \mathbb{R}$. fig:asymE

viscosity, α is the Beaver-Joseph-Saffman parameter assumed to be constant for simplicity, and $\mathbb{K}_i$ is the permeability tensor in subdomain Ω_i. We also set $\mathbf{n} = \mathbf{n}_1 = -\mathbf{n}_2$ in what follows.

2 Dimensional reduction by Fourier analysis

2.1 Elimination of the fracture by Fourier analysis

Let us set $\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}$, $\delta_i = (-1)^i \delta$, and take the Fourier transform in the y direction of the Stokes equations and of the transmission conditions. Setting in short

$$\hat{u}_i = \widehat{\mathbf{u}_i \cdot \mathbf{n}}(\delta_i, k), \quad \hat{p}_i = \hat{p}_i(\delta_i, k),$$

leads to the system

$$\begin{array}{rlrl}
-\mu \partial_{xx} \hat{u}(x,k) + \mu k^2 \hat{u}(x,k) + \partial_x \hat{p}(x,k) = 0 & x \in (-\delta, \delta), & (4) & \text{eq_momentumu} \\
-\mu \partial_{xx} \hat{v}(x,k) + \mu k^2 \hat{v}(x,k) + ik \hat{p}(x,k) = 0 & x \in (-\delta, \delta), & (5) & \text{eq_momentumv} \\
\partial_x \hat{u}(x,k) + ik \hat{v}(x,k) = 0 & x \in (-\delta, \delta), & (6) & \text{eq_divU} \\
\hat{p}_i = \hat{p}(\delta_i, k) - \mu \partial_x \hat{u}(\delta_i, k) & i = 1,2, & (7) & \text{eq_pjumpFourier} \\
(-1)^{i+1} \mu \partial_x \hat{v}(\delta_i, k) = \alpha \hat{v}(\delta_i, k) & i = 1,2, & (8) & \text{eq_BJFourier} \\
\hat{u}_i = \hat{u}(\delta_i, k) & i = 1,2. & (9) & \text{eq_fluxcontFourier}
\end{array}$$

Using that $\Delta p = 0$ yields the equation $\partial_{xx} \hat{p}(x,k) - k^2 \hat{p}(x,k) = 0$, whose solution is $\hat{p}(x,k) = C_1(k) e^{|k|x} + C_2(k) e^{-|k|x}$. We next substitute this pressure solution $\hat{p}$ into the momentum equations (4)-(5) of the previous system yielding four additional integration constants $C_j(k)$ with $j = 3,4,5,6$. These 6 integration constants can be computed using the divergence free condition (6) (providing two additional equations on these 6 constants) and the transmission conditions (7)-(8). The last two transmission conditions (9) are then used to provide the following two exact transmission conditions of the model posed on $\Omega_1 \cup \Omega_2$ eliminating the fracture model:

$$\mu|k| \begin{pmatrix} H_1^{ex}(|k|\delta) & 0 \\ 0 & H_2^{ex}(|k|\delta) \end{pmatrix} \begin{pmatrix} \hat{u}_2 + \hat{u}_2 \\ \hat{u}_1 - \hat{u}_2 \end{pmatrix} = \begin{pmatrix} \hat{p}_1 - \hat{p}_2 \\ \hat{p}_1 + \hat{p}_2 \end{pmatrix}, \tag{10}$$

trans_cond_exactes

where, setting $\xi := |k|\delta$,

$$\begin{aligned} H_1^{ex}(\xi) &= \frac{-4(1+C_\alpha \xi^2)e^{2\xi} + (2+3C_\alpha \xi)e^{4\xi} + (2-3C_\alpha \xi)}{4\xi(1+C_\alpha)e^{2\xi} + (1+2C_\alpha \xi)e^{4\xi} + (2C_\alpha \xi - 1)}, \\ H_2^{ex}(\xi) &= \frac{4(1+C_\alpha \xi^2)e^{2\xi} + (2+3C_\alpha \xi)e^{4\xi} + (2-3C_\alpha \xi)}{-4\xi(1+C_\alpha)e^{2\xi} + (1+2C_\alpha \xi)e^{4\xi} + (2C_\alpha \xi - 1)}, \end{aligned} \tag{11}$$

H_exactes

and $C_\alpha := \dfrac{\mu}{\alpha\delta}$ is a dimensionless parameter governing the Beaver-Joseph-Saffman condition (3). To simplify the presentation, we develop in the following the analysis for the case $\alpha = +\infty$, i.e. $C_\alpha = 0$, corresponding to replacing the Beaver-Joseph-Saffman condition by the no slip condition $\mathbf{u} \cdot \boldsymbol{\tau} = 0$. This approximation is valid for a wide range of not too large rock permeabilities. The discussion of the general case is postponed to Section 4.

2.2 Reduced transmission conditions

An asymptotic expansion of H_i^{ex}, $i = 1,2$, with respect to small ξ provides the reduced transmission conditions

$$\mu|k| \begin{pmatrix} H_1^{red}(|k|\delta) & 0 \\ 0 & H_2^{red}(|k|\delta) \end{pmatrix} \begin{pmatrix} \hat{u}_2 + \hat{u}_2 \\ \hat{u}_1 - \hat{u}_2 \end{pmatrix} = \begin{pmatrix} \hat{p}_1 - \hat{p}_2 \\ \hat{p}_1 + \hat{p}_2 \end{pmatrix}, \tag{12}$$

red_transmissions

with the approximation H_i^{red} of H_i^{ex} given by

$$H_1^{red}(\xi) = \xi, \qquad H_2^{red}(\xi) = \frac{3}{\xi^3}\Big(1 + \frac{4}{5}\xi^2\Big),$$

at order $O(\xi^5)$ and $O(\xi)$. Note that these orders of approximation are the highest ones providing a well-posed reduced model, i.e. such that $|k|\ H_i^{red}(|k|\delta) > 0$ for all $k > 0$. Setting for $i = 1,2$

$$\gamma_i^{\mathbf{n}} \mathbf{u}_i = \mathbf{u}_i \cdot \mathbf{n}\Big(\delta_i, \cdot\Big), \qquad \gamma_i p_i = p_1\Big(\delta_i, \cdot\Big),$$

provides the following reduced model with elimination of the fracture unknowns:

$$\begin{aligned} &\operatorname{div}(\mathbf{u}_i) = f_i && \text{on } \Omega_i,\ i = 1,2, \\ &\mathbf{u}_i = -\mathbb{K}_i \nabla p_i && \text{on } \Omega_i,\ i = 1,2, \\ &-\mu \partial_{yy} \frac{(\gamma_1^{\mathbf{n}} \mathbf{u}_1 + \gamma_2^{\mathbf{n}} \mathbf{u}_2)}{2} = \frac{\gamma_1 p_1 - \gamma_2 p_2}{2\delta} && \text{on } \mathbb{R}, \\ &\mu\Big(1 - \frac{4}{5}\delta^2\, \partial_{yy}\Big) \frac{(\gamma_1^{\mathbf{n}} \mathbf{u}_1 - \gamma_2^{\mathbf{n}} \mathbf{u}_2)}{2\delta} = -\frac{\delta^2}{3} \partial_{yy} \frac{(\gamma_1 p_1 + \gamma_2 p_2)}{2} && \text{on } \mathbb{R}. \end{aligned} \tag{13}$$

red_mat

2.3 Reconstruction along the fracture

As in [3,2], the reconstruction along the fracture starts with averaging both the Stokes unknowns and equations along the fracture width, setting

$$\hat{P} := \frac{1}{2\delta}\int_{-\delta}^{\delta} \hat{p}(x,k)dx, \quad \hat{U} := \frac{1}{2\delta}\int_{-\delta}^{\delta} \hat{u}(x,k)dx, \quad \hat{V} := \frac{1}{2\delta}\int_{-\delta}^{\delta} \hat{v}(x,k)dx.$$

From the divergence free condition (6), we obtain by integration along the fracture width the reduced material conservation equation

$$ik2\delta\,\hat{V} = \hat{u}_1 - \hat{u}_2. \tag{14}$$

eq_divu_frac

By integration of the momentum equation (4), and taking into account the pressure jump condition (7), we get that

$$\mu|k|^2 2\delta\hat{U} = (\hat{p}_1 - \hat{p}_2). \tag{15}$$

eq_u_frac

By integration of the momentum equation (5), we get the relation

$$-\mu(\partial_x\hat{v}(\delta,k) - \partial_x\hat{v}(-\delta,k)) + \mu|k|^2 2\delta\,\hat{V} + ik2\delta\,\hat{P} = 0. \tag{16}$$

eq_exact_P

Then, the classical approach developed in [3,2] amounts to make profile assumptions along the width for U, V and P in order to derive both the coupling conditions and the approximation of the wall friction term $-\mu(\partial_x\hat{v}(\delta,k) - \partial_x\hat{v}(-\delta,k))$.

In our approach the coupling conditions were already derived by Fourier analysis and asymptotic expansions. The approximation of the friction term is obtained in the same way from the Fourier expression of $\partial_x\hat{v}(x,k)$ which can be shown to lead to

$$F^{ex}(\xi) = \delta\frac{(\partial_x\hat{v}(-\delta,k) - \partial_x\hat{v}(\delta,k))}{\hat{V}} = -2\frac{\xi^2\left(4\xi e^{2\xi} + e^{4\xi} - 1\right)}{4\xi e^{2\xi} - e^{4\xi} + 1}.$$

By asymptotic expansion for small $\xi = |k|\delta$, we obtain the following approximation F^{red} of F^{ex} at order $O(\xi^4)$:

$$F^{red}(\xi) = 6 + \frac{4}{5}\xi^2,$$

which leads to

$$\frac{6\mu}{\delta}\hat{V} + \widetilde{\mu}|k|^2 2\delta\,\hat{V} + ik2\delta\,\hat{P} = 0, \tag{17}$$

eq_v_frac

with the modified tangential viscosity $\widetilde{\mu} = \left(1 + \frac{2}{5}\right)\mu$.

Equations (14)-(15)-(17) are the reconstructed equations along the fracture. These equations can be combined with (13) in order to obtain the following coupled formulation of the reduced model:

$$\begin{aligned}
&\mathrm{div}(\mathbf{u}_i) = f_i && \text{on } \Omega_i,\ i=1,2,\\
&\mathbf{u}_i = -\mathbb{K}_i \nabla p_i && \text{on } \Omega_i,\ i=1,2,\\
&2\delta\ \partial_y V = \gamma_1^{\mathbf{n}}\mathbf{u}_1 - \gamma_2^{\mathbf{n}}\mathbf{u}_2, && \text{on } \mathbb{R},\\
&-2\mu\delta\ \partial_{yy} U = \gamma_1 p_1 - \gamma_2 p_2 && \text{on } \mathbb{R},\\
&6\frac{\mu}{\delta} V - 2\widetilde{\mu}\delta\ \partial_{yy} V + 2\delta\ \partial_y P = 0 && \text{on } \mathbb{R},\\
&U = \frac{\gamma_1^{\mathbf{n}}\mathbf{u}_1 + \gamma_2^{\mathbf{n}}\mathbf{u}_2}{2} && \text{on } \mathbb{R},\\
&\frac{\mu}{\delta}\left(\gamma_1^{\mathbf{n}}\mathbf{u}_1 - \gamma_2^{\mathbf{n}}\mathbf{u}_2\right) = \gamma_1 p_1 + \gamma_2 p_2 - 2P && \text{on } \mathbb{R}.
\end{aligned} \tag{18}$$

red_coupledfracmat

Compared with the classical approach developped in [3,2] our methodology leads to a correction term which amounts to replace the tangential viscosity μ by $\widetilde{\mu}$ in the fifth equation of (18). This correction plays an essential role to obtain the error estimates shown in the next section.

3 Error estimates

We use the same setting as in [1] for the Darcy subproblems assuming for simplicity that $\mathbb{K}_1 = \mathbb{K}_2 = I$ and considering homogeneous Dirichlet conditions on $\partial\Omega_i \setminus \overline{\Gamma}$. For each subdomain $i = 1,2$, we denote by $\hat{s}_i \geq 0$ the Fourier transform of the Steklov Poincaré operator with $\hat{s}_i = |k|\coth(|k|(L_i - \delta))$, and we denote by $\widehat{R(f_i)}$ the Fourier transform of $\gamma_i^n \nabla(\Delta^{-1} f_i)$ with Δ^{-1} defined on Ω_i with homogeneous Dirichlet boundary conditions on $\partial\Omega_i$. In this section, the superscripts *red* and *ex* are used for the reduced and exact model solutions. We assume in the following that δ is such that $\delta \leq L = \min(\frac{L_1}{2}, \frac{L_2}{2})$.

3.1 Error estimates on the traces $\gamma_i p_i$ and $\gamma_i^{\mathbf{n}}\mathbf{u}_i$

sec_error1

For the exact and reduced solutions we have, with $\bullet = \mathrm{red}, \mathrm{ex}$,

$$\hat{u}_1^{\bullet} = -\hat{s}_1 \hat{p}_1^{\bullet} - \widehat{R(f_1)}, \quad \hat{u}_2^{\bullet} = \hat{s}_2 \hat{p}_2^{\bullet} - \widehat{R(f_2)}.$$

We want to provide an error estimate for the errors on the traces

$$\hat{e}_{p_i} = \hat{p}_i^{ex} - \hat{p}_i^{red}, \qquad \hat{e}_{u_i} = \hat{u}_i^{ex} - \hat{u}_i^{red},$$

for $i = 1,2$ which are linked by the relations $\hat{e}_{u_i} = (-1)^i \hat{s}_i \hat{e}_{p_i}$.

From the exact and reduced transmission conditions (10) and (12), setting

$$E_i = H_i^{ex} - H_i^{red},$$

and

$$D(k) = \left(\frac{1}{\mu|k|\hat{s}_1} + H_1^{red}\right)\left(\frac{1}{\mu|k|\hat{s}_2} + H_2^{red}\right) + \left(\frac{1}{\mu|k|\hat{s}_2} + H_1^{red}\right)\left(\frac{1}{\mu|k|\hat{s}_1} + H_2^{red}\right),$$

we obtain that

$$\hat{e}_{u_1} = \frac{-(\frac{1}{\mu|k|\hat{s}_2} + H_2^{red})E_1(\hat{u}_1^{ex} + \hat{u}_2^{ex}) - (\frac{1}{\mu|k|\hat{s}_2} + H_1^{red})E_2(\hat{u}_1^{ex} - \hat{u}_2^{ex})}{D(k)},$$
$$\hat{e}_{u_2} = \frac{-(\frac{1}{\mu|k|\hat{s}_1} + H_2^{red})E_1(\hat{u}_1^{ex} + \hat{u}_2^{ex}) + (\frac{1}{\mu|k|\hat{s}_1} + H_1^{red})E_2(\hat{u}_1^{ex} - \hat{u}_2^{ex})}{D(k)}.$$

It remains to estimate $|\hat{e}_{u_i}|$. We can establish the following bounds

$$\frac{|E_2(\xi)|}{\xi} \leq C_2, \quad \frac{|E_1(\xi)|}{\xi^5} \leq C_1, \quad \forall \xi \geq 0,$$

and

$$\left|\frac{1}{H_2^{ex}(\xi)}\right| \leq C_3\xi^3, \quad \frac{1}{H_2^{red}(\xi)} \leq C_3\xi^3, \quad k \leq \hat{s}_i(k) \leq k + \frac{1}{L}, \quad \forall \xi, k \geq 0,$$

with $C_1 = \frac{1}{45}$, $C_2 = \frac{81}{175}$, $C_3 = \frac{1}{3}$. We deduce the estimates

$$|\hat{e}_{u_i}| = \left[\mu|k|(|k| + \frac{1}{L})C_1|k|\delta|\hat{u}_1^{ex} + \hat{u}_2^{ex}| + C_2C_3|\hat{u}_1^{ex} - \hat{u}_2^{ex}|\right] |k|^4\delta^4, \tag{19}$$

and

$$|\hat{e}_{u_i}| = \left[\mu|k|(|k| + \frac{1}{L})C_1|\hat{u}_1^{ex} + \hat{u}_2^{ex}| + \frac{1}{\mu|k|}C_2(C_3)^2|k|^2\delta^2|\hat{p}_1^{ex} + \hat{p}_2^{ex}|\right] |k|^5\delta^5. \tag{20}$$

Estimates on $\hat{e}_{p_i}$ are readily deduced from the relations $\hat{e}_{u_i} = (-1)^i\hat{s}_1\hat{e}_{p_i}$. An improved estimate can also be derived on $\hat{e}_{p_1} - \hat{e}_{p_2}$ using the additional bound $|\frac{1}{\hat{s}_1} - \frac{1}{\hat{s}_2}| \leq \frac{1}{|k|(L|k|+1)}$:

$$|\hat{e}_{p_1} - \hat{e}_{p_2}| \leq \mu\left[2(|k| + \frac{1}{L})C_1|\hat{u}_1^{ex} + \hat{u}_2^{ex}| + \frac{1}{2L}C_2C_3|\hat{u}_1^{ex} - \hat{u}_2^{ex}|\right] |k|^5\delta^5. \tag{21}$$

3.2 Error estimates on the fracture mean values U, V and P

Let us proceed with the error estimates on the fracture mean values $\hat{V}$, $\hat{U}$ and $\hat{P}$. For the error $\hat{e}_V = \hat{V}^{ex} - \hat{V}^{red}$, we have from (14) the bound

$$|\hat{e}_V| \leq \frac{1}{|k|2\delta}|\hat{e}_{u_1} - \hat{e}_{u_2}|,$$

then, it suffices to apply (19) or (20) providing respectively an $O(\delta^3)$ or an $O(\delta^4)$ error estimate.

Similarly, for the error $\hat{e}_U = \hat{U}^{ex} - \hat{U}^{red}$, we have from (15) the bound

$$|\hat{e}_U| \leq \frac{1}{\mu 2\delta|k|^2}|\hat{e}_{p_1} - \hat{e}_{p_2}|.$$

Then, it suffices to apply (21) providing an $O(\delta^4)$ error estimate.

To estimate the error on the mean pressure, it can be shown that there exists $C_4 = \frac{22}{175}$ such that

$$\frac{|F^{ex}(\xi) - F^{red}(\xi)|}{\xi^4} \leq C_4, \quad \forall \xi \geq 0.$$

Then, we deduce from (16) and the definition of F^{ex} the following error estimate for $\hat{e}_P = \hat{P}^{ex} - \hat{P}^{red}$:

$$|\hat{e}_P| \leq \mu \Big[\Big((1 + \frac{2}{15C_3})|k| + \frac{1}{C_3}|k|^{-1}\delta^{-2} \Big) |\hat{e}_V| + \frac{C_4}{2}|k|^3\delta^2|\hat{V}^{ex}| \Big],$$

of order $O(\delta^2)$.

4 Extension to the general Beaver Joseph Saffman condition

In the general case, the functions H_i^{ex} and F^{ex} depend on two dimensionless parameters, namely $|k|\delta$ and $C_\alpha = \frac{\mu}{\alpha\delta}$. The extension distinguishes two cases, first $\alpha > 0$ (including the previous case $\alpha = +\infty$ i.e. $C_\alpha = 0$) and second $\alpha = 0$. In the first case, the asymptotic expansions of H_i^{ex} and F^{ex} are done for small values of $|k|\delta$ at given $C_\alpha < +\infty$. This choice permits to recover the proper wall friction term in the V momentum equation (22). We obtain the same model as in (18) with modified coefficients for the fifth equation:

$$\frac{6\frac{\mu}{\delta}}{1+3C_\alpha} V - 2\widetilde{\mu}\delta\ \partial_{yy}V + 2\delta\ \partial_y P = 0. \tag{22}$$

eq_VCalpha

The tangential viscosity $\widetilde{\mu} = \Big(1 + \frac{2}{5(3C_\alpha+1)^2}\Big)\mu$ is again corrected compared with the classical model reduction approach for which $\widetilde{\mu} = \mu$. The error estimates are the same as in Subsections (3.1) and (3.2) with constants C_i, $i \in \{1,2,3,4\}$ depending on C_α.

In the second case, for $\alpha = 0$ corresponding to $C_\alpha = +\infty$, the expansions of H_i^{ex} are done w.r.t. small values of $|k|\delta$ and $F^{ex} = F^{red} = 0$. We obtain the following reduced model:

$$\begin{aligned}
&\mathrm{div}(\mathbf{u}_i) = f_i && \text{on } \Omega_i,\ i = 1,2,\\
&\mathbf{u}_i = -\mathbb{K}_i \nabla p_i && \text{on } \Omega_i,\ i = 1,2,\\
&2\delta\ \partial_y V = \gamma_1^{\mathbf{n}}\mathbf{u}_1 - \gamma_2^{\mathbf{n}}\mathbf{u}_2 && \text{on } \mathbb{R},\\
&-2\mu\delta\ \partial_{yy}U = \gamma_1 p_1 - \gamma_2 p_2 && \text{on } \mathbb{R},\\
&-\mu\ \partial_{yy}V + \partial_y P = 0 && \text{on } \mathbb{R},\\
&U = \frac{\gamma_1^{\mathbf{n}}\mathbf{u}_1 + \gamma_2^{\mathbf{n}}\mathbf{u}_2}{2} && \text{on } \mathbb{R},\\
&\frac{\mu}{\delta}\Big(1 - \frac{\delta^2}{6}\partial_{yy}\Big)\Big(\gamma_1^{n}\mathbf{u}_1 - \gamma_2^{\mathbf{n}}\mathbf{u}_2\Big) = \gamma_1 p_1 + \gamma_2 p_2 - 2P && \text{on } \mathbb{R},
\end{aligned} \tag{23}$$

red_coupledfracmatalph

which differs in the last equation from the model obtained by the classical model reduction approach [3] providing the equation $\frac{\mu}{\delta}\Big(\gamma_1^{\mathbf{n}}\mathbf{u}_1 - \gamma_2^{\mathbf{n}}\mathbf{u}_2\Big) = \gamma_1 p_1 + \gamma_2 p_2 - 2P$. The error estimates for the case $\alpha = 0$ differ from the ones of Subsections (3.1) and (3.2). Setting $C_1 = \frac{2}{15}$ and $C_2 = \frac{2}{945}$, we obtain

$$|\hat{e}_{u_i}| \leq \varepsilon |k|(|k| + \frac{1}{L})|k|^5 \Big(C_1 |\hat{u}_1^{ex} + \hat{u}_2^{ex}| + C_2 |\hat{u}_1^{ex} - \hat{u}_2^{ex}| \Big) \delta^5, \quad i = 1,2,$$

and

$$|\hat{e}_V| \leq \frac{1}{|k|} \frac{|\hat{e}_{u_1} - \hat{e}_{u_2}|}{2\delta} \leq \varepsilon (|k| + \frac{1}{L})|k|^5 \Big(C_1 |\hat{u}_1^{ex} + \hat{u}_2^{ex}| + C_2 |\hat{u}_1^{ex} - \hat{u}_2^{ex}| \Big) \delta^4,$$

$$|\hat{e}_P| \leq \varepsilon |k| |\hat{e}_V| \leq \varepsilon^2 (|k| + \frac{1}{L})|k|^6 \Big(C_1 |\hat{u}_1^{ex} + \hat{u}_2^{ex}| + C_2 |\hat{u}_1^{ex} - \hat{u}_2^{ex}| \Big) \delta^4,$$

$$|\hat{e}_U| \leq \frac{1}{\varepsilon 2\delta |k|^3} (|\hat{e}_{u_1}| + |\hat{e}_{u_2}|) \leq (|k| + \frac{1}{L})|k|^3 \Big(C_1 |\hat{u}_1^{ex} + \hat{u}_2^{ex}| + C_2 |\hat{u}_1^{ex} - \hat{u}_2^{ex}| \Big) \delta^4.$$

5 Conclusions

subsec:concl

This work extends the dimensional reduction methodology based on Fourier analysis developed in [1] to the case of a Darcy-Stokes matrix fracture coupled model. This analysis leads to correction terms which cannot be a priori obtained by the classical technique based on averaging along the fracture width combined with profile assumptions on the velocities and pressure in the fracture [3,2]. More precisely, the new mixed-dimensional model exhibits a correction of the tangential viscosity along the fracture in the case $\alpha > 0$ and a second order correction term in the second closure equation in the case $\alpha = 0$. These terms play an essential role in the error estimates between the equi and mixed-dimensional models derived by the Fourier analysis. Numerical tests are ongoing in order to assess numerically these results.

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