On Nilpotent Subdomain Iterations

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1 Introduction and model problem

Subdomain iterations which lead to a nilpotent iteration operator converge in a finite number of steps, and thus are equivalent to direct solvers. Such methods have led to very powerful new algorithms over the last few years, like the sweeping preconditioner of Engquist and Ying [4, 5], or the source transfer domain decomposition method of Chen and Xiang [1, 2]. Their underlying mathematical structure are optimal Schwarz methods, see [14, 6, 7] and references therein[1].

We study here under which conditions the classical Neumann-Neumann, Dirichlet-Neumann and optimal Schwarz method can be nilpotent for the model problem

$$\eta u - \partial_{xx} u = f \text{ in } \Omega := (0,1), \quad u(0) = u(1) = 0, \tag{1}$$

and a decomposition of the domain into J subdomains, $\Omega_j := (x_{j-1}, x_j)$, with $0 = x_0 < x_1 < \ldots < x_J = 1$ and subdomain length $\ell_j := x_j - x_{j-1}$. For two subdomains, we show that they all can be made nilpotent. For three subdomains, Neumann-Neumann can not be made nilpotent any more, but Dirichlet-Neumann can. For four subdomains, also Dirichlet-Neumann can not be made nilpotent any more for general decompositions, but for decompositions with subdomains of equal size, Dirichlet-Neumann can be made nilpotent for an arbitrary number of subdomains. Optimal Schwarz methods are always nilpotent for an arbitrary number of subdomains, even unequal ones. Our results indicate that for more general problems and more than two subdomains, only the optimal Schwarz method will be nilpotent.

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[1] Optimal here is not in the sense of scalable, but really optimal: faster convergence is not possible

2 The Neumann-Neumann algorithm

For two subdomains, $J = 2$, the Neumann-Neumann algorithm applied to (1) is

$$\begin{cases} \eta u_j^{(n)} - \partial_{xx} u_j^{(n)} = f \text{ in } \Omega_j, \\ u_j^{(n)}(x_1) = h^{(n)}, \end{cases} \quad \begin{cases} \eta \psi_j^{(n)} - \partial_{xx} \psi_j^{(n)} = 0 \text{ in } \Omega_j, \\ \partial_{n_j} \psi_j^{(n)}(x_1) = \partial_{n_1} u_1^{(n)}(x_1) + \partial_{n_2} u_2^{(n)}(x_1), \end{cases}$$
$$h^{(n+1)} := h^{(n)} - \theta(\psi_1^{(n)}(x_1) + \psi_2^{(n)}(x_1)), \tag{2}$$

with $h^{(0)}$ an initial guess, θ a relaxation parameter, and in each iteration $u_1^{(n)}(0) = u_2^{(n)}(1) = 0$ and $\psi_1^{(n)}(0) = \psi_2^{(n)}(1) = 0$.

Since the problem is linear, it suffices to consider the homogeneous case of equation (1) and analyze the convergence of (2) to the zero solution. For $\eta > 0$ and $f = 0$, the differential equations in (2) can readily be solved[2], and we obtain for the relaxation after a short calculation the relation

$$h^{(n+1)} = (1 - \theta(2 + \varphi(\eta)))h^{(n)}, \quad \varphi(t) := \frac{\tanh(\sqrt{t}\ell_1)}{\tanh(\sqrt{t}\ell_2)} + \frac{\tanh(\sqrt{t}\ell_2)}{\tanh(\sqrt{t}\ell_1)}, \ t > 0. \tag{3}$$

Proposition 1. *For two subdomains, the Neumann-Neumann algorithm* (2) *is convergent iff* $0 < \theta < \theta_\eta^*$, $\theta_\eta^* := \frac{2}{2+\varphi(\eta)}$. *Moreover, convergence is reached after two iterations for* $\theta := \frac{\theta_\eta^*}{2}$, *which in the symmetric case (i.e.* $x_1 = \frac{1}{2}$*) becomes* $\theta := \frac{1}{4}$, *i.e. the method is then nilpotent.*

Proof. The convergence factor of the Neumann-Neumann algorithm (2) is $\rho_{\theta,\eta} := |1 - \theta(2 + \varphi(\eta))|$, and thus the algorithm is convergent iff $\rho_{\theta,\eta} < 1$, which is equivalent to requiring that $0 < \theta < \theta_\eta^*$. Moreover, $\rho_{\theta,\eta}$ vanishes when $\theta := \frac{\theta_\eta^*}{2}$, which makes the algorithm nilpotent.

Proposition 2. *For three subdomains, it is not possible to make the Neumann-Neumann algorithm nilpotent in general.*

Proof. We consider the analogous definition of the Neumann-Neumann algorithm from (2) for three equal subdomains, i.e. $x_0 = 0$, $x_1 = \frac{1}{3}$, $x_2 = \frac{2}{3}$, $x_3 = 1$, and obtain after a short calculation as in Proposition 1 with explicit subdomain solutions

$$\begin{pmatrix} h_1^{(n+1)} \\ h_2^{(n+1)} \end{pmatrix} = \begin{pmatrix} 1 - \theta_1(4 + \frac{1}{s^2}) & -\frac{\theta_1}{cs^2} \\ -\frac{\theta_2}{cs^2} & 1 - \theta_2(4 + \frac{1}{s^2}) \end{pmatrix} \begin{pmatrix} h_1^{(n)} \\ h_2^{(n)} \end{pmatrix}, \tag{4}$$

where $s := \sinh(\sqrt{\eta}/3)$ and $c := \cosh(\sqrt{\eta}/3)$. Convergence in a finite number of iterations is possible iff the spectral radius of the iteration matrix in (4) vanishes, which means that the characteristic polynomial must be a monomial of degree 2. The fact that the other coefficients must vanish implies that the relaxation parameters θ_1 and θ_2 must satisfy the system of equations

[2] all our results remain valid also for $\eta = 0$ by taking limits

$$(4+\frac{1}{s^2})\theta_1+(4+\frac{1}{s^2})\theta_2=2 \quad \text{and} \quad (4+\frac{1}{s^2})^2\theta_1\theta_2=\alpha, \tag{5}$$

where $\alpha := \frac{(4+\frac{1}{s^2})^2}{(4+\frac{1}{s^2})^2-(\frac{1}{s^2c})^2} > 1$. Now (5) has no real solution, since the associated characteristic equation $\lambda^2-2\lambda+\alpha=0$ does not admit one. It is thus not possible in general to obtain a nilpotent iteration for the Neumann-Neumann algorithm with three subdomains.

We will see in the numerical section that also for more than three subdomains, it is not possible in general to make the Neumann-Neumann algorithm nilpotent, and we will even get divergent iterations.

3 The Dirichlet-Neumann algorithm

The Dirichlet-Neumann algorithm applied to (1) for two subdomains is

$$\begin{cases} \eta u_1^{(n)}-\partial_{xx}u_1^{(n)}=f \text{ in } \Omega_1, \\ u_1^{(n)}(x_1)=h^{(n)}, \end{cases} \quad \begin{cases} \eta u_2^{(n)}-\partial_{xx}u_2^{(n)}=f \text{ in } \Omega_2, \\ \partial_x u_2^{(n)}(x_1)=\partial_x u_1^{(n)}(x_1), \end{cases} \\ h^{(n+1)} := (1-\theta)h^{(n)}+\theta u_2^{(n)}(x_1), \tag{6}$$

with $h^{(0)}$ an initial guess, θ a relaxation parameter, and $u_1^{(n)}(0)=u_2^{(n)}(1)=0$. As for the Neumann-Neumann algorithm, we study the homogeneous part of eq. (1), and obtain after a short calculation using the explicitly available subdomain solutions

$$h^{(n+1)}=(1-\theta(1+\psi(\eta)))h^{(n)}, \quad \psi(t):=\frac{\tanh(\sqrt{t}\ell_2)}{\tanh(\sqrt{t}\ell_1)},\ t>0. \tag{7}$$

Proposition 3. *The Dirichlet-Neumann algorithm* (6) *is convergent for two subdomains iff* $0<\theta<\theta_\eta^*$, $\theta_\eta^* := \frac{2}{1+\psi(\eta)}$. *Moreover, convergence is reached after two iterations for* $\theta := \frac{\theta_\eta^*}{2}$, *which in the symmetric case (i.e.* $x_1=\frac{1}{2}$*) becomes* $\theta := \frac{1}{2}$, *i.e. the algorithm is then nilpotent.*

Proof. The proof is similar to the proof of Proposition 1.

Proposition 4. *For three subdomains, the Dirichlet-Neumann algorithm converges in three iterations if either*

$$(\theta_1^*,\theta_2^*)=\left(\frac{1-\sqrt{1-\alpha}}{1+\frac{c_1s_2}{s_1c_2}},\frac{1+\sqrt{1-\alpha}}{1+\frac{s_2s_3}{c_2c_3}}\right) \quad \text{or} \quad (\theta_1^*,\theta_2^*)=\left(\frac{1+\sqrt{1-\alpha}}{1+\frac{c_1s_2}{s_1c_2}},\frac{1-\sqrt{1-\alpha}}{1+\frac{s_2s_3}{c_2c_3}}\right), \tag{8}$$

where $s_i := \sinh(\sqrt{\eta}\ell_i)$, $c_i := \cosh(\sqrt{\eta}\ell_i)$, $i=1,\ldots,3$, *and* $\alpha := \frac{(1+\frac{c_1s_2}{s_1c_2})(1+\frac{s_2s_3}{c_2c_3})}{1+\frac{c_1s_2}{s_1c_2}+\frac{s_2s_3}{c_2c_3}+\frac{c_1s_3}{s_1c_3}}$.

Proof. With the analogously to (6) defined Dirichlet-Neumann algorithm for three subdomains, and solving the subdomain problems explicitly, we obtain after a short calculation

$$\begin{pmatrix} h_1^{(n+1)} \\ h_2^{(n+1)} \end{pmatrix} = \begin{pmatrix} 1-\theta_1(1+\frac{c_1 s_2}{s_1 c_2}) & \frac{\theta_1}{c_2} \\ -\theta_2\frac{c_1 s_3}{s_1 c_2 c_3} & 1-\theta_2(1+\frac{s_2 s_3}{c_2 c_3}) \end{pmatrix} \begin{pmatrix} h_1^{(n)} \\ h_2^{(n)} \end{pmatrix}, \tag{9}$$

and the matrix is nilpotent iff its spectral radius vanishes, i.e.

$$\theta_1(1+\frac{c_1 s_2}{s_1 c_2})+\theta_2(1+\frac{s_2 s_3}{c_2 c_3}) = 2, \quad (1+\frac{c_1 s_2}{s_1 c_2})(1+\frac{s_2 s_3}{c_2 c_3})\theta_1\theta_2 = \alpha. \tag{10}$$

This system admits the real solutions given in (8), since $0<\alpha<1$.

Proposition 5. *For four subdomains, convergence of the Dirichlet-Neumann algorithm in a finite number of iterations can not always be achieved.*

Proof. We focus for simplicity on the case $\eta=0$ and obtain for the analogously to (6) defined Dirichlet-Neumann algorithm for four subdomains after a short calculation

$$\begin{pmatrix} h_1^{(n+1)} \\ h_2^{(n+1)} \\ h_3^{(n+1)} \end{pmatrix} = \begin{pmatrix} 1-\left(\frac{\ell_2}{\ell_1}+1\right)\theta_1 & \theta_1 & 0 \\ -\frac{\theta_2\ell_3}{\ell_1} & 1-\theta_2 & \theta_2 \\ -\frac{\theta_3\ell_4}{\ell_1} & 0 & 1-\theta_3 \end{pmatrix} \begin{pmatrix} h_1^{(n)} \\ h_2^{(n)} \\ h_3^{(n)} \end{pmatrix}. \tag{11}$$

For nilpotence, the spectral radius of (11) must vanish, which means that the characteristic polynomial must be a monomial of degree 3. The fact that the other coefficients must vanish implies after a short calculation that θ_1, θ_2 and θ_3 must satisfy the system of equations $(1+\frac{\ell_2}{\ell_1})\theta_1+\theta_2+\theta_3=3,\ (1+\frac{\ell_2+\ell_3}{\ell_1})\theta_1\theta_2+(1+\frac{\ell_2}{\ell_1})\theta_1\theta_3+\theta_2\theta_3=3,\ (1+\frac{\ell_2+\ell_3+\ell_4}{\ell_1})\theta_1\theta_2\theta_3=1$. Substituting the first equation into the second one we obtain $\frac{\ell_1+\ell_2+\ell_3}{\ell_1}\theta_1\theta_2+\theta_3(3-\theta_3)=3 \implies \frac{(1-\ell_4)}{\ell_1}\theta_1\theta_2+\theta_3(3-\theta_3)=3$, and replacing $\theta_1\theta_2$ by $\frac{\ell_1}{\theta_3}$ yields $1-\ell_4+\theta_3^2(3-\theta_3)=3\theta_3 \implies (\theta_3-1)^3=-\ell_4 \implies \theta_3^*=1-\sqrt[3]{\ell_4}$. We therefore get

$$(1+\frac{\ell_2}{\ell_1})\theta_1+\theta_2 = 3-\theta_3^*, \qquad (1+\frac{\ell_2}{\ell_1})\theta_1\theta_2 = (1+\frac{\ell_2}{\ell_1})\frac{\ell_1}{\theta_3^*}. \tag{12}$$

The system (12) has real solutions if and only if the discriminant is non negative,

$$\Delta := \left(-3\ell_4-4\ell_3+3\ell_4^{2/3}\right)\left(\sqrt[3]{\ell_4}-1\right)^{-1} \geq 0, \tag{13}$$

which is equivalent to $-3\ell_4-4\ell_3+3\ell_4^{2/3}\leq 0$, and hence if this condition is not satisfied, the algorithm can not be made nilpotent.

We will see in Section 5 that for subdomains of equal size, Dirichlet-Neumann can be made nilpotent also for a larger number of subdomains.

4 The Optimal Schwarz algorithm

A non-overlapping Schwarz algorithm for (1) with two subdomains is

$$\begin{cases} \eta u_1^{(n+1)} - \partial_{xx} u_1^{(n+1)} = f \text{ in } \Omega_1, \\ (\partial_x + p_1^+) u_1^{(n+1)}(x_1) = (\partial_x + p_1^+) u_2^{(n)}(x_1), \end{cases} \quad \begin{cases} \eta u_2^{(n+1)} - \partial_{xx} u_2^{(n+1)} = f \text{ in } \Omega_2, \\ (\partial_x - p_2^-) u_2^{(n+1)}(x_1) = (\partial_x - p_2^-) u_1^{(n)}(x_1), \end{cases} \tag{14}$$

with $p_1^+, p_2^- > 0$ and $u_1^{(n)}(0) = u_2^{(n)}(1) = 0$. A direct computations shows that an optimal Schwarz method converging in two iterations is obtained for an arbitrary initial guess if $p_1^+ = \sqrt{\eta}\coth(\sqrt{\eta}\ell_2)$ and $p_2^- = \sqrt{\eta}\coth(\sqrt{\eta}\ell_1)$, and we even have

Proposition 6. *For J subdomains, let $\ell_j^+ := x_J - x_j$, $j = 1\dots,J-1$ and $\ell_j^- := x_{j-1} - x_0$, $j = 2,\dots,J$. Then setting $p_j^- := \sqrt{\eta}\coth(\sqrt{\eta}\ell_j^-)$ and $p_j^+ := \sqrt{\eta}\coth(\sqrt{\eta}\ell_j^+)$ in an analogously to* (14) *defined algorithm with $J \geq 2$ subdomains, an optimal Schwarz method converging in J iterations is obtained.*

Proof. By linearity, we again study convergence to the zero solution. Let $u_j^{(n)}$ be the approximate solution in each Ω_j at iteration n. First we prove that if

$$\begin{aligned} \partial_x u_j^{(n)} + p_j^+ u_j^{(n)} = 0 \text{ at } x = x_j &\implies \partial_x u_j^{(n)} + p_{j-1}^+ u_j^{(n)} = 0 \text{ at } x = x_{j-1}, \\ \partial_x u_j^{(n)} - p_j^- u_j^{(n)} = 0 \text{ on } x = x_{j-1} &\implies \partial_x u_j^{(n)} - p_{j+1}^- u_j^{(n)} = 0 \text{ on } x = x_j. \end{aligned} \tag{15}$$

To see this, suppose that $\partial_x u_j^{(n)} + p_j^+ u_j^{(n)} = 0$ on $x = x_j$, and let v be defined by $v(x) := u_j^{(n)}(x_{j-1})\frac{\sinh(\sqrt{\eta}(x_J - x))}{\sinh(\sqrt{\eta}\ell_{j-1}^+)}$. Then $\partial_x v + p_j^+ v = 0$ at $x = x_j$, and by construction $v(x_{j-1}) = u_j^{(n)}(x_{j-1})$. Hence v satisfies

$$\begin{gathered} (\eta - \partial_{xx})(u_j^{(n)} - v) = 0 \text{ in } (x_{j-1}, x_j), \\ (\partial_x + p_j^+)(u_j^{(n)} - v) = 0 \text{ at } x = x_j, \quad u_j^{(n)} - v = 0 \text{ at } x = x_{j-1}. \end{gathered} \tag{16}$$

Therefore, by uniqueness of the solution we must have $u_j^{(n)} = v$ on (x_{j-1}, x_j) and thus $\partial_x u_j^{(n)} + p_{j-1}^+ u_j^{(n)}$ at $x = x_{j-1}$, as it holds for v. The proof for the second line in (15) is similar.

Now since $\partial_x u_1^{(1)} - p_2^- u_1^{(1)} = 0$, we have from the transmission condition $\partial_x u_2^{(2)} - p_2^- u_2^{(2)} = \partial_x u_1^{(1)} - p_2^- u_1^{(1)} = 0$, which gives $\partial_x u_2^{(2)} - p_3^- u_2^{(2)} = 0$, and using the transmission condition again we get $\partial_x u_3^{(3)} - p_3^- u_3^{(3)} = \partial_x u_2^{(2)} - p_3^- u_2^{(2)} = 0$, and so on, until $\partial_x u_J^{(J)} - p_J^- u_J^{(J)} = 0$ and a similar argument holds for p_j^+. Hence, after J iterations the interior iterates $u_j^{(J)}$ satisfy

$$\begin{gathered} (\eta - \partial_{xx})(u_j^{(J)}) = 0 \text{ in } (x_{j-1}, x_j), \\ (\partial_x + p_j^+) u_j^{(J)} = 0 \text{ at } x = x_j, \quad (\partial_x - p_j^-) u_j^{(J)} = 0 \text{ at } x = x_{j-1}, \end{gathered} \tag{17}$$

and on the domains on the left and right, we get

$$\begin{aligned} (\eta-\partial_{xx})(u_1^{(J)}) &= 0 \text{ in } (x_0,x_1), & (\eta-\partial_{xx})(u_J^{(J)}) &= 0 \text{ in } (x_{J-1},x_J),\\ (\partial_x+p_1^+)u_1^{(J)} &= 0 \text{ at } x=x_1, & (\partial_x-p_J^-)u_J^{(J)} &= 0 \text{ at } x=x_{J-1}.\\ u_1^{(J)} &= 0 \text{ at } x=x_0. & u_J^{(J)} &= 0 \text{ at } x=x_J, \end{aligned} \tag{18}$$

Hence, $u_j^{(J)}=0$, for all $j=1,\dots,J$, which concludes the proof.

One can show that this result still holds in higher dimensions for a decomposition into strips, provided one uses the then non-local Dirichlet to Neumann operators in the transmission conditions, see [14]. One can however also obtain a nilpotent iteration with less restrictions, which also holds for higher dimensions just by replacing the transmission parameters below by the Dirichlet to Neumann operators again.

Proposition 7. *For J subdomains and $1<d<J$,*[3] *choosing p_j^- for $j=2,\dots,d$ and p_j^+ for $j=d,\dots J-1$ as in Proposition 6, optimal Schwarz will converge in $2J^*-1$ iterations where $J^*:=\max(d,J-d+1)$, independently of the choice of the remaining p_j^-,p_j^+.*

Proof. Following the proof of Proposition 6, after $j^*:=\max(d,J-d+1)$ iterations, the $u_d^{(j^*)}$ satisfy

$$\begin{aligned} &(\eta-\partial_{xx})(u_d^{(j^*)}) = 0 \text{ in } (x_{d-1},x_d),\\ (\partial_x-p_d^-)u_d^{(j^*)} = 0 \text{ at } x=x_{d-1}, &\quad (\partial_x+p_d^+)u_d^{(j^*)} = 0 \text{ at } x=x_d. \end{aligned} \tag{19}$$

Hence $u_d^{(j^*)}$ vanishes in (x_{d-1},x_d) and it follows that $u_j^{(j^*+j-d)}=0$ for $j=d+1,\dots J$, and $u_{d-j}^{(j^*+j)}=0$ for $j=1,\dots d-1$. Thus optimal Schwarz will converge after $j^*+\max(d-1,J-d)=2\max(d,J-d+1)-1$ iterations, which concludes the proof.

5 Numerical experiments

We discretize our model problem (1) using finite differences with a mesh size $\Delta x = 10^{-5}$ and chose the right hand side such that the exact solution is $\sin(\pi x)$ for the parameter $\eta=1$. We decompose the domain into $J=2,3,\dots,10$ equal subdomains, and start the iterations with a random initial guess. For each algorithm, we use the best possible relaxation parameters, i.e. the ones that minimize the spectral radius of the iteration operator, and we plot the error versus iteration on a semi-log scale. In Figure 1 we see on the left that Neumann-Neumann is nilpotent for 2 subdomains,

[3] Even the case $d=1$ and $d=J$ can be handled by changing one of the Robin conditions into a Dirichlet one

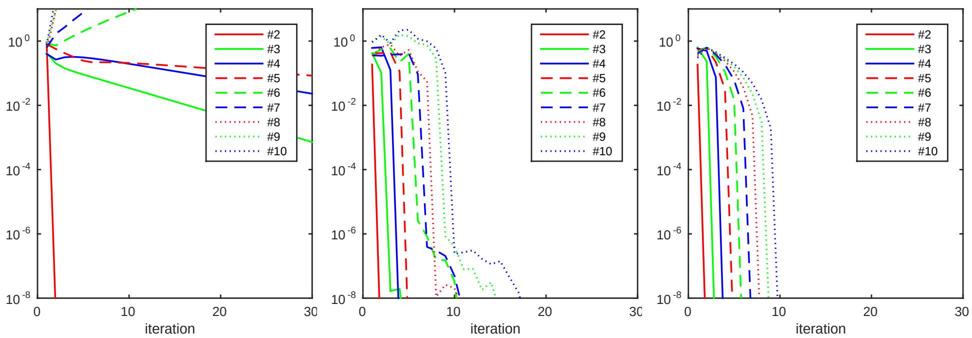

Fig. 1 Error versus number of iterations for Neumann-Neumann (left), Dirichlet-Neumann (middle), and optimal Schwarz (right) for different numbers of subdomains $J = 2, 3, \ldots, 10$ using the best possible relaxation parameters at the interfaces.

as shown in Proposition 1. For 3, 4 and 5 subdomains, Neumann-Neumann still converges, but is not nilpotent, see Proposition 2, and for more than 5 subdomains, the iterations even diverge. One can show that the convergence factor of Neumann-Neumann for this model problem with optimized relaxation parameters behaves like $\mathscr{O}(\frac{1}{\ell^2})$ where ℓ is the subdomain size, so divergence will always set in at some point. For Dirichlet-Neumann in the middle of Figure 1, we see nilpotence for all J in this special case of equal sized subdomains, but this would not be the case for general decompositions, see Proposition 5. The optimal Schwarz method on the right of Figure 1 always converges in J iterations, as expected from Proposition 6.

6 Conclusion

We showed for a one dimensional model problem that the Neumann-Neumann method can only be nilpotent for a decomposition into two general subdomains; the Dirichlet-Neumann method can be nilpotent also for a decomposition into 3 general subdomains, but not any more for a decomposition into four general subdomains. We expect that for subdomains of equal size, Dirichlet-Neumann can be made nilpotent for an arbitrary number of subdomains. The optimal Schwarz method is nilpotent for a decomposition into an arbitrary number of subdomains, also of unequal size and in higher spatial dimensions, and this even if one does not use systematically the Dirichlet to Neumann operators, see our new result in Proposition 7. Our negative results for Neumann-Neumann and Dirichlet-Neumann methods in one spatial dimension imply that these algorithms can not be nilpotent in higher spatial dimensions either. For the Dirichlet-Neumann method and equal subdomains, our result indicates that nilpotence is also possible in higher dimensions for a strip decomposition, provided that the relaxation parameters become non-local operators. Optimal Schwarz methods are nilpotent in higher dimensions without any restrictions. Such nilpotent iterations have led to some of the best solvers for Helmholtz problems recently, see [11, 12, 4, 5, 1, 2, 15], and have been important in the development of optimized Schwarz methods [13, 3, 6, 7]. Well chosen coarse corrections can make a domain decomposition method also nilpotent, see the very recent discoveries in [8, 9, 10].

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