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AILU for Helmholtz problems: A new Preconditioner Based on the Analytic Parabolic Factorization.*

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We investigate a new type of preconditioner which is based on the analytic factorization of the operator into two parabolic factors. Approximate analytic factorizations lead to new block ILU preconditioners. We analyze the preconditioner at the continuous level where it is possible to optimize its performance. Numerical experiments illustrate the effectiveness of the new approach.

1. Introduction

Given the Helmholtz operator $\mathcal{L} = -\omega^2 - \Delta$ acting on $u : \mathbf{R}^n \longrightarrow \mathbf{R}$, $n = 2$ we wish to solve the elliptic partial differential equation

$$\mathcal{L}(u) = f \tag{1.1}$$

in a given domain $\Omega \subset \mathbf{R}^n$ with appropriate boundary conditions. Discretizing the elliptic operator with a finite element or finite difference method on a structured grid, we obtain a large system of linear equations

$$K\boldsymbol{u} = \boldsymbol{f} \tag{1.2}$$

*Presented at ICTCA'99 the 4^{th} International Conference on Theoretical and Computational Acoustics, May 1999, Trieste, Italy where the discrete elliptic operator K has the block structure

$$K = \begin{bmatrix} D_1 & L_{1,2} & & \\ L_{2,1} & D_2 & \ddots & \\ & \ddots & \ddots & L_{n-1,n} \\ & & L_{n,n-1} & D_n \end{bmatrix}. \tag{1.3}$$

The diagonal blocks D_i represent in our notation the discretization of the y part of the elliptic operator $\mathcal{L}$ and include on the diagonal a part of the discretization of the x part of the elliptic operator, the rest being contained in $L_{i,j}$. Since K is sparse, it is interesting to solve (1.2) by an iterative method and it is necessary to precondition the system to obtain the solution efficiently.

Domain decomposition methods can be used as preconditioners, see [6], [3], [5] or [2]. This approach is not considered here. Instead, we focus on the two main groups of matrix based preconditioners: approximate inverses and incomplete factorizations [1]. Both techniques do not relate to the underlying differential operator in general. We pursue a different approach here by deriving an ILU preconditioner from the analytic factorization of the differential operator itself before it is discretized. Such a preconditioner is a good approximation of the continuous operator. In addition the continuous analysis allows us to optimize the preconditioner for the given elliptic PDE.

Although the parabolic factorization of elliptic operators has been a topic of interest for a while [10] and [4] the first use of this approach for iterative solvers was proposed by Nataf in [8] and extended in [9]. The idea was also picked up by Giladi and Keller, motivated by an asymptotic analysis in [7]. The main difficulties remaining in this approach are the low quality of the approximate factorization and thus the limited applicability. In all the previous work the factored operator was non symmetric and the approximate factorization was only considered for small diffusion coefficients which simplifies this type of approximation. We consider here symmetric operators and using a link between the analytic factorization and the exact block LU decomposition we obtain approximate factorizations of high quality. Our approach is related to earlier work at the discrete level by Wittum in [12,13] extended later by Wagner [11].

2. Analytic Parabolic Factorization

Given an elliptic operator $\mathcal{L}(u)$ we write the operator as a product of two parabolic operators,

$$\mathcal{L}(u) = -(\partial_x + \Lambda_1)(\partial_x - \Lambda_2)(u) \tag{2.4}$$

where Λ_1 and Λ_2 are positive operators up to a compact operator. The first factor represents a parabolic operator acting in the positive x direction and the second one a parabolic operator acting in the negative x direction.

In the sequel we restrict ourselves for the analysis to the case of $\mathcal{L} = (-\omega^2 - \Delta)$, where Δ denotes the Laplacian in two dimensions and $\omega \geq 0$.

Our results are based on Fourier analysis. We take a Fourier transform of $\mathcal{L} = (-\omega^2 - \Delta)$ in y to obtain

$$\mathcal{F}_y(-\omega^2 - \Delta) = -\partial_{xx} + k^2 - \omega^2 = -(\partial_x + \sqrt{k^2 - \omega^2})(\partial_x - \sqrt{k^2 - \omega^2}) \tag{2.5}$$

and thus we have the continuous parabolic factorization

$$(-\omega^2 - \Delta) = -(\partial_x + \Lambda_1)(\partial_x - \Lambda_2) \tag{2.6}$$

where $\Lambda_1 = \Lambda_2 = \mathcal{F}_y^{-1}(\sqrt{k^2 - \omega^2})$. Note that the Λ_i are nonlocal operators in y because of the square root.

To relate this parabolic factorization to the exact block LU decomposition of the discrete matrix operator, we discretize the x direction of $(-\omega^2 - \Delta)$ and compute the analytic factorization (2.5) for the semi discrete operator $(-\omega^2 - \Delta_h)$. We have

$$\Delta_h = D_x^- D_x^+ + \partial_{yy}$$

where $D_x^+(u) := (u_{i+1} - u_i)/h$ and $D_x^-(u) := (u_i - u_{i-1})/h$ represent the discrete derivatives on a given mesh. Taking a Fourier transform in y of $-\omega^2 - \Delta_h$ as before we obtain the factored form

$$\mathcal{F}_y(-\omega^2 - \Delta_h) = -\frac{1}{h^2\tau}(D_x^- + \lambda_1)(D_x^+ - \lambda_2) = -\frac{1}{h^2\tau}(D_x^- D_x^+ - \lambda_2 D_x^- + \lambda_1 D_x^+ - \lambda_1\lambda_2),$$

with the unknowns λ_1, λ_2 and the additional parameter τ introduced because of the discretization. Using $D_x^+ - D_x^- = hD_x^- D_x^+$ to replace the term with D_x^- we find for

$$\tau = \frac{1}{h^2} + \frac{-\omega^2 + k^2}{2} + \frac{1}{2h}\sqrt{(-\omega^2 + k^2)^2 h^2 + 4(-\omega^2 + k^2)}, \tag{2.7}$$

where we choose the positive root and $\sqrt{x}$ is defined to have $\Re(x) \geq 0$, since we defined λ_1 and λ_2 to be positive operators for $|k| > \omega$. Similarly, we find

$$\lambda_1 = \lambda_2 = \tau h - \frac{1}{h} = h\frac{-\omega^2 + k^2}{2} + \frac{1}{2}\sqrt{(-\omega^2 + k^2)^2 h^2 + 4(-\omega^2 + k^2)}$$

which are positive for $|k| > \omega$. The semi discrete analytic parabolic factorization is thus given by

$$\mathcal{F}_y(-\omega^2 - \Delta_h) = -\left(D_x^- + (\tau h - \frac{1}{h})\right)\frac{1}{h^2\tau}\left(D_x^+ - (\tau h - \frac{1}{h})\right). \tag{2.8}$$

Note that as we take the limit for $h \longrightarrow 0$ in (2.8) we recover again the continuous parabolic factorization (2.5) since the middle term disappears in the limit. For discrete problems it is however important to include the middle factor, which was not the case in previous work on continuous parabolic factorizations. One can show that (2.8) corresponds to the exact block-LU decomposition of the fully discrete matrix operator (1.3).

Indeed, the symmetric matrix K admits the exact block LU decomposition

$$K = \begin{bmatrix} T_1 & & & \\ L_{2,1} & T_2 & & \\ & \ddots & \ddots & \\ & & L_{n,n-1} & T_n \end{bmatrix} \begin{bmatrix} T_1^{-1} & & & \\ & T_2^{-1} & & \\ & & \ddots & \\ & & & T_n^{-1} \end{bmatrix} \begin{bmatrix} T_1 & L_{1,2} & & \\ & T_2 & \ddots & \\ & & \ddots & L_{n-1,n} \\ & & & T_n \end{bmatrix} \tag{2.9}$$

where the matrices T_i are given by the recurrence relation

$$T_i = \begin{cases} D_1 & i = 1, \\ D_i - L_{i,i-1}T_{i-1}^{-1}L_{i-1,i} & 1 < i \leq n. \end{cases} \tag{2.10}$$

To relate this decomposition to the semi-discrete parabolic factorization (2.8), we formulate the exact block LU decomposition for $-\omega^2 - \Delta_h$ on an infinite mesh in the x component of the operator and Fourier transform the y component to compare with the parabolic factors in (2.8). We obtain the infinite matrix

$$\hat{K} = \begin{bmatrix} \ddots & \ddots & & \\ \ddots & \hat{D} & \hat{L} & \\ & \hat{L} & \hat{D} & \ddots \\ & & \ddots & \ddots \end{bmatrix}$$

where the entries are given by

$$\hat{D} = -\omega^2 + k^2 + \frac{2}{h^2}, \quad \hat{L} = -\frac{1}{h^2}.$$

The exact block LU decomposition of $\hat{K}$ is given by

$$\hat{K} = \begin{bmatrix} \ddots & & & \\ \ddots & \hat{T}_\infty & & \\ & \hat{L} & \hat{T}_\infty & \\ & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \ddots & & & \\ & \hat{T}_\infty^{-1} & & \\ & & \hat{T}_\infty^{-1} & \\ & & & \ddots \end{bmatrix} \begin{bmatrix} \ddots & \ddots & & \\ & \hat{T}_\infty & \hat{L} & \\ & & \hat{T}_\infty & \ddots \\ & & & \ddots \end{bmatrix} \tag{2.11}$$

where $\hat{T}_\infty$ is a solution of

$$\hat{T}_\infty = \hat{D} - \hat{L}\hat{T}_\infty^{-1}\hat{L}, \tag{2.12}$$

which is also the limit of the recurrence relation (2.10). Note that we talk about a block decomposition here, because the scalar entries in the matrix contain Fourier components which will become matrices, once the y direction is discretized. We have the following

Theorem 1 (Equivalence of Block LU and Parabolic Factorization) *The block LU decomposition (2.11) of the semi-discretized elliptic operator $-\omega^2 - \Delta_h$ on an unbounded domain is identical to the analytic parabolic factorization (2.8) of the same operator and tends to the parabolic factorization (2.5) of the continuous elliptic operator in the limit as h goes to zero.*

Proof.

□.Since $\hat{T}_\infty$ satisfies (2.12) it is a solution of

$$\hat{T}_\infty^2 - (\eta + k^2 + \frac{2}{h^2})\hat{T}_\infty + \frac{1}{h^4} = 0, \tag{2.13}$$

so that $\hat{T}_\infty = \tau$ given by (2.7). The exact block LU factorization of K in the y-Fourier transformed domain is therefore

$$\hat{K} = \begin{bmatrix} \ddots & & & \\ \ddots & \hat{T} & & \\ & \hat{L} & \hat{T} & \\ & & \ddots & \ddots \end{bmatrix} \frac{1}{\hat{T}} \begin{bmatrix} \ddots & \ddots & & \\ & \hat{T} & \hat{L} & \\ & & \hat{T} & \ddots \\ & & & \ddots \end{bmatrix}.$$

By rewriting each entry $\hat{T}$ in the form $1/h^2+\hat{T}-1/h^2$ and observing that $\hat{L}=-1/h^2$, each row of the first matrix in the above factorization can be expressed using a finite difference operator, $D_x^-/h+\hat{T}-1/h^2$. Similarly for the last matrix in the above factorization, we find $-D_x^+/h+\hat{T}-1/h^2$ for each row. Hence the above factorization can be written entirely in terms of finite differences,

$$\hat{K}=\left(\frac{D_x^-}{h}+\hat{T}-\frac{1}{h^2}\right)\frac{1}{\hat{T}}\left(-\frac{D_x^+}{h}+\hat{T}-\frac{1}{h^2}\right) \tag{2.14}$$

which coincides after rearranging with the analytic parabolic factorization found in (2.8) and hence establishes convergence to the parabolic factorization (2.5) as h goes to zero.

3. The AILU Preconditioner

One could use directly the parabolic factorization given in (2.8) to solve the original problem (1.2). Instead of solving the linear system, one would have to solve two lower dimensional parabolic problems, one in the positive and one in the negative x direction, corresponding to a forward and a backward solve of the exact block LU decomposition. This is however not advisable since the parabolic factorization contains nonlocal operators in y. We therefore approximate the parabolic factorization by local operators and use the factorization as a preconditioner corresponding to a new type of ILU preconditioner we call AILU (Analytic ILU). We replace the nonlocal operator τ in (2.8) by a local approximation of the form

$$\tau_{app}=\frac{1}{h^2}+\frac{-\omega^2+k^2}{2}+\frac{1}{2h}(p+qk^2), \quad p,q\in\mathbf{C},\ \Re(q)>0,$$

where the square root in (2.7) is replaced by $p+qk^2$. This leads to a classical linear second order parabolic problem. Since we have the analytic parabolic factorization, we can use the parameters p and q to optimize the performance of the AILU preconditioner. We insert the approximation τ_{app} into the factorization (2.8) and obtain the operator resulting from the approximate factorization of $-\omega^2+k^2-D_x^+D_x^-$ in the form

$$\mathcal{L}_{app}=-D^-D^++\tau_{app}+\frac{1}{\tau_{app}h^4}-\frac{2}{h^2}.$$

The complex numbers p and q are to be chosen so that $\mathcal{L}_{app}^{-1}\mathcal{L}$ is as close as possible to the identity except perhaps for a few frequencies which will be taken into account by the Krylov method. We find after some calculation that formally the symbol of $\mathcal{L}_{app}^{-1}\mathcal{L}$ is

$$\frac{k_x^2-\omega^2+k^2}{k_x^2+\dfrac{(h\frac{-\omega^2+k^2}{2}+(p+qk^2))^2}{1+\frac{h^2}{2}(-\omega^2+k^2)+\frac{h}{2}(p+qk^2)}}$$

where k_x is a Fourier variable along the x direction. In a numerical setting, the frequency parameters k,k_x can not vary arbitrarily. For Dirichlet boundary conditions, they are bounded from below by the size of the domain, $k,k_x>\frac{\pi}{L}$ where L denotes the size of the domain in the x or y direction. From above, k,k_x are bounded by the mesh size h, $k,k_x<\frac{\pi}{h}$. We choose p so that the preconditioned operator $\mathcal{L}_{app}^{-1}\mathcal{L}$ has a symbol equal to one for $k=\pi/L$ and any k_x. The other parameter q is chosen so that the symbol of $\mathcal{L}_{app}^{-1}\mathcal{L}$

	Iteration count				Solution process only in Mflops			
ω	QMR	ILU('0')	ILU(1e-2)	AILU	QMR	ILU('0')	ILU(1e-2)	AILU
5	197	60	22	23	120.1	60.4	28.3	28.3
10	737	370	80	36	1858.2	1489.3	421.4	176.2
15	1775	> 2000	220	43	10185.2	> 18133.0	2615.1	475.9
20	> 2000	—	> 2000	64	> 20335.0	—	> 42320.0	1260.1
30	—	—	—	90	—	—	—	3984.2
40	—	—	—	135	—	—	—	10625.7
50	—	—	—	285	—	—	—	24000.1

Table 1: Comparison of iteration count, flop count in mega flops for the solution process.

	Precond. cost in Mflops		
ω	ILU('0')	ILU(1e-2)	AILU
5	0.21	5.1	0.3
10	0.86	38.6	0.9
15	1.96	127.5	2.0
20	3.49	298.2	3.6
30	7.88	996.1	8.1
40	–	–	14.4
50	–	–	22.5

Table 2: Comparison of flop count in mega flops for computing the preconditioner.

is as close as possible to one except for the frequencies $k_x^2 + k^2 \simeq \omega^2$ where the symbol of $\mathcal{L}_{app}^{-1}\mathcal{L} = 0$ for any parameter q.

4. Numerical Experiments

We first consider a two dimensional problem,

$$-\omega^2 u - \partial_{xx}u - \partial_{yy}u = \delta(x - \frac{1}{2})\delta(y - \frac{1}{2}), \quad 0 < x, y < 1$$

with homogeneous Dirichlet boundary conditions on the top, bottom and right boundaries. On the left boundary ($x = 0$), we impose an absorbing boundary condition $(-\partial_x + i\omega)(u) = 0$. The computational domain is an open cavity. The right hand side corresponds to a point source in the middle of the cavity. We start QMR with the initial guess $u^{(0)} = 0$ and we compare the new preconditioner with the unpreconditioned QMR algorithm, and the preconditioners ILU('0') and ILU(1e-2) using the QMR algorithm and a tolerance of 1e-6. Tables 1-2 shows the results obtained from numerical experiments for different values of ω with ~ 62 points per wavelength. The number of points in each direction is $\omega \times 10$.

The new AILU preconditioner shows a good reduction of the iteration count. In addition it permits the solution of the given problem in fewer flops than any of the other methods tested. For big problems the savings become significant especially when the other methods do not converge after 2000 iterations. Note that for $\omega = 40$ and 50 we were not able to compute ILU('0') and ILU(1e-2) because of time limitations (more than 2 hours).

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