On the Definition of Dirichlet and Neumann Conditions for the Biharmonic Equation and Its Impact on Associated Schwarz Methods

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1 Introduction

We are interested in formulating and analyzing Schwarz methods for the biharmonic equation

$$\Delta^2 u = f \quad \text{in } \Omega, \tag{1}$$

where Δ denotes the Laplacian, f is a source term and Ω is a domain in $\mathbb{R}^2$. The biharmonic equation is quite different from the Laplace equation, since it requires two boundary conditions, and not just one.

A classical clamped boundary condition would impose the value and normal derivative at the boundary,

$$\mathcal{D}_1(u) := \begin{bmatrix} u \\ \frac{\partial u}{\partial n} \end{bmatrix}, \tag{2}$$

and a two level additive Schwarz method with this "Dirichlet" boundary condition at the interfaces between subdomains was studied in [1], where a condition number estimate of order $1 + (\frac{H}{\delta})^4$ was proved for large overlap and order $1 + (\frac{H}{\delta})^3$ for small overlap. A non-overlapping Schwarz preconditioner for a discontinuous Galerkin discretization was introduced in [8], with a condition number estimate of order $(1 + \frac{H}{h})^3$. The convergence rate for the classical Schwarz method with "Dirichlet" condition (2) was also studied in [15].

Considering (2) as "Dirichlet" condition, there are two corresponding possibilities for the associated "Neumann" conditions, depending on which functional minimization led to the necessary optimality condition in (1). If the problem comes from a Stokes formulation [4], the variational derivative leads

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for the "Neumann" conditions to

$$\mathcal{N}_1(u) := \begin{bmatrix} \Delta u \\ -\partial_n \Delta u \end{bmatrix}. \tag{3}$$

If one however uses the energy functional of a thin plate, see [11] and references therein, the "Neumann" condition associated with (2) is

$$\mathcal{N}_2(u) := \begin{bmatrix} \Delta u - (1-\sigma)\partial_{\tau\tau} u \\ -\partial_n \Delta u - (1-\sigma)\partial_\tau(\partial_{n\tau} u) \end{bmatrix}, \tag{4}$$

where ∂_τ is the tangential derivative along the boundary and $\sigma \in (0,1)$ is a material constant. While condition (3) does not always lead to a well posed problem for the biharmonic equation, condition (4), which can be interpreted as the freely supported boundary condition for the plate problem, is always well posed up to a linear function, analogously to the Neumann condition for the Laplace equation. A FETI method using (2) and (4) was proposed and studied in [7], and later in [13], where continuity of the transverse displacements is enforced at substructure cross points, and a condition number estimate of order $(1+\log \frac{H}{h})^3$ was obtained. An optimized Schwarz waveform relaxation method based on combining the "Dirichlet" condition (2) with the "Neumann" condition (3) was introduced in [14] for the corresponding time dependent problem, and an optimized choice of the combining parameters in the transmission conditions was illustrated by numerical experiments.

The clamped condition (2) is however not the only possible choice for a "Dirichlet" condition. Instead of (2) and (3), one could also consider

$$\mathcal{D}_3(u) := \begin{bmatrix} u \\ \Delta u \end{bmatrix} \tag{5}$$

as the "Dirichlet" condition, and then naturally the corresponding "Neumann" condition would be

$$\mathcal{N}_3(u) := \begin{bmatrix} \partial_n u \\ -\partial_n \Delta u \end{bmatrix}, \tag{6}$$

see for example [5, 17]. Similarly, in the thin plate case, instead of (2) and (4), another choice for the "Dirichlet" condition would be

$$\mathcal{D}_4(u) := \begin{bmatrix} u \\ \Delta u - (1-\sigma)\partial_{\tau\tau} u \end{bmatrix}, \tag{7}$$

and then the corresponding "Neumann" condition would be

$$\mathcal{N}_4(u) := \begin{bmatrix} \partial_n u \\ -\partial_n \Delta u - (1-\sigma)\partial_\tau(\partial_{n\tau} u) \end{bmatrix}. \tag{8}$$

When the boundary is flat, conditions (5) and (7) are essentially equivalent, since imposing u also imposes $\partial_{\tau\tau}$. Similarly also conditions (6) and (8) are equivalent for flat boundaries. For curved boundaries however, and as transmission conditions, these conditions are different.

Because of these different choices for the “Dirichlet” conditions, the classical Schwarz methods studied in [1] and [15] are not the only possible ones for the biharmonic equation, and similarly there are also more possibilities for optimized Schwarz methods than the one in [14]. We will show that a different choice of “Dirichlet” conditions in the classical Schwarz method permits the removal of the typical power of 3 in the convergence estimates, and leads to faster methods, while optimized Schwarz methods are robust with respect to which condition is chosen to be the “Dirichlet” one.

2 Classical Schwarz Methods

Because of the three different possibilities for the “Dirichlet” conditions in (2), (5) and (7), we get three classical Schwarz methods which we index by $j \in \{1, 3, 4\}$. To simplify the description and analysis, we consider an unbounded domain $\Omega = \mathbb{R}^2$ and solutions u decaying at infinity. We assume that Ω is divided into two subdomains $\Omega_1 = (-\infty, L) \times \mathbb{R}$ and $\Omega_2 = (0, +\infty) \times \mathbb{R}$, where $L \geq 0$ denotes the overlap.

Given an initial approximation u_2^0, the three classical alternating Schwarz methods indexed by $j \in \{1, 3, 4\}$ compute for $n = 1, 2, \ldots$

$$\begin{array}{llll} \Delta^2 u_1^n = f_1 & \text{in } \Omega_1, & \Delta^2 u_2^n = f_2 & \text{in } \Omega_2, \\ \mathcal{D}_j(u_1^n) = \mathcal{D}_j(u_2^{n-1}) & \text{at } x = L, & \mathcal{D}_j(u_2^n) = \mathcal{D}_j(u_1^n) & \text{at } x = 0. \end{array} \tag{9}$$

Taking a Fourier transform in the y direction with Fourier symbol k, and assuming that the relevant numerical Fourier frequencies $|k|$ lie in the interval $[k_{min}, k_{max}]$ with $k_{min}, k_{max} > 0$, we obtain by a direct computation (see also [15] for $j = 1$):

Theorem 1. *If $L > 0$, the convergence factors ρ_j for the Algorithm (9) are*

$$\begin{aligned} \rho_1(L) &= (k_{\min}L + \sqrt{k_{\min}^2 L^2 + 1})^2 e^{-2k_{\min}L} \sim 1 - \frac{1}{3}k_{\min}^3 L^3, \\ \rho_{3,4}(L) &= e^{-2k_{\min}L} \sim 1 - 2k_{\min}L. \end{aligned}$$

We see that the classical clamped “Dirichlet” transmission condition (2) leads to a convergence factor depending on the overlap L cubed, whereas using the other two possible “Dirichlet” conditions (5) or (7), the convergence factor only depends linearly on L. This substantially improved convergence factor, which is now like for Laplace’s equation [9], is illustrated for an example in Figure 1 on the left.

3 Optimal and Optimized Schwarz Methods

Optimized Schwarz methods [9] use a combination of Dirichlet and Neumann conditions as transmission conditions, and allowing a non-local operator for this combination can lead to optimal Schwarz methods which converge in a finite number of steps (two in the case of two subdomains, see [9] and references therein). Letting $\mathcal{D}_2 := \mathcal{D}_1$, such a method, again indexed by $j \in \{1,2,3,4\}$, computes for an initial approximation u_2^0 and $n = 1,2,\ldots$

$$\begin{array}{rll} \Delta^2 u_1^n &= f_1 & \text{in } \Omega_1, \\ (\mathcal{N}_j + P_j\mathcal{D}_j)(u_1^n) &= (\mathcal{N}_j + P_j\mathcal{D}_j)(u_2^{n-1}) & \text{at } x = L, \\ \Delta^2 u_2^n &= f_2 & \text{in } \Omega_2, \\ (\mathcal{N}_j + P_j\mathcal{D}_j)(u_2^n) &= (\mathcal{N}_j + P_j\mathcal{D}_j)(u_1^n) & \text{at } x = 0, \end{array} \tag{10}$$

where P_j is a two by two matrix to be chosen for best performance of the method, depending on the choice of "Dirichlet" and "Neumann" conditions $\mathcal{D}_j$ and $\mathcal{N}_j$ we made. The following result can be obtain by a direct but lengthy calculation using Fourier analysis.

Theorem 2. *If the symbols of the elements in the matrix P_j for variant j of Algorithm* (10) *are chosen in the Fourier domain as*

$$\begin{array}{ll} \hat{P}_1 = \begin{bmatrix} 2|k|^2 & 2|k| \\ 2|k|^3 & 2|k|^2 \end{bmatrix}, & \hat{P}_2 = \begin{bmatrix} (1+\sigma)|k|^2 & 2|k| \\ 2|k|^3 & (1+\sigma)|k|^2 \end{bmatrix}, \\ \hat{P}_3 = \begin{bmatrix} |k| & \frac{1}{2|k|} \\ 0 & -|k| \end{bmatrix}, & \hat{P}_4 = \begin{bmatrix} \frac{1}{2}(1+\sigma)|k| & \frac{1}{2|k|} \\ \frac{1}{2}(1-\sigma)(\sigma+3)|k|^3 & -\frac{1}{2}(1+\sigma)|k| \end{bmatrix}, \end{array} \tag{11}$$

then the resulting optimal Schwarz method converges in two iterations.

Remark 1. The choice of the matrix P_j, $j \in \{1,2,3,4\}$ in Theorem 2 leads in each case to the transparent boundary condition, and the associated algorithm can be interpreted as an exact factorization independently of the PDE one considers, see [12] and references therein, and also the more recent variants [6, 2, 16, 3]. Such factorizations are theoretically still possible in the presence of cross points, see [10].

The optimal choice of $\hat{P}_j$ in Theorem 2 corresponds to a non-local operator once back-transformed using the inverse Fourier transform, and thus is often approximated using an absorbing boundary condition or perfectly matched layers to obtain a more practical algorithm. Theorem 2 also indicates a very simple, structurally consistent local approximation: replacing $|k|$ by a constant $p \geq 0$ will make the approximation exact for precisely this frequency $|k|$, and leads to the following results.

Theorem 3. *With the structural consistent approximations for $p \geq 0$,*

$$P_1^a = \begin{bmatrix} 2p^2 & 2p \\ 2p^3 & 2p^2 \end{bmatrix}, \quad P_3^a = \begin{bmatrix} p & \frac{1}{2p} \\ 0 & -p \end{bmatrix}, \tag{12}$$

the convergence factor of the optimized Schwarz algorithm (10) *is*

$$\rho(L) = \left(\frac{p-|k|}{p+|k|}\right)^2 e^{-2|k|L} < 1. \tag{13}$$

With overlap, $L > 0$, the optimal choice for p for best performance, and the associated contraction factor are for L small

$$p \sim \left(\frac{k_{\min}^2}{2L}\right)^{1/3}, \quad \rho(L) \sim 1 - 4(2k_{\min})^{1/3}L^{1/3}, \tag{14}$$

where $k_{\min}$ is an estimate for the lowest frequency along the interface. Without overlap, $L = 0$, and with $k_{\max}$ an estimate for the largest frequency along the interface, one obtains

$$p = \sqrt{k_{\min}k_{\max}}, \ \rho(0) = \left(\frac{\sqrt{k_{max}} - \sqrt{k_{min}}}{\sqrt{k_{max}} + \sqrt{k_{min}}}\right)^2 \sim 1 - 4\sqrt{\frac{k_{\min}}{k_{\max}}}, \ k_{\max} \text{ large.} \tag{15}$$

Proof. The convergence factor (13) can be obtained by a direct computation, and noticing that it is identical to the case of the Laplace equation, the results from [9] can then be used to obtain (14) and (15).

Theorem 4. *With the structural consistent approximations for $p \geq 0$,*

$$P_2^a = \begin{bmatrix} (1+\sigma)p^2 & 2p \\ 2p^3 & (1+\sigma)p^2 \end{bmatrix}, \ P_4^a = \begin{bmatrix} \frac{1}{2}(1+\sigma)p & \frac{1}{2p} \\ \frac{1}{2}(1-\sigma)(\sigma+3)p^3 & -\frac{1}{2}(1+\sigma)p \end{bmatrix}, \tag{16}$$

the convergence factor of the optimized Schwarz algorithm (10) *for $j = 2$ and $j = 4$ coincide. With overlap, $L > 0$, the optimal choice of p for best performance, and the associated contraction factor are for L small*

$$p \sim \frac{1}{2^{1/3}}\left(\frac{6k_{\min}^4}{(1-\sigma^2)L}\right)^{1/5}, \quad \rho(L) \sim 1 - \frac{16}{3}\frac{(6^2k_{\min}^3(1-\sigma^2))^{1/5}}{3-2\sigma-\sigma^2}L^{3/5}. \tag{17}$$

Without overlap, one obtains for $k_{\max}$ large

$$p \sim \sqrt{k_{\min}k_{\max}}, \quad \rho(0) \sim 1 - \frac{16k_{\min}^{3/2}}{3-2\sigma-\sigma^2}\frac{1}{k_{\max}^{3/2}}. \tag{18}$$

The proof of Theorem 4 requires a detailed asymptotic analysis and is too long for this short manuscript. We see however that the constant σ from the plate problem enters the convergence factor, and the convergence of algorithm (10) for $j \in \{2,4\}$ is worse than in the case $j \in \{1,3\}$. Theorem 3 and Theorem 4 also show that the optimized Schwarz algorithms have the same performance, independently of the choice of "Dirichlet" condition, in contrast to the classical Schwarz method.

Table 1 Iteration numbers for classical Schwarz (9) and optimized Schwarz (10).

	Classical Schwarz $j = 1$				Classical Schwarz $j = 3$				Optimized Schwarz $j = 1, 3$			
$L \setminus h$	1/16	1/32	1/64	1/128	1/16	1/32	1/64	1/128	1/16	1/32	1/64	1/128
h	853	6469	50906	>200000	34	68	134	267	6	9	12	14
$2h$	235	1655	12819	101157	18	35	67	135	5	8	11	14
$4h$	53	305	2189	16971	9	17	34	67	4	7	9	13

One might be wondering what the importance is of the structural consistent choice of the approximate transmission condition in Theorem 3 and Theorem 4. Our next result answers this question for one particular case.

Theorem 5. *For algorithm* (10) *in the case* $j = 1$ *without overlap, if we permit the general matrix*

$$P_1^g = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}, \tag{19}$$

then the optimal choice of the parameters is

$$p_{11} = p_{22} \geq 0, \quad p_{12}p_{21} = p_{11}^2, \quad \frac{p_{21}}{p_{12}} = k_{min}k_{max}. \tag{20}$$

Therefore, the structural choice in Theorem 3 is optimal.

The proof of Theorem 5 is technical and too long for this short paper.

4 Numerical Results

We solve the biharmonic equation (1) numerically on the unit square domain $\Omega = (0, 1) \times (0, 1)$ with the homogeneous "Dirichlet" conditions $\mathcal{D}_1(u) = 0$ on $\partial\Omega$, and choose for the right hand side $f := 24y^2(1-y)^2 + 24x^2(1-x)^2 + 8[(1-2x)^2 - 2(x-x^2)][(1-2y)^2 - 2(y-y^2)]$, so that the exact solution is $u = x^2(1-x)^2y^2(1-y)^2$. We discretize (1) using a standard 13-point finite difference scheme obtained by taking the square of the standard five point Laplacian, see [11]. We divide the domain into two equal overlapping subdomains Ω_1 and Ω_2. We stop the Schwarz iteration when $\frac{\|u^n - u\|_{l^2}}{\|u\|_{l^2}} \leq 10^{-6}$, where u^n denotes the discrete approximation at iteration n, and u is the discrete solution obtained by a direct method.

We compare for $j = 1, 3$ the classical Schwarz algorithm (9) to the optimized Schwarz algorithm (10). The results in Table 1 clearly show how the good choice of "Dirichlet" greatly improves the performance, and also the superiority of the optimized Schwarz method, as one would expect from the contraction factor plot in Figure 1 on the left. In Figure 1 on the right we show the plot corresponding to Table 1, and we can clearly see the asymptotic difference in behavior as predicted by Theorem 1 and Theorem 3.

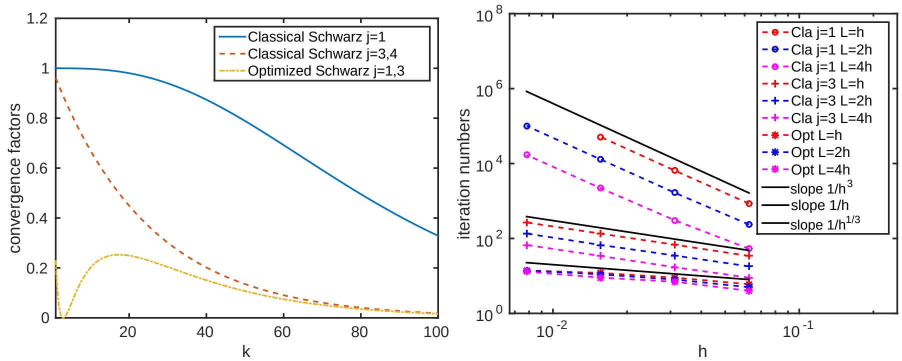

Fig. 1 Left: convergence factors corresponding to an overlap $L = 1/50$ for the biharmonic equation and various Schwarz algorithms. Right: graphical representation of the results from Table 1, and theoretical prediction from Theorem 1 and Theorem 3.

5 Conclusions

We showed that using the classical clamped boundary conditions as "Dirichlet" transmission conditions for a Schwarz algorithm applied to the biharmonic equation leads to a convergence that depends on the overlap cubed, see also [1, 15]. A better choice of "Dirichlet" conditions involving a Laplacian leads to a convergence that only depends linearly on the overlap, like in the case of Laplace's equation, without additional computational cost, since the Laplacian appearing in this new "Dirichlet" condition is naturally available, for example in a mixed formulation. We then proved that optimized Schwarz methods do not depend on the choice of what the "Dirichlet" condition is, and they all lead to a still substantially better convergence behavior than the classical Schwarz method with the best "Dirichlet" condition. We also found that transmission conditions based on the thin plate model ($\mathcal{D}_j$ and $\mathcal{N}_j$ for $j = 2, 4$) are inferior in performance compared to the ones coming from the Stokes model ($\mathcal{D}_j$ and $\mathcal{N}_j$ for $j = 1, 3$).

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