An Asymptotic Approach to Compare Coupling Mechanisms for Different Partial Differential Equations

Martin J. Gander[1] and Véronique Martin[2]

1 Introduction

In many applications the viscous terms become only important in parts of the computational domain. A typical example is the flow of air around the wing of an airplane. It can then be desirable to use an expensive viscous model only where the viscosity is essential for the solution and an inviscid one elsewhere. This leads to the interesting problem of coupling partial differential equations of different types.

The purpose of this paper is to explain several coupling strategies developed over the last decades, and to introduce a systematic way to compare them. We will use the following simple model problem to do so:

$$\begin{aligned} \mathcal{L}_{ad}u := -\nu u'' + au' + cu &= f \qquad \text{in } \Omega = (-L_1, L_2),\\ \mathcal{B}_1 u &= g_1 \qquad \text{on } x = -L_1,\\ \mathcal{B}_2 u &= g_2 \qquad \text{on } x = L_2, \end{aligned} \tag{1}$$

where ν and c are strictly positive constants, $a, g_1, g_2 \in \mathbb{R}$, $f \in L^2(\Omega)$, $L_1, L_2 > 0$ and $\mathcal{B}_j$, $j = 1, 2$ are suitable boundary operators of Dirichlet, Neumann or Robin type. If in part of Ω, the diffusion plays only a minor role, one would like to replace the viscous solution u by an inviscid approximation, which leads to two separate problems: a viscous problem on, say, $\Omega^- := (-L_1, x_0 + \delta)$, where δ stands for the size of the overlap and x_0 the position of the interface,

$$\begin{aligned} \mathcal{L}_{ad}u_{ad} &= f \qquad \text{in } \Omega^-,\\ \mathcal{B}_1 u_{ad} &= g_1 \qquad \text{on } x = -L_1, \end{aligned} \tag{2}$$

Section de Mathématiques, Université de Genève, 2-4 rue du Lièvre, CP 64, CH-1211 Genève, SWITZERLAND. martin.gander@unige.ch · LAMFA UMR 6140 Université de Picardie Jules Verne, Amiens, FRANCE. veronique.martin@u-picardie.fr

and a pure advection reaction problem on $\Omega^+ := (x_0, L_2)$,

$$\mathcal{L}_a u_a := a u_a' + c u_a = f \qquad \text{in } \Omega^+. \tag{3}$$

Coupling conditions for (2) and (3) need then to be chosen to connect the two subproblems. In order to compare the quality of the various coupling strategies, we propose to use as a measure how close the coupled solution is to the fully viscous solution of (1). The idea behind this quality measure is that in principle the viscosity should be taken into consideration everywhere, and so it is the viscous solution that we are interested in. However, for computational savings, one would like to use a simpler, non-viscous model whenever the viscosity does not play an important role.

We describe in this paper in detail several coupling strategies, and compare them by testing how close the coupled solution is to the fully viscous one: in Section 2 we present an overlapping coupling method based on optimization. In Section 3 we present several non-overlapping coupling strategies based on coupling conditions at the interface between the two regions. In both sections, the position of the interface needs to be known a priori. This is in contrast to Section 4, where we present an adaptive coupling strategy which detects the partition into viscous and non-viscous regions automatically.

2 Methods based on overlap and optimization

In this section, we present a very general overlapping coupling strategy that was proposed in Dinh et al. [1988], where the authors considered as the viscous model the incompressible Navier-Stokes equations, while the inviscid model was the potential equation (the assumption of a small vorticity is made).

For the model problem (1), the coupling strategy works as follows: in each subdomain, we solve the corresponding equation with a Dirichlet condition at the artificial interface,

$$u_{ad}(x_0 + \delta) = \lambda_1 \text{ and } \text{ if } a > 0,\, u_a(x_0) = \lambda_2,$$

and then determine (λ_1, λ_2) to be a solution of the optimization problem

$$J_0(\lambda_1, \lambda_2) := \|u_{ad} - u_a\|^2_{L^2(x_0, x_0+\delta)} \ \longrightarrow\ \min.$$

The authors in Dinh et al. [1988] solve this optimization problem using a gradient type method, so that the adjoint equation also needs to be computed.

This coupling strategy based on optimization has been studied mathematically in Gervasio et al. [2001] and Agoshkov et al. [2006] for our model problem in 2D, see also Discacciati et al. [2010] for a complete description of the algorithms for the model problem, and also for the coupling of Navier-Stokes equations with a Darcy model, or the coupling of the Stokes and potential equations. In Agoshkov et al. [2006] other cost functionals to be minimized are proposed, namely

$$J_\alpha(\lambda_1,\lambda_2) = \|u_{ad} - u_a\|^2_{L^2(x_0,x_0+\delta)} + \frac{\alpha}{2}(\lambda_1^2 + \sigma(a)\lambda_2^2),$$

where $\sigma(a)$ is 0 for $a < 0$ and 1 for $a > 0$, or

$$G_3(\lambda_1,\lambda_2,\lambda_3) = \|u_{ad} - u_a\|^2_{L^2(x_0,x_0+\delta)} + \lambda_1^2 + \sigma(a)\lambda_2^2 + \|\lambda_3\omega\|^2_{L^2(x_0,x_0+\delta)},$$

where the equation $\mathcal{L}_a u_a = f + \omega\lambda_3$ is solved in (x_0, L_2) with ω a smooth function such that $0 \le \omega(x) \le 1$ in Ω and $\omega \equiv 0$ outside the overlap.

In order to compare the quality of these coupling strategies, we compute numerically the error between the viscous and the coupled solution as a function of the viscosity for the case $L_1 = L_2 = 1$, $x_0 = -0.6$, $f(x) = e^{-1000(x+1)^2}$ and $c = 1$. We use a centered finite difference scheme to discretize the two differential operators, with mesh size 2×10^{-5}. We consider the case of a positive velocity, $a = 1$, with $g_1 = 0$, $g_2 = 0$, $\mathcal{B}_1 = Id$ and $\mathcal{B}_2 = \partial_x - (a - \sqrt{a^2 + 4\nu c})/2\nu$ (the absorbing boundary operator) and the case of a negative velocity, $a = -1$, with $g_1 = 0$, $g_2 = 0$, $\mathcal{B}_1 = Id$ and $\mathcal{B}_2 = Id$. In all experiments, the error in the advection domain $\|u - u_a\|_{\Omega^+}$ is $\mathcal{O}(\nu)$ for all coupling strategies, which is natural, since the advection equation is used instead of the advection-diffusion equation. The numerical error estimates for the overlapping techniques in the viscous domain Ω^- are summarized in Table 1. We see that for $a > 0$ and small overlap, the cost functions J_0 and G_3

	$a > 0$		$a < 0$	
	Small Overlap	Large Overlap	Small Overlap	Large Overlap
J_0	$\mathcal{O}(\nu^{5/2})$	$\mathcal{O}(\nu^{3/2})$	$\mathcal{O}(\nu)$	$\mathcal{O}(\nu)$
J_α	$\mathcal{O}(\nu^{1/2})$	$\mathcal{O}(\nu^{3/2})$	$\mathcal{O}(\nu)$	$\mathcal{O}(\nu)$
G_3	$\mathcal{O}(\nu^{5/2})$	$\mathcal{O}(\nu^{3/2})$	$\mathcal{O}(\nu)$	$\mathcal{O}(\nu)$

Table 1 Overlapping coupling with optimization: numerically computed error estimate for $\|u - u_{ad}\|_{\Omega^-}$ (we chose $\alpha = 0.5$ in J_α)

provide a better coupling, since they lead to coupled solutions substantially closer to the viscous one than J_α. For large overlap, the accuracy is similar for all cost functions used. For $a < 0$, all coupling strategies give a result $\mathcal{O}(\nu)$, since information is coming from the inviscid approximation in Ω^+ to Ω^-, and in Ω^+ the error $\|u - u_a\|_{\Omega^+}$ is $\mathcal{O}(\nu)$.

The non overlapping case $\delta = 0$ is also considered in Gervasio et al. [2001], namely

$$J_{\alpha,\beta}(\lambda_1,\lambda_2) = \alpha(u_{ad}(x_0) - u_a(x_0))^2 + \beta(\phi_1 - \phi_2)^2,$$

where $\phi_1 = -\nu u'_{ad}(x_0) + a u_{ad}(x_0)$ and $\phi_2 = a u_a(x_0)$ (see Section 3.1) and α, β are 0 or 1. Using the same numerical setting, we obtain for ν small the

	$a > 0$	$a < 0$
$\sigma(a)(u_{ad}(x_0) - u_a(x_0))^2 + (\phi_1 - \phi_2)^2$	$\mathcal{O}(\nu^{3/2})$	$\mathcal{O}(\nu)$
$(\phi_1 - \phi_2)^2$	$\mathcal{O}(\nu^{1/2})$	$\mathcal{O}(\nu)$
$(u_{ad}(x_0) - u_a(x_0))^2$	$\mathcal{O}(\nu^{1/2})$	$\mathcal{O}(\nu)$

Table 2 Non overlapping case with optimization: numerically computed error estimates for $\|u - u_{ad}\|_{\Omega^-}$

error estimates shown in Table 2. We observe that for positive advection, the coupled solution obtained without overlap and optimization is also dependent on the functional used, and in general inferior to the case with overlap.

3 Methods based on coupling conditions

From now on we assume that there is no overlap, $\delta = 0$. The coupling techniques in this section are based on coupling conditions, and we will present three strategies: the first one is based on singular perturbation, the second one on boundary layer corrections, and the last one on the factorization of the operator.

3.1 Coupling conditions from singular perturbation

In Gastaldi and Quarteroni [1989/90] the authors propose to find coupling conditions for (2) and (3) by introducing a regularization of the inviscid problem using a small artificial viscosity ϵ. They thus consider

$$\begin{aligned} -\nu w_\epsilon'' + a w_\epsilon' + c w_\epsilon &= f \quad \text{on } (-L_1, x_0), \\ -\epsilon v_\epsilon'' + a v_\epsilon' + c v_\epsilon &= f \quad \text{on } (x_0, L_2). \end{aligned} \tag{4}$$

This coupling problem which involves two elliptic equations needs to be completed by two boundary conditions. The first one simply states continuity of the solution: $w_\epsilon(x_0) = v_\epsilon(x_0)$. For the second one, two choices are possible : we can impose the continuity of the normal flux, $\nu {w_\epsilon}'(x_0) = \epsilon {v_\epsilon}'(x_0)$ (such boundary conditions are called variational conditions) or we impose the continuity of the normal derivative, ${w_\epsilon}'(x_0) = {v_\epsilon}'(x_0)$ (called non variational conditions). Letting ϵ tend to 0, it has been rigorously proved in Gastaldi and Quarteroni [1989/90] that w_ϵ (resp. v_ϵ) tends to u_{ad} (resp. u_a). At the boundary, with the variational conditions, the limiting solution satisfies

$$\begin{aligned} &(-\nu u_{ad}' + a u_{ad})(x_0) = a u_a(x_0), \quad u_{ad}(x_0) = u_a(x_0) \quad &&\text{for} \quad a > 0, \\ &(-\nu u_{ad}' + a u_{ad})(x_0) = a u_a(x_0), &&\text{for} \quad a < 0, \end{aligned} \tag{5}$$

while the non variational conditions lead to

$$\begin{aligned} &u_{ad}(x_0) = u_a(x_0), \quad u'_{ad}(x_0) = u'_a(x_0), \quad &\text{for} \quad a > 0, \\ &u_{ad}(x_0) = u_a(x_0), &\text{for} \quad a < 0. \end{aligned} \tag{6}$$

Rigorous error estimates comparing the coupled solutions obtained with these approaches were obtained in Gander et al. [2009], and they are summarized in Table 3, where we observe that the non variational conditions lead to a better coupled solution for positive advection than the variational ones, while for negative advection, again there is no difference between the two approaches. Finally, it has been proved in Discacciati et al. [2010] that the coupling problem with variational conditions is equivalent to the problem using optimization on $\sigma(a)(u_{ad}(0) - u_a(0))^2 + (\phi_1 - \phi_2)^2$; our observation is thus consistent. Note that the other non-overlapping coupling conditions based on optimization yield less accurate coupled solutions.

	$a > 0$	$a < 0$
Variational Conditions	$\mathcal{O}(\nu^{3/2})$	$\mathcal{O}(\nu)$
Non Variational Conditions	$\mathcal{O}(\nu^{5/2})$	$\mathcal{O}(\nu)$

Table 3 Variational versus non-variational coupling conditions: theoretical error estimates for $\|u - u_{ad}\|_{\Omega^-}$

3.2 Coupling through boundary layer correction

A different approach, only adding a correction for the boundary layer, was proposed in Coclici et al. [2000]. Here, the authors define the coupled solution of interest to be the solution of the regularized problem (4), and they consider the variational solution obtained from (5) as a first approximation of the regularized one. More precisely the coupled solution is represented as a perturbation of the variational solution in the form

$$\begin{aligned} w_\epsilon(x) &= u_{ad}(x) + r_\epsilon(x), \\ v_\epsilon(x) &= u_a(x) + l_\epsilon(x) + s_\epsilon(x), \end{aligned}$$

where l_ϵ is a boundary layer function and r_ϵ and s_ϵ are the remainders of the asymptotic expansion. The boundary layer term can be computed analytically, but integrals that are involved are then approximated numerically. The numerical solution does not take into account the remainders r_ϵ and s_ϵ and thus, compared to the solution obtained with (5), the pure advection solution in Ω^+ is the only one to be corrected.

3.3 Coupling conditions from operator factorization

A very accurate set of coupling conditions can be derived from an operator factorization, see Gander et al. [2009], and requires the solution of a modified advection equation: if we introduce $\lambda^{\pm} = (a \pm \sqrt{a^2 + 4\nu c})/2\nu$, the advection diffusion equation can be factored, i.e.

$$\mathcal{L}_{ad} u = (\partial_x - \lambda^+)(\partial_x - \lambda^-)u = f,$$

which gives after integration on (x_0, L_2)

$$(\partial_x - \lambda^-)u(x_0) = (\partial_x - \lambda^-)u(L_2)e^{-\lambda^+ L_2} + \int_{x_0}^{L_2} f(\sigma)e^{-\lambda^+ \sigma} d\sigma.$$

Introducing the new advection equation $(\partial_x - \lambda^+)\tilde{u}_a = f$, we find that the viscous solution satisfies

$$(\partial_x - \lambda^-)u(x_0) = \tilde{u}_a(x_0) + ((\partial_x - \lambda^-)u(L_2) - \tilde{u}_a(L_2))e^{-\lambda^+ L_2}. \quad (7)$$

Solving the advection-diffusion equation in Ω^- with the boundary condition (7) (replacing u by u_{ad} on the left hand side) would thus yield the exact coupled solution, i.e. $u_{|\Omega^-} = u_{ad}$. However the term in L_2 can not be used directly, and one chooses instead $\tilde{u}_a(L_2)$ to be an expansion of $(\partial_x - \lambda^-)u(L_2)$ for ν small, so that the proposed coupling condition is

$$(\partial_x - \lambda^-)u_{ad}(x_0) = \tilde{u}_a(x_0). \quad (8)$$

This leads to the coupling procedure

1. Solve the new advection equation $(\partial_x - \lambda^+)\tilde{u}_a = f$ on (x_0, L_2) with $\tilde{u}_a(L_2) = z_0 + z_1\nu + \cdots + \mathcal{O}(\nu^m)$.
2. Solve the advection-diffusion equation on $(-L_1, x_0)$ with the transmission condition (8).
3. Solve the advection equation (3) on (x_0, L_2) with the condition $u_{ad}(x_0) = u_a(x_0)$ if $a > 0$.

For our model problem, rigorous error estimates obtained in Gander et al. [2009] are shown in Table 4. We see that this coupling strategy leads to a coupled solution which is much closer to the fully viscous one than any of the other strategies. Even in the case of negative advection, one can now obtain approximations more accurate than $\mathcal{O}(\nu)$. Note however that $\lambda^{\pm}$ are simple constants only in the stationary one dimensional case. In the case of

	$a > 0$	$a < 0$
Factorization of the operator	$\mathcal{O}(e^{-a/\nu})$	$\mathcal{O}(\nu^m)$

Table 4 Coupling based on factorization: theoretical error estimates for $\|u - u_{ad}\|_{\Omega^-}$

evolution, or for higher dimensions, the $\lambda^{\pm}$ need to be approximated (see for example Gander et al. [2011]).

4 The χ-formulation

A very different approach for coupling viscous and inviscid problems is proposed in Brezzi et al. [1989]: the method called χ-formulation decides automatically where the viscous model and where the inviscid one needs to be used, and solves the equation

$$\begin{aligned} -\nu\chi(u'') + au' + cu &= f \qquad \text{on } (-L_1, L_2),\\ u &= g_1 \qquad \text{on } x = -L_1,\\ \mathcal{B}u &= 0 \qquad \text{on } x = L_2, \end{aligned}$$

where the χ function is defined by

$$\chi(s) = \begin{cases} 0 & 0 \le s < \delta - \sigma,\\ (s - \delta + \sigma)\frac{\delta}{\sigma} & \delta - \sigma \le s \le \delta,\\ s & s > \delta, \end{cases}$$

so that the diffusion term is neglected as soon as it is small enough. This leads however to a non-linear equation, even if the underlying models are linear, which requires a Newton type algorithm.

In Brezzi et al. [1989], the method is studied for the model problem at the continuous level, and well posedness is proved. Several years later, in Achdou and Pironneau [1993] and Lai et al. [1998], this strategy is used to solve the Navier-Stokes equations. Note that other cut-off functions can also be considered. We show in Table 5 numerically computed error estimates for the χ-formulation applied to our model problem.

	$a > 0$	$a < 0$
χ-formulation	$\mathcal{O}(\nu^{5/2})$	$\mathcal{O}(\nu)$

Table 5 χ-formulation: numerically computed error estimate for $\|u - u_{ad}\|_{\Omega^-}$

5 Conclusions

For a positive velocity a, among all the strategies presented in this paper, the best coupling condition is provided by the factorization of the operator in the non overlapping case: the error between the corresponding coupled solution and the full viscous solution is exponentially small. Good algebraically small errors of $\mathcal{O}(\nu^{5/2})$ can also be obtained: in the overlapping case by optimization on J_0 and G_3 with a small overlap, and in the non overlapping case using the non variational conditions (6), or with the χ-formulation. The other strategies yield less accurate error estimates. When $a < 0$, the factorization method is the only one to provide a better estimate than $\mathcal{O}(\nu)$.

References

Y. Achdou and O. Pironneau. The χ-method for the Navier-Stokes equations. *IMA J. Numer. Anal.*, 13(4):537–558, 1993.

V. Agoshkov, P. Gervasio, and A. Quarteroni. Optimal control in heterogeneous domain decomposition methods for advection-diffusion equations. *Mediterr. J. Math.*, 3(2):147–176, 2006.

F. Brezzi, C. Canuto, and A. Russo. A self-adaptive formulation for the Euler/Navier-Stokes coupling. *Comput. Methods Appl. Mech. Engrg.*, 73 (3):317–330, 1989.

C. A. Coclici, G. Moroşanu, and W. L. Wendland. The coupling of hyperbolic and elliptic boundary value problems with variable coefficients. *Math. Methods Appl. Sci.*, 23(5):401–440, 2000.

Q. V. Dinh, R. Glowinski, J. Périaux, and G. Terrasson. On the coupling of viscous and inviscid models for incompressible fluid flows via domain decomposition. In *First International Symposium on Domain Decomposition Methods for Partial Differential Equations (Paris, 1987)*, pages 350–369. SIAM, Philadelphia, PA, 1988.

M. Discacciati, Gervasio P., and A. Quarteroni. Heterogeneous mathematical models in fluid dynamics and associated solution algorithms. *Tech. Report MOX 04/2010*, 2010.

M. J. Gander, L. Halpern, C. Japhet, and V. Martin. Viscous problems with inviscid approximations in subregions: a new approach based on operator factorization. In *CANUM 2008*, volume 27 of *ESAIM Proc.*, pages 272–288. EDP Sci., Les Ulis, 2009.

M. J. Gander, L. Halpern, and V. Martin. How close to the fully viscous solution can one get with inviscid approximations in subregions ? In *Domain Decomposition Methods in Science and Engineering XIX*, volume 78 of *Lect. Notes Comput. Sci. Eng.*, pages 237–244. Springer, 2011.

F. Gastaldi and A. Quarteroni. On the coupling of hyperbolic and parabolic systems: analytical and numerical approach. *Appl. Numer. Math.*, 6(1-2): 3–31, 1989/90. Spectral multi-domain methods (Paris, 1988).

P. Gervasio, J.-L. Lions, and A. Quarteroni. Heterogeneous coupling by virtual control methods. *Numer. Math.*, 90(2):241–264, 2001.

C.-H. Lai, A. M. Cuffe, and K. A. Pericleous. A defect equation approach for the coupling of subdomains in domain decomposition methods. *Comput. Math. Appl.*, 35(6):81–94, 1998.