Optimal coarse spaces for FETI and their approximation

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Abstract. One-level iterative domain decomposition methods share only information between neighboring subdomains, and are thus not scalable in general. For scalability, a coarse space is thus needed. This coarse space can however do more than just make the method scalable: there exists an optimal coarse space in the sense that we have convergence after exactly one coarse correction, and thus the method becomes a direct solver. We introduce and analyze here a new such optimal coarse space for the FETI method for the positive definite Helmholtz equation in one and two space dimensions for strip domain decompositions. We then show how one can approximate the optimal coarse space using optimization techniques. Computational results illustrating the performance and effectiveness of this new coarse space and its approximations are also presented.

1 Introduction

Domain decomposition techniques are widely used for solving algebraic systems resulting from the discretization of partial differential equations. The basic idea of these methods is to decompose the original domain Ω into subdomains $\Omega_1, \ldots, \Omega_N$ which may or may not overlap, and then to solve local subproblems in each subdomain Ω_i, which are coupled by an iteration through artificial boundary conditions at the interfaces between subdomains. Such one level methods are in general not scalable, since the communication between subdomains is local, and hence convergence deteriorates when the number of subdomains N grows. A natural remedy for this is to introduce an additional coarse correction to these methods which will allow global communication between subdomains, and hence improve their convergence behavior.

In this paper, we study the one-level FETI (Finite Element Tearing and Interconnect) iterative method, also known as Dirichlet-Dirichlet method, see [2,3] and references therein. Following ideas in [1,4,5], we show how an optimal coarse space for FETI can be developed and approximated in an optimized way. We give an error analysis for strip decompositions in 1D and 2D, and illustrate the performance of our new coarse spaces by numerical experiments. Our approximate coarse spaces are related to the ones developed in [6], which were obtained differently, namely by adding coarse space

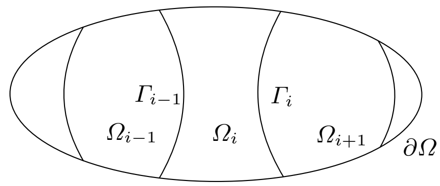

Fig. 1. Strip decomposition of the domain Ω.

components to improve an estimate in the general convergence proof of FETI methods.

2 One- and two-level FETI algorithms

Let Ω be an open domain in $\mathbb{R}^d$, $d = 1, 2$. We consider the model problem

$$\eta u - \Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1}$$

where $f \in L^2(\Omega)$, and $\eta > 0$. We decompose the domain Ω into non-overlapping subdomains $(\Omega_i)_{1\leq i\leq N}$ with interfaces $(\Gamma_i)_{1\leq i\leq N-1}$ as shown in Figure 1, a so called strip decomposition. We denote by n_i the outward normal corresponding to Γ_i as seen from subdomain Ω_i, and introduce

Algorithm 1 (One-level FETI algorithm)

1. Initialize $\lambda^0_{i,j} = -\lambda^0_{j,i} = \lambda^0_i$ on each interface $\partial\Omega_i \cap \partial\Omega_j$.
2. Until convergence
   (a) Solve the Neumann followed by the Dirichlet problems

$$\begin{aligned} \eta u_i^n - \Delta u_i^n &= f \text{ in } \Omega_i, & \eta \psi_i^n - \Delta \psi_i^n &= 0 \text{ in } \Omega_i, \\ \frac{\partial u_i^n}{\partial n_i} &= \lambda^n_{i,j} \text{ on } \partial\Omega_i \cap \partial\Omega_j, & \psi_i^n &= \frac{1}{2}\left(u_i^n - u_j^n\right) \text{ on } \partial\Omega_i \cap \partial\Omega_j, \\ u_i^n &= 0 \text{ on } \partial\Omega_i \cap \partial\Omega, & \psi_i^n &= 0 \text{ on } \partial\Omega_i \cap \partial\Omega. \end{aligned} \tag{2}$$

   (b) Update the Neumann traces by

$$\lambda^{n+1}_{i,j} = \lambda^n_{i,j} - \frac{1}{2}\left(\frac{\partial \psi_i^n}{\partial n_i} + \frac{\partial \psi_j^n}{\partial n_i}\right) \text{ on } \partial\Omega_i \cap \partial\Omega_j. \tag{3}$$

This algorithm has no mechanism for global communication and is therefore not scalable, as we will see in Section 3, so a coarse correction needs to be added. Note that when convergence is reached, the local corrections ψ_i^n are zero, meaning that the iterates u_i^n have no Dirichlet jump. Therefore, a good coarse correction should be capable to reduce the jump between the iterates in order to speed-up convergence. As in [1] for optimized Schwarz methods, we define the quadratic functional

$$q : \prod_{i=1}^{N} H^1(\Omega_i) \mapsto \mathbb{R}^+, \quad (u_i)_{1\le i\le N} \mapsto \sum_{\partial\Omega_i\cap\partial\Omega_j\neq\emptyset} \int_{\partial\Omega_i\cap\partial\Omega_j} |u_i - u_j|^2 \,\mathrm{d}s,$$

where $H^1(\Omega_i)$ denotes the Sobolev space that consists of square integrable functions on Ω_i with square integrable gradient. Since the iterates u_i^n of FETI have continuous normal derivatives at interfaces, we choose a coarse space with continuous normal derivatives at interfaces, namely,

$$X_d = \left\{ v \in \prod_{i=1}^{N} H^1(\Omega_i) : \forall i, \eta v_i - \Delta v_i = 0, \left[\frac{\partial v_i}{\partial n_i}\right]_{\Gamma_i} = 0, v_{i|\partial\Omega_i\cap\partial\Omega} = 0 \right\}. \tag{4}$$

This leads to the optimal two level FETI algorithm given by

Algorithm 2 (Two-level FETI algorithm with optimal coarse space)

1. Initialize $\lambda_{i,j}^0 = -\lambda_{j,i}^0 = \lambda_i^0$ on each interface $\partial\Omega_i \cap \partial\Omega_j$.
2. Until convergence
   (a) Solve the Neumann followed by the Dirichlet problems

$$\begin{aligned} \eta u_i^n - \Delta u_i^n &= f \text{ in } \Omega_i, & \eta \psi_i^n - \Delta \psi_i^n &= 0 \text{ in } \Omega_i,\\ \frac{\partial u_i^n}{\partial n_i} &= \lambda_{i,j}^n \text{ on } \partial\Omega_i \cap \partial\Omega_j, & \psi_i^n &= \frac{1}{2}\left(\tilde{u}_i^n - \tilde{u}_j^n\right) \text{ on } \partial\Omega_i \cap \partial\Omega_j,\\ u_i^n &= 0 \text{ on } \partial\Omega_i \cap \partial\Omega, & \psi_i^n &= 0 \text{ on } \partial\Omega_i \cap \partial\Omega, \end{aligned} \tag{5}$$

   where $q\left((\tilde{u}_i^n)_{1\le i\le N}\right) = \min_{v\in X_d} q\left((u_i^n)_{1\le i\le N} + v\right)$.

   (b) Update the Neumann traces by

$$\lambda_{i,j}^{n+1} = \frac{\partial \tilde{u}_i^n}{\partial n_i} - \frac{1}{2}\left(\frac{\partial \psi_i^n}{\partial n_i} + \frac{\partial \psi_j^n}{\partial n_i}\right) \quad \text{on } \partial\Omega_i \cap \partial\Omega_j. \tag{6}$$

3 Convergence analysis

Since problem (1) is linear, it suffices to set $f = 0$ and to analyze the convergence of $(\lambda_i^n)_{1\le i\le N-1}$ to zero. We start in the 1D case, see Figure 2. Using explicit solutions, we obtain

Lemma 1. *Let* $\boldsymbol{\lambda}^n := \left[\lambda_1^n, \lambda_2^n, \ldots, \lambda_{N-1}^n\right]^T \in \mathbb{R}^{N-1}$. *Then for* $N \ge 3$, *Algorithm 1 for* $f = 0$ *is equivalent to the iteration* $\boldsymbol{\lambda}^n = T\boldsymbol{\lambda}^{n-1}$, *where the*

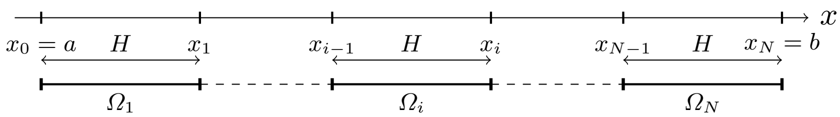

Fig. 2. One-dimensional geometry

matrix $T \in \mathbb{R}^{(N-1)\times(N-1)}$ *is given by*

$$T := \frac{1}{s^2}\begin{bmatrix} -\frac{1}{4} & 0 & \frac{1}{4} & 0 & \cdots & \cdots & 0 \\ \frac{1}{4c} & -\frac{1}{2} & 0 & \frac{1}{4} & \ddots & & \vdots \\ \frac{1}{4} & 0 & -\frac{1}{2} & & \frac{1}{4} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \frac{1}{4} & 0 & -\frac{1}{2} & 0 & \frac{1}{4} \\ \vdots & & \ddots & \frac{1}{4} & 0 & -\frac{1}{2} & \frac{1}{4c} \\ 0 & \cdots & \cdots & 0 & \frac{1}{4} & 0 & -\frac{1}{4} \end{bmatrix},$$

with $c := \cosh\left(\sqrt{\eta}H\right)$ *and* $s := \sinh\left(\sqrt{\eta}H\right)$.

Proof. Solving for u_i^n in (2) in one spatial dimension, we get

$$u_i^n(x) = \lambda_i^n \frac{\cosh\left(\sqrt{\eta}(x - x_{i-1})\right)}{\sqrt{\eta}s} - \lambda_{i-1}^n \frac{\cosh\left(\sqrt{\eta}(x_i - x)\right)}{\sqrt{\eta}s}, \quad i = 2, \ldots, N-1,$$
$$u_1^n(x) = \lambda_1^n \frac{\sinh\left(\sqrt{\eta}(x - x_0)\right)}{\sqrt{\eta}c},\ u_N^n(x) = -\lambda_{N-1}^n \frac{\sinh\left(\sqrt{\eta}(x_N - x)\right)}{\sqrt{\eta}c}.$$

We then obtain for ψ_i^n in (2)

$$\psi_i^n(x) = A_i \frac{\sinh(\sqrt{\eta}(x - x_{i-1}))}{2\sqrt{\eta}s} - A_{i-1}\frac{\sinh(\sqrt{\eta}(x_i - x))}{2\sqrt{\eta}s}, \quad i = 3, \ldots, N-2,$$
$$\psi_1^n(x) = \tilde{A}_1 \frac{\sinh(\sqrt{\eta}(x - x_0))}{2\sqrt{\eta}s},$$
$$\psi_2^n(x) = A_2 \frac{\sinh(\sqrt{\eta}(x - x_1))}{2\sqrt{\eta}s} - \tilde{A}_1 \frac{\sinh(\sqrt{\eta}(x_2 - x))}{2\sqrt{\eta}s},$$
$$\psi_{N-1}^n(x) = \tilde{A}_{N-1} \frac{\sinh(\sqrt{\eta}(x - x_{N-2}))}{2\sqrt{\eta}s} - A_{N-2}\frac{\sinh(\sqrt{\eta}(x_{N-1} - x))}{2\sqrt{\eta}s},$$
$$\psi_N^n(x) = -\tilde{A}_{N-1}\frac{\sinh(\sqrt{\eta}(x_N - x))}{2\sqrt{\eta}s},$$

where $A_i := -\frac{1}{s}\lambda_{i+1}^n + \frac{2c}{s}\lambda_i^n - \frac{1}{s}\lambda_{i-1}^n$, $\tilde{A}_1 := -\frac{1}{s}\lambda_2^n + \left(\frac{c}{s} + \frac{s}{c}\right)\lambda_1^n$, $\tilde{A}_{N-1} := -\frac{1}{s}\lambda_{N-2}^n + \left(\frac{c}{s} + \frac{s}{c}\right)\lambda_{N-1}^n$. Using then formula (3), we get the desired result.

Theorem 2. *If $H > \frac{\ln(1+\sqrt{2})}{\sqrt{\eta}}$, then the one-level Algorithm 1 is convergent in 1D, and satisfies the convergence estimate*

$$\max_{1\leq i\leq N-1} |\lambda_i^n| \leq \frac{1}{\sinh^{2n}(\eta H)} \max_{1\leq i\leq N-1} |\lambda_i^0|. \tag{7}$$

Proof. Using Lemma 1, we have that

$$\|T\|_\infty = \max\left\{\frac{1}{2s^2}, \frac{3}{4s^2} + \frac{1}{4cs^2}, \frac{1}{s^2}\right\} = \frac{1}{s^2},$$

since $c \geq 1$, which completes the proof.

Theorem 3. *Algorithm 2 converges in 1D after one iteration.*

Proof. Since it is possible with the chosen coarse space to cancel the jump, i.e. $(q((\tilde{u}_i^n)_{1\leq i\leq N}) = 0)$, the iterate $\tilde{u}^n$ defined as $\tilde{u}^n_{|\Omega_i} = \tilde{u}_i^n$ is continuous in both Dirichlet and Neumann traces, and hence it is precisely the monodomain solution, which concludes the proof.

Theorem 3 shows that the coarse space X_d is optimal, i.e. it is the smallest coarse space leading to Theorem 3. In 1D, the coarse space X_d is of dimension precisely $N-1$, which is needed to remove the $N-1$ constraints at the interfaces.

In 2D, we restrict the analysis to the case where the subdomains are rectangles as shown in Figure 3, and expand u_j^n and ψ_j^n in a sine series,

$$u_j^n(x,y) = \sum_{m=1}^{\infty} \hat{u}_j^n(x,m)\sin(k_m y), \quad \psi_j^n(x,y) = \sum_{m=1}^{\infty} \hat{\psi}_j^n(x,m)\sin(k_m y), \tag{8}$$

where $k_m := \frac{m\pi}{L}$. This allows us to study the convergence based on the Fourier coefficients.

Lemma 4. *Let $\hat{\boldsymbol{\lambda}}^n(m) := \left[\hat{\lambda}_1^n(m), \hat{\lambda}_2^n(m), \ldots, \hat{\lambda}_{N-1}^n(m)\right]^T \in \mathbb{R}^{N-1}$. Then for $N \geq 3$ Algorithm 1 is equivalent to $\hat{\boldsymbol{\lambda}}^n(m) = T_m\hat{\boldsymbol{\lambda}}^{n-1}(m)$, where the matrix $T_m \in \mathbb{R}^{(N-1)\times(N-1)}$ is given by*

$$T_m := \frac{1}{s_m^2}\begin{bmatrix} -\frac{1}{4} & 0 & \frac{1}{4} & 0 & \cdots & \cdots & 0 \\ \frac{1}{4c_m} & -\frac{1}{2} & 0 & \frac{1}{4} & \ddots & & \vdots \\ \frac{1}{4} & 0 & -\frac{1}{2} & & \frac{1}{4} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \frac{1}{4} & 0 & -\frac{1}{2} & 0 & \frac{1}{4} \\ \vdots & & \ddots & \frac{1}{4} & 0 & -\frac{1}{2} & \frac{1}{4c_m} \\ 0 & \cdots & \cdots & 0 & \frac{1}{4} & 0 & -\frac{1}{4} \end{bmatrix}, \tag{9}$$

with $s_m := \sinh\left(\sqrt{\eta + k_m^2}H\right)$ and $c_m := \cosh\left(\sqrt{\eta + k_m^2}H\right)$.

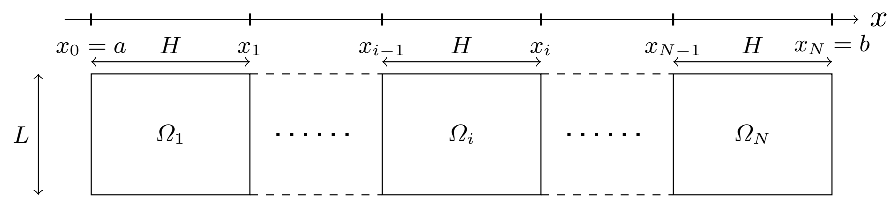

Fig. 3. Two-dimensional geometry.

Proof. Since the Fourier coefficients $\hat{u}_j^n(x,m)$ and $\hat{\psi}_j^n(x,m)$ satisfy a sequence of one-dimensional problems on each Ω_i,

$$\begin{aligned} (\eta + k_m^2)\hat{u}_i^n - \partial_{xx}\hat{u}_i^n &= 0, & (\eta + k_m^2)\hat{\psi}_i^n - \partial_{xx}\hat{\psi}_i^n &= 0, \\ \partial_x \hat{u}_i^n(x_{i-1},m) &= \hat{\lambda}_{i-1}^n(m), & \hat{\psi}_i^n(x_{i-1},m) &= \tfrac{\hat{u}_i^n(x_{i-1},m) - \hat{u}_{i-1}^n(x_{i-1},m)}{2}, \\ \partial_x \hat{u}_i^n(x_i,m) &= \hat{\lambda}_i^n(m), & \hat{\psi}_i^n(x_i,m) &= \tfrac{\hat{u}_i^n(x_i,m) - \hat{u}_{i+1}^n(x_i,m)}{2}, \end{aligned}$$

the iteration matrix T_m becomes the matrix T of Lemma 1 if we replace η by $\eta + k_m^2$, which concludes the proof.

Theorem 5. *If $H > \frac{\ln(1+\sqrt{2})}{\sqrt{\eta + k_1^2}}$, then the one level Algorithm 1 is convergent in 2D, and satisfies the convergence estimate*

$$\max_{1 \le i \le N-1} \|\lambda_i^n\|_2 \le \frac{1}{\sinh^{2n}\left(\sqrt{\eta + k_1^2}H\right)} \max_{1 \le i \le N-1} \|\lambda_i^0\|_2. \qquad (10)$$

Proof. We define the sequences $\Lambda_i^n = \left\{\hat{\lambda}_i^n(m)\right\}_{m \ge 1}$. By Lemma 4, we have

$$\hat{\lambda}_i^{n+1}(m) = \frac{1}{s_m^2}\left(\frac{1}{4}\hat{\lambda}_{i-2}^n(m) - \frac{1}{2}\hat{\lambda}_i^n(m) + \frac{1}{4}\hat{\lambda}_{i+2}^n(m)\right)$$

for each $m \ge 1$. Using then Parseval's identity $\|\lambda_i^n\|_2^2 = \frac{L}{2}\sum_{m=1}^{\infty}\hat{\lambda}_i^n(m)^2 = \frac{L}{2}\|\Lambda_i^n\|_2^2$, we have for $i = 3, \ldots, N-3$

$$\begin{aligned} \|\lambda_i^{n+1}\|_2 &\le \frac{\sqrt{L}}{\sqrt{2}s_1^2}\left(\frac{1}{4}\|\Lambda_{i-2}^n\|_2 + \frac{1}{2}\|\Lambda_i^n\|_2 + \frac{1}{4}\|\Lambda_{i+2}^n\|_2\right) \\ &\le \frac{1}{s_1^2}\max_{1 \le i \le N-1}\|\lambda_i^n\|_2 \\ &= \frac{1}{\sinh^2\left(\sqrt{\eta + k_1^2}H\right)}\max_{1 \le i \le N-1}\|\lambda_i^n\|_2, \end{aligned}$$

where we used the triangle inequality, and the monotonicity of $1/s_m^2$ in m to bound it with $1/s_1^2$. Similarly one can show that the same bound also holds for the remaining subdomains $i = 1, 2, N-2, N-1$, and hence we get the stated result.

Algorithm 2 still converges after one iteration in two spatial dimension provided that the optimal coarse space X_d is used. However, this coarse space is now infinite dimensional, and even though it becomes finite dimensional after discretization, it is in practice too big to be used. To obtain an effective approximation, note that according to Theorem 5, low frequencies are responsible for the deterioration of the algorithm. Thus, the choice of the approximate coarse space can be optimized choosing to include the most slowly converging low frequency modes. We thus propose as approximation of the optimal coarse space X_d the optimized coarse space

$$\widetilde{X}_d := \Big\{ v \in X_d : \forall i, \partial_x v_i(x_i, y) \in \operatorname{span}\Big\{ \sin(k_m y), m = 1, \ldots, J \Big\}\Big\}, \quad (11)$$

where $J \geq 1$ can be suitably chosen to enrich the coarse space.

Theorem 6. *Algorithm 2 with the approximate coarse space $\widetilde{X}_d$ satisfies the convergence estimate*

$$\max_{1\leq i\leq N-1} \|\lambda_i^n\|_2 \leq \frac{1}{\sinh^{2n}\left(\sqrt{\eta + k_{J+1}^2}H\right)} \max_{1\leq i\leq N-1} \|\lambda_i^0\|_2. \quad (12)$$

Proof. Since at each iteration n, we have $u_i^n(x,y) = \sum_{m=1}^{\infty} \hat{u}_i^n(x,m)\sin(k_m y)$, using Parseval's identity, we have for $v \in \widetilde{X}_d$

$$\int_0^L \left|(u_{i+1}^n + v)(x_i^+, y) - (u_i^n + v)(x_i^-, y)\right|^2 \mathrm{d}y =$$
$$\frac{L}{2}\sum_{m=1}^{J}\left|(\hat{u}_{i+1}^n + \hat{v}_{i+1})(x_i^+, m) - (\hat{u}_i^n + \hat{v}_i)(x_i^-, m)\right|^2 + \frac{L}{2}\sum_{m=J+1}^{\infty}\left|\hat{u}_{i+1}^n(x_i^+, m) - \hat{u}_i^n(x_i^-, m)\right|^2,$$

we obtain

$$\begin{aligned} U^n &= \operatorname*{argmin}_{v\in\widetilde{X}_d} \sum_{i=1}^{N-1}\int_0^L \left|(u_{i+1}^n + v_{i+1})(x_i^+, y) - (u_i^n + v_i)(x_i^-, y)\right|^2 \mathrm{d}y \\ &= \operatorname*{argmin}_{v\in\widetilde{X}_d} \sum_{i=1}^{N-1}\sum_{m=1}^{J} \left|(\hat{u}_{i+1}^n + \hat{v}_i)(x_i^+, m) - (\hat{u}_i^n + \hat{v}_i)(x_i^-, m)\right|^2, \end{aligned}$$

and since we are now minimizing a finite dimensional quadratic, the quantity can be canceled. Hence, $\hat{\lambda}_i^{n+1}(m) = 0$ for $m \leq J$, and following the proof of Theorem 5, we get the stated bound.

4 Numerical Experiments

We start with the one dimensional problem on $\Omega = (-1, 1)$ with $\eta = 2$ and $f(x) = 1$. We discretize Algorithm 1 and 2 using centered finite differences

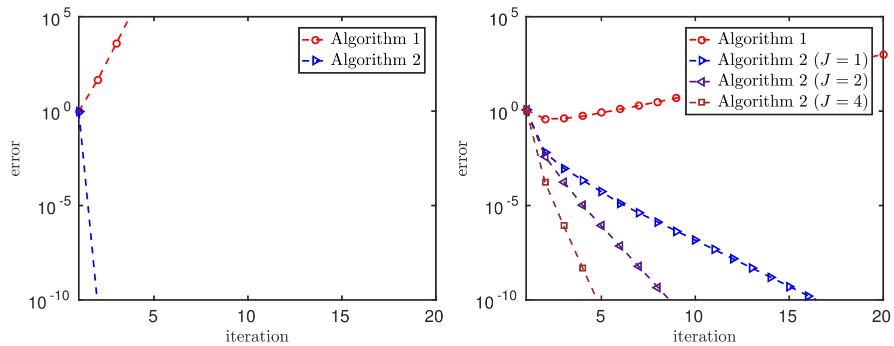

Fig. 4. Convergence of Algorithm 1 and Algorithm 2 in 1D (left) and 2D (right)

with mesh size $\Delta x = 10^{-4}$, and run the one-level algorithm with 30 equally sized subdomains. As expected, Figure 4 (left) shows that without coarse correction the algorithm fails to converge, and with the optimal coarse correction we get convergence after one iteration.

In 2d on $\Omega = (-1, 1)^2$ divided into 10 equally sized subdomains, we run Algorithm 2 on the error equation $\eta e - \Delta e = 0$ with $\eta = 2$, discretized by centered finite differences using the mesh size $\Delta x = \Delta y = 5 \cdot 10^{-3}$. We can see in Figure 4 (right) that approximations of the optimal coarse space are enough to obtain a convergent iterative FETI method, and the more low frequency components we include in the optimized coarse space, the better the convergence becomes.

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