

OPTIMIZED SCHWARZ METHODS FOR THE TIME-HARMONIC MAXWELL EQUATIONS WITH DAMPING*

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Abstract. In a previous paper, two of the authors have proposed and analyzed an entire hierarchy of optimized Schwarz methods for Maxwell's equations in both the time-harmonic and the time-domain case. The optimization process has been performed in a particular situation where the electric conductivity was neglected. Here, we take into account this physical parameter which leads to a fundamentally different analysis and a new class of algorithms for this more general case. From the mathematical point of view, the approach is different, since the algorithm does not encounter the pathological situations of the zero-conductivity case and thus the optimization problems are different. We analyze one of the algorithms in this class in detail and provide asymptotic results for the remaining ones. We illustrate our analysis with numerical results.

Key words. Schwarz algorithms, optimized transmission conditions, Maxwell's equations

AMS subject classifications. 65M55, 65F10, 65N22

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1. Introduction. Finding robust solvers for indefinite problems, such as Helmholtz and the time-harmonic version of Maxwell's equations, has always been a challenging research topic. From the pioneering work of Després [5], where the first provably convergent domain decomposition algorithm for Helmholtz equations was proposed and then extended to Maxwell's equations in [6], other studies followed with the declared purpose of improving the performance of these algorithms. The first attempts to obtain better algorithms for this kind of equations were given in [4] and then [3], where the first inspiring ideas of optimized Schwarz methods can be found. Then, in [15] and in particular in [14], the advantage of the optimization process was used at its best for the Helmholtz equations to design an algorithm without overlap; for the second order Maxwell system, see [1]. But, for the first time, an entire hierarchy of optimized transmission conditions for the first order Maxwell equations was given in [9], with complete asymptotic results. Applications to real-life problems using a discontinuous Galerkin method can be found in [10, 11]. For finite element-based nonoverlapping and nonconforming domain decomposition methods for the computation of multiscale electromagnetic radiation and scattering problems, we refer the reader to [16, 18, 17].

This paper is organized as follows: in section 2, we introduce the mathematical formulation of the problem to be solved, and in section 3, the classical and optimized Schwarz algorithms are presented. These algorithms are analyzed by computing their convergence factor in Fourier space in section 4. The main part of the paper, and the core of this study, is given in section 5, where an optimization problem is solved for

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one of the optimized Schwarz methods in all detail, and asymptotic results are given for all optimized Schwarz algorithms considered in this paper.

2. Presentation of the problem. The system of Maxwell's equations describes mathematically the propagation of electromagnetic waves

$$(2.1) \quad \varepsilon \frac{\partial \mathcal{E}}{\partial t} - \operatorname{curl} \mathcal{H} + \sigma \mathcal{E} = -\mathcal{J}, \quad \mu \frac{\partial \mathcal{H}}{\partial t} + \operatorname{curl} \mathcal{E} = 0,$$

where $\mathcal{E}$ and $\mathcal{H}$ denote the electric and magnetic field, ε is the electric permittivity, and μ is the magnetic permeability. Here σ denotes the conductivity and $\mathcal{J}$ the applied current density.

We are interested in solving the time-harmonic Maxwell equations, which are obtained from their time-domain counterpart (2.1) by assuming that the electric field $\mathcal{E}$ and the magnetic field $\mathcal{H}$ follow a harmonic dependence on time (as a result of the time-harmonic dependence of the current density $\mathcal{J}(\mathbf{x}, t) = \operatorname{Re}(\mathbf{J}(\mathbf{x}) \exp(i\omega t))$),

$$\mathcal{E}(\mathbf{x}, t) = \operatorname{Re}(\mathbf{E}(\mathbf{x}) \exp(i\omega t)), \quad \mathcal{H}(\mathbf{x}, t) = \operatorname{Re}(\mathbf{H}(\mathbf{x}) \exp(i\omega t)),$$

where the positive real parameter ω is the pulsation of the harmonic wave. The unknown complex-valued vector fields $\mathbf{E}$ and $\mathbf{H}$ are the solutions of the time-harmonic Maxwell equations

$$(2.2) \quad i\omega\varepsilon\mathbf{E} - \operatorname{curl} \mathbf{H} + \sigma\mathbf{E} = -\mathbf{J}, \quad i\omega\mu\mathbf{H} + \operatorname{curl} \mathbf{E} = \mathbf{0}.$$

Equations (2.2) are solved in a bounded domain Ω . On the boundary $\partial\Omega = \Gamma_a \cup \Gamma_m$, the following boundary conditions are imposed:

- a perfect electric conductor condition on Γ_m : $\mathbf{n} \times \mathbf{E} = 0$,
- an impedance condition on Γ_a : $\mathcal{B}_{\mathbf{n}}(\mathbf{E}, \mathbf{H}) = \mathcal{B}_{\mathbf{n}}(\mathbf{E}_{\text{inc}}, \mathbf{H}_{\text{inc}})$,

where the vector $(\mathbf{E}^{\text{inc}}, \mathbf{H}^{\text{inc}})$ represents an incident electromagnetic wave and $\mathbf{n}$ denotes the unit outward normal at any point of $\partial\Omega$. The impedance operator is given by

$$\mathcal{B}_{\mathbf{n}}(\mathbf{E}, \mathbf{H}) = \mathbf{n} \times \frac{\mathbf{E}}{Z} - \mathbf{n} \times (\mathbf{H} \times \mathbf{n}).$$

Note that this condition is equivalent to imposing Dirichlet conditions on the incoming characteristics if we consider the hyperbolic nature of the underlying time-dependent system, and can also be seen as an approximation of a transparent boundary condition.

3. Schwarz methods for Maxwell's equations. We introduce here a domain decomposition method based on a Schwarz algorithm. In order to simplify the presentation we consider a decomposition of the computational domain Ω with boundary $\partial\Omega$ into two subdomains Ω_i , $i = 1, 2$, such that $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$. We set $\Gamma_{ij} = \partial\Omega_i \cap \bar{\Omega}_j \setminus \partial\Omega$, where $\partial\Omega_i$ is the boundary of Ω_i . For this decomposition (as shown in Figure 3.1), the classical Schwarz algorithm consists in computing $(\mathbf{E}^{j,n+1}, \mathbf{H}^{j,n+1})$ from $(\mathbf{E}^{j,n}, \mathbf{H}^{j,n})$, for $j = 1, 2$, by the iterative process

$$(3.1) \quad \begin{array}{ll} -i\omega\varepsilon\mathbf{E}^{1,n} + \operatorname{curl} \mathbf{H}^{1,n} - \sigma\mathbf{E}^{1,n} = \mathbf{J} & \text{in } \Omega_1, \\ i\omega\mu\mathbf{H}^{1,n} + \operatorname{curl} \mathbf{E}^{1,n} = \mathbf{0} & \text{in } \Omega_1, \\ \mathcal{B}_{\mathbf{n}_1}(\mathbf{E}^{1,n}, \mathbf{H}^{1,n}) = \mathcal{B}_{\mathbf{n}_1}(\mathbf{E}^{2,n-1}, \mathbf{H}^{2,n-1}) & \text{on } \Gamma_{12}, \\ -i\omega\varepsilon\mathbf{E}^{2,n} + \operatorname{curl} \mathbf{H}^{2,n} - \sigma\mathbf{E}^{2,n} = \mathbf{J} & \text{in } \Omega_2, \\ i\omega\mu\mathbf{H}^{2,n} + \operatorname{curl} \mathbf{E}^{2,n} = \mathbf{0} & \text{in } \Omega_2, \\ \mathcal{B}_{\mathbf{n}_2}(\mathbf{E}^{2,n}, \mathbf{H}^{2,n}) = \mathcal{B}_{\mathbf{n}_2}(\mathbf{E}^{1,n-1}, \mathbf{H}^{1,n-1}) & \text{on } \Gamma_{21}. \end{array}$$

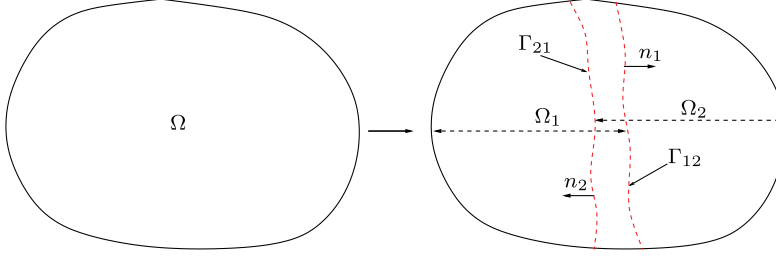
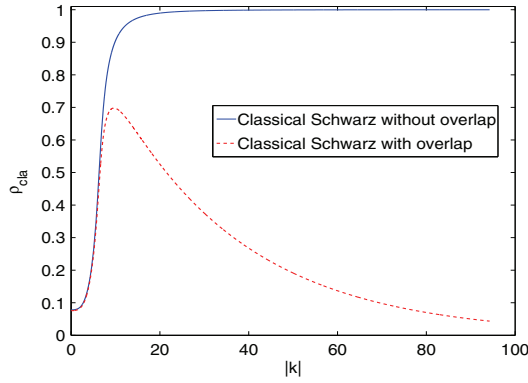


FIG. 3.1. Domain decomposition into two subdomains.

FIG. 3.2. Convergence factor ρ_{cla} of the classical Schwarz method with ($L = \delta$) and without overlap ($L = 0$) as a function of $|\mathbf{k}|$, $\omega = 2\pi$, $\sigma = 2$, and $\mu = \varepsilon = 1$.

For the classical Schwarz method, we see that at the interfaces (artificial boundaries between subdomains), characteristic information is exchanged. This type of conditions has been proposed for the first time in [6] for the second order version of the time-harmonic Maxwell equations.

In the constant coefficient case for the case of two half-spaces $\Omega_1 = (-\infty, L) \times \mathbb{R}^2$, $\Omega_2 = (0, \infty) \times \mathbb{R}^2$, an estimate of the convergence factor of this algorithm has been obtained in [9] using Fourier analysis. The convergence factor (the reduction of the error between two successive iterations) for each Fourier mode $\mathbf{k} = (k_y, k_z)$ is given by

$$\rho_{cla}(\mathbf{k}, \tilde{\omega}, \sigma, Z, L) = \left| \frac{\sqrt{|\mathbf{k}|^2 - \tilde{\omega}^2 + i\tilde{\omega}\sigma Z - i\tilde{\omega}}}{\sqrt{|\mathbf{k}|^2 - \tilde{\omega}^2 + i\tilde{\omega}\sigma Z + i\tilde{\omega}}} e^{-\sqrt{|\mathbf{k}|^2 - \tilde{\omega}^2 + i\tilde{\omega}\sigma Z} L} \right|.$$

Here $\tilde{\omega} = \omega \sqrt{\varepsilon \mu} = \frac{\omega}{c}$ denotes the adimensionalized frequency (or the wave number), $c = \frac{1}{\sqrt{\varepsilon \mu}}$ is the wave propagation speed, and L is the size of the overlap between the subdomains. From this result, we see that if $\sigma > 0$, the method converges also without overlap ($L = 0$). But for the high frequencies, the convergence factor is close to one, and therefore the algorithm is not very efficient; see Figure 3.2.

The convergence can be improved by modifying the transmission conditions in the classical Schwarz algorithm (3.1), namely

$$(3.2) \quad \begin{aligned} -i\omega\varepsilon\mathbf{E}^{1,n} + \operatorname{curl} \mathbf{H}^{1,n} - \sigma\mathbf{E}^{1,n} &= \mathbf{J} && \text{in } \Omega_1, \\ i\omega\mu\mathbf{H}^{1,n} + \operatorname{curl} \mathbf{E}^{1,n} &= \mathbf{0} && \text{in } \Omega_1, \\ (\mathcal{B}_{\mathbf{n}_1} + \mathcal{S}_1\mathcal{B}_{\mathbf{n}_2})(\mathbf{E}^{1,n}, \mathbf{H}^{1,n}) &= (\mathcal{B}_{\mathbf{n}_1} + \mathcal{S}_1\mathcal{B}_{\mathbf{n}_2})(\mathbf{E}^{2,n-1}, \mathbf{H}^{2,n-1}) && \text{on } \Gamma_{12}, \\ -i\omega\varepsilon\mathbf{E}^{2,n} + \operatorname{curl} \mathbf{H}^{2,n} - \sigma\mathbf{E}^{2,n} &= \mathbf{J} && \text{in } \Omega_2, \\ i\omega\mu\mathbf{H}^{2,n} + \operatorname{curl} \mathbf{E}^{2,n} &= \mathbf{0} && \text{in } \Omega_2, \\ (\mathcal{B}_{\mathbf{n}_2} + \mathcal{S}_2\mathcal{B}_{\mathbf{n}_1})(\mathbf{E}^{2,n}, \mathbf{H}^{2,n}) &= (\mathcal{B}_{\mathbf{n}_2} + \mathcal{S}_2\mathcal{B}_{\mathbf{n}_1})(\mathbf{E}^{1,n-1}, \mathbf{H}^{1,n-1}) && \text{on } \Gamma_{21}, \end{aligned}$$

where $\mathcal{S}_j$, $j = 1, 2$, are tangential, possibly pseudodifferential, operators. For $\sigma = 0$, different choices of $\mathcal{S}_j$, $j = 1, 2$, have been developed in [9] and lead to different optimized Schwarz algorithms for Maxwell's equations. In the next section, we develop different choices of $\mathcal{S}_j$ for the case $\sigma > 0$. This leads to optimization problems which are very different from the case $\sigma = 0$. The min-max problems can now have interior maxima, which complicates the mathematical analysis substantially, but their solution is necessary to obtain optimized transmission conditions in the case $\sigma > 0$.

4. Convergence analysis for nonzero electric conductivity. We present here an analysis of algorithm (3.2) for the case where the electric conductivity is nonzero, $\sigma > 0$. In [9], it was proved that if the operators $\mathcal{S}_l$, $l = 1, 2$, have the Fourier symbol

$$(4.1) \quad \mathcal{F}(\mathcal{S}_l) = \frac{1}{(\lambda + i\tilde{\omega})(\lambda + i\tilde{\omega} + \sigma Z)} \begin{bmatrix} k_y^2 - k_z^2 - \lambda\sigma Z & -2k_y k_z \\ -2k_y k_z & k_z^2 - k_y^2 - \lambda\sigma Z \end{bmatrix},$$

where $\lambda = \sqrt{|\mathbf{k}|^2 - \tilde{\omega}^2 + i\tilde{\omega}\sigma Z}$, then algorithm (3.2) converges in two iterations. It was also shown that these symbols can be written in different forms,

$$\begin{aligned} \mathcal{F}(\mathcal{S}_l) &= \frac{1}{(\lambda + i\tilde{\omega})(\lambda + i\tilde{\omega} + \sigma Z)} M = \frac{1}{|\mathbf{k}|^2 + \lambda\sigma Z} \frac{\lambda - i\tilde{\omega}}{\lambda + i\tilde{\omega}} M \\ &= \frac{1}{|\mathbf{k}|^2 - \lambda\sigma Z} \frac{\lambda - i\tilde{\omega} - \sigma Z}{\lambda + i\tilde{\omega} + \sigma Z} M = (\lambda - i\tilde{\omega})(\lambda - i\tilde{\omega} - \sigma Z) \tilde{M}^{-1}, \end{aligned}$$

where M and $\tilde{M}$ are given by

$$M = \begin{bmatrix} k_y^2 - k_z^2 - \lambda\sigma Z & -2k_y k_z \\ -2k_y k_z & k_z^2 - k_y^2 - \lambda\sigma Z \end{bmatrix}, \quad \tilde{M} = \begin{bmatrix} k_y^2 - k_z^2 + \lambda\sigma Z & -2k_y k_z \\ -2k_y k_z & k_z^2 - k_y^2 + \lambda\sigma Z \end{bmatrix}$$

and $\lambda = \sqrt{|\mathbf{k}|^2 - \tilde{\omega}^2 + i\omega\sigma Z}$. This motivates different approximations of the transparent conditions in the context of optimized Schwarz methods. In the following we will denote by M_s and $\tilde{M}_s$ the matrices M and $\tilde{M}$ where we replaced the nonlocal operator λ by a constant s . Note that after this approximation is performed we get local differential operators: for example, k_y^2 is the symbol of the partial differential operator $-\partial_{yy}$.

COROLLARY 4.1. *For different symbols of the operators $\mathcal{S}_j$ (which approximate the nonlocal operators (4.1)) we get the following convergence factors.*

Algorithm 1. *If $\mathcal{S}_j$, $j = 1, 2$, have the symbols*

$$(4.2) \quad \sigma_j = \mathcal{F}(\mathcal{S}_j) = 0,$$

then the (classical) Schwarz algorithm (3.2) has the convergence factor

$$(4.3) \quad \rho_1(|\mathbf{k}|, \tilde{\omega}, \sigma, Z, L) = \left| \frac{\sqrt{|\mathbf{k}|^2 - \tilde{\omega}^2 + i\tilde{\omega}\sigma Z - i\tilde{\omega}}}{\sqrt{|\mathbf{k}|^2 - \tilde{\omega}^2 + i\tilde{\omega}\sigma Z + i\tilde{\omega}}} e^{-\sqrt{|\mathbf{k}|^2 - \tilde{\omega}^2 + i\tilde{\omega}\sigma Z} L} \right|.$$

Algorithm 2. If S_j , $j = 1, 2$, have the Fourier symbols

$$(4.4) \quad \sigma_j = \mathcal{F}(S_j) = \frac{s - i\tilde{\omega}}{(s + i\tilde{\omega})(|\mathbf{k}|^2 + s\sigma Z)} M_s, \quad s \in \mathbb{C},$$

then the Schwarz algorithm (3.2) has the convergence factor

$$(4.5) \quad \rho_2(|\mathbf{k}|, \tilde{\omega}, \sigma, Z, L, s) = \left| \left(\frac{\sqrt{|\mathbf{k}|^2 - \tilde{\omega}^2 + i\tilde{\omega}\sigma Z - s}}{\sqrt{|\mathbf{k}|^2 - \tilde{\omega}^2 + i\tilde{\omega}\sigma Z + s}} \right) e^{-\sqrt{|\mathbf{k}|^2 - \tilde{\omega}^2 + i\tilde{\omega}\sigma Z} L} \right|.$$

Algorithm 3. If S_j , $j = 1, 2$, have the Fourier symbols

$$(4.6) \quad \sigma_j = \mathcal{F}(S_j) = \frac{1}{|\mathbf{k}|^2 - 2\tilde{\omega}^2 + 2i\tilde{\omega}\sigma Z + (2i\tilde{\omega} + \sigma Z)s} M_s, \quad s \in \mathbb{C},$$

then (3.2) has the convergence factor

$$(4.7) \quad \rho_3(|\mathbf{k}|, \tilde{\omega}, \sigma, Z, L, s) = \left| \left(\frac{\sqrt{|\mathbf{k}|^2 - \tilde{\omega}^2 + i\tilde{\omega}\sigma Z - i\tilde{\omega}}}{\sqrt{|\mathbf{k}|^2 - \tilde{\omega}^2 + i\tilde{\omega}\sigma Z - i\tilde{\omega}}} \right) \right| \rho_2(|\mathbf{k}|, \tilde{\omega}, \sigma, Z, L, s).$$

Algorithm 4. If S_j , $j = 1, 2$, have the Fourier symbols

$$(4.8) \quad \sigma_j = \mathcal{F}(S_j) = \frac{s_j - i\tilde{\omega}}{(s_j + i\tilde{\omega})(|\mathbf{k}|^2 + s_j\sigma Z)} M_{s_j}, \quad s_j \in \mathbb{C},$$

then the Schwarz algorithm (3.2) has the convergence factor

$$(4.9) \quad \rho_4(|\mathbf{k}|, \tilde{\omega}, \sigma, Z, L, s_1, s_2) = \left| \prod_{l=1}^2 \frac{\sqrt{|\mathbf{k}|^2 - \tilde{\omega}^2 + i\tilde{\omega}\sigma Z - s_l}}{\sqrt{|\mathbf{k}|^2 - \tilde{\omega}^2 + i\tilde{\omega}\sigma Z + s_l}} e^{-2\sqrt{|\mathbf{k}|^2 - \tilde{\omega}^2 + i\tilde{\omega}\sigma Z} L} \right|^{\frac{1}{2}}.$$

Algorithm 5. If S_j , $j = 1, 2$, have the Fourier symbols

$$(4.10) \quad \sigma_j = \mathcal{F}(S_j) = \frac{1}{|\mathbf{k}|^2 - 2\tilde{\omega}^2 + 2i\tilde{\omega}\sigma Z + (2i\tilde{\omega} + \sigma Z)s_j} M_{s_j}, \quad s_j \in \mathbb{C},$$

then the Schwarz algorithm (3.2) has the convergence factor

$$(4.11) \quad \rho_5(|\mathbf{k}|, \tilde{\omega}, \sigma, Z, L, s_1, s_2) = \left| \left(\frac{\sqrt{|\mathbf{k}|^2 - \tilde{\omega}^2 + i\tilde{\omega}\sigma Z - i\tilde{\omega}}}{\sqrt{|\mathbf{k}|^2 - \tilde{\omega}^2 + i\tilde{\omega}\sigma Z + i\tilde{\omega}}} \right) \right| \rho_4(|\mathbf{k}|, \tilde{\omega}, \sigma, Z, L, s_1, s_2).$$

Proof. The convergence factor for Algorithm 1 was already computed in [9]. The interesting cases for the computation are Algorithms 4 and 5 (since Algorithms 2 and 3 are particular cases of the former, when the parameters are identical for both subdomains). In what follows we will illustrate in detail the proof for Algorithm 4, the treatment of the remaining cases being very similar. In order to be self-consistent

we will recall here the elements of the technique based on Fourier transform already found in [9].

Because of the linearity of the equations, it suffices to analyze the convergence to the zero solution when the right-hand side vanishes. Performing a Fourier transform of system (2.2) in the y and z directions, the first and the fourth equation provide an algebraic expression for $\hat{E}_1$ and $\hat{H}_1$, which is in agreement with the fact that these are the characteristic variables associated with the null eigenvalue. Inserting these expressions into the remaining Fourier transformed equations, we obtain the first order system

$$(4.12) \quad \partial_x \begin{pmatrix} \hat{E}_2 \\ \hat{E}_3 \\ \hat{H}_2 \\ \hat{H}_3 \end{pmatrix} + \begin{bmatrix} 0 & 0 & -\frac{k_y k_z}{i\tilde{\omega}/Z + \sigma} & \frac{-\tilde{\omega}^2 + k_y^2 + i\tilde{\omega}\sigma Z}{i\tilde{\omega}/Z + \sigma} \\ 0 & 0 & \frac{\tilde{\omega}^2 - k_z^2 - i\tilde{\omega}\sigma Z}{i\tilde{\omega}/Z + \sigma} & \frac{k_y k_z}{i\tilde{\omega}/Z + \sigma} \\ \frac{k_y k_z}{i\tilde{\omega}Z} & \frac{\tilde{\omega}^2 - k_y^2 - i\tilde{\omega}\sigma Z}{i\tilde{\omega}Z} & 0 & 0 \\ \frac{-\tilde{\omega}^2 + k_z^2 + i\tilde{\omega}\sigma Z}{i\tilde{\omega}Z} & -\frac{k_y k_z}{i\tilde{\omega}Z} & 0 & 0 \end{bmatrix} \begin{pmatrix} \hat{E}_2 \\ \hat{E}_3 \\ \hat{H}_2 \\ \hat{H}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The eigenvalues of the matrix in (4.12) and their corresponding eigenvectors are

$$(4.13) \quad \lambda_{1,2}^{TH} = -\sqrt{|\mathbf{k}|^2 - \tilde{\omega}^2 + i\tilde{\omega}\sigma Z}, \quad \mathbf{v}_1 = \begin{pmatrix} \frac{k_y k_z}{(i\tilde{\omega}/Z + \sigma)\lambda} \\ \frac{-\tilde{\omega}^2 + k_z^2 + i\tilde{\omega}\sigma Z}{(i\tilde{\omega}/Z + \sigma)\lambda} \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} \frac{\tilde{\omega}^2 - k_y^2 - i\tilde{\omega}\sigma Z}{(i\tilde{\omega}/Z + \sigma)\lambda} \\ -\frac{k_y k_z}{(i\tilde{\omega}/Z + \sigma)\lambda} \\ 0 \\ 1 \end{pmatrix},$$

and

$$(4.14) \quad \lambda_{3,4}^{TH} = \sqrt{|\mathbf{k}|^2 - \tilde{\omega}^2 + i\tilde{\omega}\sigma Z}, \quad \mathbf{v}_3 = \begin{pmatrix} -\frac{k_y k_z}{(i\tilde{\omega}/Z + \sigma)\lambda} \\ \frac{\tilde{\omega}^2 - k_z^2 - i\tilde{\omega}\sigma Z}{(i\tilde{\omega}/Z + \sigma)\lambda} \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} \frac{k_y^2 - \tilde{\omega}^2 + i\tilde{\omega}\sigma Z}{(i\tilde{\omega}/Z + \sigma)\lambda} \\ \frac{k_y k_z}{(i\tilde{\omega}/Z + \sigma)\lambda} \\ 0 \\ 1 \end{pmatrix},$$

where we set $\lambda := \sqrt{|\mathbf{k}|^2 - \tilde{\omega}^2 + i\tilde{\omega}\sigma Z}$. Because of the radiation condition, the solutions of system (4.12) in Ω_l , $l = 1, 2$, are given by

$$(4.15) \quad \begin{aligned} (\hat{E}_2^1; \hat{E}_3^1; \hat{H}_2^1; \hat{H}_3^1) &= (\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2) e^{\lambda(x-L)}, \\ (\hat{E}_2^2; \hat{E}_3^2; \hat{H}_2^2; \hat{H}_3^2) &= (\beta_1 \mathbf{v}_3 + \beta_2 \mathbf{v}_4) e^{-\lambda x}, \end{aligned}$$

where the coefficients α_j and β_j ($j = 1, 2$) are uniquely determined by the transmission conditions. At the n th step of the optimized Schwarz algorithm, the coefficients $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ in (4.15) satisfy

$$(4.16) \quad \alpha^n = \bar{A}_1^{-1} \bar{A}_2 e^{-\lambda L} \beta^{n-1}, \quad \beta^n = \bar{B}_1^{-1} \bar{B}_2 e^{-\lambda L} \alpha^{n-1},$$

where the matrices $\bar{A}_l$ and $\bar{B}_l$, $l = 1, 2$, are given by

$$\bar{A}_1 = A_1 + \sigma_1 A_2, \quad \bar{A}_2 = A_2 + \sigma_1 A_1, \quad \bar{B}_1 = A_1 + \sigma_2 A_2, \quad \bar{B}_2 = A_2 + \sigma_2 A_1,$$

with A_l , $l = 1, 2$, defined as

$$\begin{aligned}
(4.17) \quad A_1 &= \begin{bmatrix} -k_y k_z & k_y^2 - \tilde{\omega}^2 + i\tilde{\omega}\lambda + \sigma Z(\lambda + i\tilde{\omega}) \\ k_z^2 - \tilde{\omega}^2 + i\tilde{\omega}\lambda + \sigma Z(\lambda + i\tilde{\omega}) & -k_y k_z \end{bmatrix}, \\
A_2 &= \begin{bmatrix} k_y k_z & -k_y^2 + \tilde{\omega}^2 + i\tilde{\omega}\lambda + \sigma Z(\lambda - i\tilde{\omega}) \\ -k_z^2 + \tilde{\omega}^2 + i\tilde{\omega}\lambda + \sigma Z(\lambda - i\tilde{\omega}) & k_y k_z \end{bmatrix}.
\end{aligned}$$

A complete double iteration of the Schwarz algorithm therefore leads to

$$\alpha^{n+1} = \bar{A}_1^{-1} \bar{A}_2 \bar{B}_1^{-1} \bar{B}_2 e^{-2\lambda L} \alpha^{n-1}, \quad \beta^{n+1} = \bar{B}_1^{-1} \bar{B}_2 \bar{A}_1^{-1} \bar{A}_2 e^{-2\lambda L} \beta^{n-1}.$$

We first replace σ_j from (4.8) into (4.16), and we compute the eigenvalues of the iteration matrices $\bar{A}_1^{-1} \bar{A}_2 \bar{B}_1^{-1} \bar{B}_2$ or equivalently $\bar{B}_1^{-1} \bar{B}_2 \bar{A}_1^{-1} \bar{A}_2$:

$$\begin{aligned}
(4.18) \quad \lambda_1 &= -\frac{\lambda - s_1}{\lambda + s_1} \cdot \frac{\lambda - s_2}{\lambda + s_2}, \\
\lambda_2 &= \lambda_1 \cdot \frac{\lambda s_1 \sigma Z - i\tilde{\omega}|\mathbf{k}|^2}{\lambda s_1 \sigma Z + i\tilde{\omega}|\mathbf{k}|^2} \cdot \frac{\lambda s_2 \sigma Z - i\tilde{\omega}|\mathbf{k}|^2}{\lambda s_2 \sigma Z + i\tilde{\omega}|\mathbf{k}|^2}.
\end{aligned}$$

The convergence factor of the algorithm will be given by

$$\rho_4(|\mathbf{k}|, \tilde{\omega}, \sigma, Z, L, s_1, s_2) = \sqrt{\max\{|\lambda_1|, |\lambda_2|\}} e^{-\lambda L}.$$

We can see that as soon as the imaginary part of λs_j is positive, λ_1 is bigger in modulus than λ_2 . This is usually the case, since a simple computation shows that $\Re \lambda, \Im \lambda > 0$ and the optimized parameter is chosen as $s_j = p_j(1 + i)$, $p_j > 0$ (see [9] and the references therein for details). Therefore the convergence factor is

$$\rho_4(|\mathbf{k}|, \tilde{\omega}, \sigma, Z, L, s_1, s_2) = \sqrt{|\lambda_1|} = \left| \frac{\lambda - s_1}{\lambda + s_1} \cdot \frac{\lambda - s_2}{\lambda + s_2} e^{-2\lambda L} \right|^{1/2},$$

and this ends the proof. $\square$

The different symbols σ_j depend on the choice of the parameters s , s_1 , and s_2 . In order to obtain an efficient algorithm, we will choose σ_j , $j = 1, 2$, such that ρ_l , $l = 2, \dots, 5$, is minimum over a range of frequencies. Therefore the parameters are solutions of the min-max problems

$$\begin{aligned}
(4.19) \quad & \min_{s \in \mathbb{C}} \max_{|\mathbf{k}| \in K} \rho_j(|\mathbf{k}|, \tilde{\omega}, \sigma, Z, L, s), \quad j = 2, 3, \\
& \min_{s_1, s_2 \in \mathbb{C}} \max_{|\mathbf{k}| \in K} \rho_j(|\mathbf{k}|, \tilde{\omega}, \sigma, Z, L, s_1, s_2), \quad j = 4, 5,
\end{aligned}$$

where K denotes the set of relevant numerical frequencies. In the next section, we will analyze these min-max problems for each of the algorithms in Corollary 4.1.

5. Optimized transmission conditions. In this section, we solve the various min-max problems seen in (4.19). The fundamental difference with the case $\sigma = 0$ is that here we do not need to exclude the resonance frequencies (see [9]). This changes the nature of the min-max problems. We have more local maxima in $|\mathbf{k}|$, and balancing these maxima by the equioscillation principle is more difficult. In a numerical implementation, the range of frequencies is bounded, i.e., $|\mathbf{k}| \in K := [k_{\min}, k_{\max}]$, where the minimum frequency $k_{\min} > 0$ is a constant depending on the

geometry, and the maximum numerical frequency that can be represented on a mesh is $k_{\max} = \frac{C}{h}$, where C is a constant and h is the mesh size.

Before solving the min-max problems in (4.19), we give an asymptotic expression for the maximum of the convergence factor (4.3) of the classical Schwarz algorithm over $|\mathbf{k}| \in K$. This allows us to see the behavior of algorithm (3.1) when h goes to zero.

COROLLARY 5.1. *The asymptotic convergence factor of the classical Schwarz method (Algorithm 1) for small mesh size h is*

$$(5.1) \quad \max_{|\mathbf{k}| \in K} \rho_1(|\mathbf{k}|, \tilde{\omega}, \sigma, Z, L) = \begin{cases} 1 - \frac{4}{3} (9\omega^4 \sigma^2 \mu^3 \varepsilon C_L^6)^{\frac{1}{8}} h^{\frac{3}{4}} + O(h^{\frac{5}{4}}), & L = C_L h, \\ 1 - \frac{\omega^2 \sigma \sqrt{\mu^3 \varepsilon}}{C^3} h^3 + O(h^5), & L = 0. \end{cases}$$

Proof. The proof is obtained by expanding the maximum of ρ_1 over $|\mathbf{k}| \in K$ for h small. $\square$

5.1. Optimization of Algorithm 2. We look for s under the form $s = p + iq$, with $(p, q) \in \mathbb{R}_+^2$ and $p = q$, such that p is the solution of the min-max problem

$$(5.2) \quad \min_{p \geq 0} \left(\max_{|\mathbf{k}| \in K} \rho_2(|\mathbf{k}|, \tilde{\omega}, \sigma, Z, L, p(1+i)) \right).$$

We choose $p = q$ in order to simplify the computations, this choice being justified for the Helmholtz equation case in [15]. The case where $p \neq q$ is discussed in [12].

The non-overlapping case. Using the change of variables

$$\xi(|\mathbf{k}|) := \Re \left(\sqrt{|\mathbf{k}|^2 - \tilde{\omega}^2 + i\sigma\tilde{\omega}Z} \right), \quad y := \sigma\tilde{\omega}Z,$$

the convergence factor simplifies to

$$\rho_2(\tilde{\omega}, Z, \sigma, 0, |\mathbf{k}|, p(1+i)) = \sqrt{\frac{4\xi^2(\xi-p)^2 + (y-2\xi p)^2}{4\xi^2(\xi+p)^2 + (y+2\xi p)^2}} =: R(\xi, y, p).$$

Since the mapping $|\mathbf{k}| \mapsto \xi(|\mathbf{k}|)$ is increasing, the optimization problem (5.2) can be rewritten as

$$\min_{p \geq 0} \left(\max_{k_{\min} \leq |\mathbf{k}| \leq k_{\max}} \rho_2(\tilde{\omega}, Z, \sigma, 0, |\mathbf{k}|, p(1+i)) \right) = \min_{p \geq 0} \left(\max_{\xi_{\min} \leq \xi \leq \xi_{\max}} R(\xi, y, p) \right),$$

where $\xi_{\min} = \xi(k_{\min})$ and $\xi_{\max} = \xi(k_{\max})$. We can prove the following result.

THEOREM 5.2. *In the nonoverlapping case, the solution of the problem*

$$\min_{p \geq 0} \left(\max_{\xi_{\min} \leq \xi \leq \xi_{\max}} R(\xi, y, p) \right)$$

is given by

$$(5.3) \quad p^* = \begin{cases} \min(p_2, p_{23}) & \text{if } y < 2\xi_{\max} \xi_{\min}, \\ \min(p_1, p_{12}) & \text{if } y \geq 2\xi_{\max} \xi_{\min}, \end{cases}$$

where p_1 , p_2 , p_{12} , and p_{23} are constants defined by

$$p_{12} = \frac{y^{\frac{1}{4}} \sqrt{\xi_{\min} (y + 2\xi_{\min}^2 + \xi_{\min} \sqrt{2y})}}{2^{\frac{3}{4}} \xi_{\min}}, \quad p_{23} = \frac{y^{\frac{1}{4}} \sqrt{\xi_{\max} (y + 2\xi_{\max}^2 + \xi_{\max} \sqrt{2y})}}{2^{\frac{3}{4}} \xi_{\max}},$$

$$p_1 = \frac{\sqrt{8\xi_{\min}^4 + 2y^2}}{4\xi_{\min}}, \quad p_2 = \frac{\sqrt{8\xi_{\max}^4 + 2y^2}}{4\xi_{\max}}.$$

Proof. We start by computing the partial derivative of R with respect to p in order to restrict the range of p in the min-max problem,

$$(5.4) \quad E_1 := \frac{\partial R}{\partial p}(\xi, y, p) = \frac{1}{R(\xi, y, p)} \frac{(32\xi^3 y + 64\xi^5) \left(p^2 - \frac{1}{8} \frac{4\xi^4 + y^2}{\xi^2}\right)}{(4\xi^4 + 8\xi^3 p + 8\xi^2 p^2 + y^2 + 4y\xi p)^2}.$$

E_1 is negative for $p \in [0, \frac{\sqrt{8\xi^4 + 2y^2}}{4\xi}]$ and positive for $p \in [\frac{\sqrt{8\xi^4 + 2y^2}}{4\xi}, +\infty)$. Moreover, the minimum of the mapping $\xi \mapsto \frac{\sqrt{8\xi^4 + 2y^2}}{4\xi}$ is $\sqrt{\frac{y}{2}}$. This implies that for $p \leq \sqrt{\frac{y}{2}}$, R decreases with p for all $\xi \in [\xi_{\min}, +\infty)$. Hence we can restrict the range of p to the interval $[\sqrt{\frac{y}{2}}, +\infty)$.

We now look for the eventual extrema of $\xi \mapsto R(\xi, y, p)$ by computing the zeros of the derivative of R with respect to ξ ,

$$(5.5) \quad \frac{\partial R}{\partial \xi}(\xi, y, p) = \frac{4p(\xi^2 - \frac{1}{2}y)(8\xi^4 + 16(y - p^2)\xi^2 + 2y^2)}{R(\xi, y, p)(4\xi^4 + 8\xi^3 p + 8\xi^2 p^2 + y^2 + 4y\xi p)^2}.$$

The polynomial $P(\xi) = (\xi^2 - \frac{1}{2}y)(8\xi^4 + 16(y - p^2)\xi^2 + 2y^2)$ has at most three positive roots, and $\xi_2 = \sqrt{\frac{y}{2}}$ is always a root. The other roots are given by

$$\xi_{1,3} = \frac{1}{2} \sqrt{-4y + 4p^2 \mp 2\sqrt{(3y - 2p^2)(y - 2p^2)}}.$$

We now show that for $p \geq \sqrt{3y/2}$, ξ_2 is a maximum and the other two roots of P cannot be maxima. The second partial derivative of R with respect to ξ at ξ_2 is

$$\frac{\partial^2 R}{\partial \xi^2}(\xi_2, y, p) = -4 \sqrt{\frac{2}{y}} \frac{(2p^2 - 3y)p}{|\sqrt{2}p - \sqrt{y}|(\sqrt{2}p + \sqrt{y})^3},$$

which shows that ξ_2 is a local maximum if $p < \sqrt{3y/2}$. Moreover, in this case, the other two roots are real and these are minima because $\xi_1 \leq \xi_2 \leq \xi_3$. In the case $p \geq \sqrt{3y/2}$, ξ_2 is a local minimum and the other two roots are not real. Therefore the maximum of R is either at $\xi_{\min}$, ξ_2 , or $\xi_{\max}$.

From the expression (5.4) we can derive the following properties of R :

- (a) The mapping $p \mapsto R(\xi_{\min}, y, p)$ is decreasing on $[\sqrt{\frac{y}{2}}, p_1]$ and increasing on $[p_1, +\infty)$.
- (b) The mapping $p \mapsto R(\xi_2, y, p)$ is increasing on $[\sqrt{\frac{y}{2}}, +\infty)$.
- (c) The mapping $p \mapsto R(\xi_{\max}, y, p)$ is decreasing on $[\sqrt{\frac{y}{2}}, p_2]$ and increasing on $[p_2, +\infty)$.

To conclude the proof of the theorem, we need the following lemmas.

LEMMA 5.3. Let p_{13} be the constant defined by

$$p_{13} = \frac{\sqrt{2\xi_{\min}\xi_{\max}(y^2 + 2y\xi_{\min}^2 + 4y\xi_{\min}\xi_{\max} + 2y\xi_{\max}^2 + 4\xi_{\min}^2\xi_{\max}^2)}}{4\xi_{\min}\xi_{\max}}.$$

Then the following properties are verified:

- (i) p_{12} is the unique solution of the equation $R(\xi_{\min}, y, p) = R(\xi_2, y, p)$ on $\mathbb{R}_+ \setminus \{0\}$.
- (ii) p_{13} is the unique solution of the equation $R(\xi_{\min}, y, p) = R(\xi_{\max}, y, p)$ on $\mathbb{R}_+ \setminus \{0\}$.
- (iii) p_{23} is the unique solution of the equation $R(\xi_2, y, p) = R(\xi_{\max}, y, p)$ on $\mathbb{R}_+ \setminus \{0\}$.
- (iv) We have

$$\max(p_1, p_2) = \begin{cases} p_2 & \text{if } y < 2\xi_{\max}\xi_{\min}, \\ p_1 & \text{if } y \geq 2\xi_{\max}\xi_{\min}. \end{cases}$$

- (v) We have

$$\max(p_{12}, p_{23}) = \begin{cases} p_{23} & \text{if } y < 2\xi_{\max}\xi_{\min}, \\ p_{12} & \text{if } y \geq 2\xi_{\max}\xi_{\min}. \end{cases}$$

- (vi) We have $\sqrt{\frac{3y}{2}} \leq p_{12}, p_{23} \leq p_{13}$.

- (vii) We have

$$\begin{aligned} & \max_{\xi_{\min} \leq \xi \leq \xi_{\max}} R(\xi, y, p) \\ &= \begin{cases} R(\xi_{\max}, y, p) & \text{if } y < 2\xi_{\max}\xi_{\min} \text{ and } p \in [\sqrt{\frac{y}{2}}, p_{23}], \\ R(\xi_2, y, p) & \text{if } y < 2\xi_{\max}\xi_{\min} \text{ and } p \in [p_{23}, +\infty), \\ R(\xi_{\min}, y, p) & \text{if } y \geq 2\xi_{\max}\xi_{\min} \text{ and } p \in [\sqrt{\frac{y}{2}}, p_{12}], \\ R(\xi_2, y, p) & \text{if } y \geq 2\xi_{\max}\xi_{\min} \text{ and } p \in [p_{12}, +\infty). \end{cases} \end{aligned}$$

Proof. To prove the first three properties, we compute

$$\begin{aligned} (5.6) \quad & R(\xi_b, y, p)^2 - R(\xi_m, y, p)^2 \\ &= \frac{8p(\xi_m - \xi_b)(2\xi_m\xi_b - y)(8\xi_b\xi_m p^2 - 4\xi_b^2\xi_m^2 - 2\xi_b^2y - 4y\xi_m\xi_b - 2y\xi_m^2 - y^2)}{(4\xi_b^2(\xi_b + p)^2 + (y + 2\xi_b p)^2)(4\xi_m^2(\xi_m + p)^2 + (y + 2\xi_m p)^2)}. \end{aligned}$$

This quantity is zero if

$$p = \frac{\sqrt{2\xi_b\xi_m(y^2 + 2y\xi_b^2 + 4y\xi_b\xi_m + 2y\xi_m^2 + 4\xi_b^2\xi_m^2)}}{4\xi_b\xi_m}.$$

By successively replacing the pairs $\{\xi_b, \xi_m\}$ by $\{\xi_{\min}, \xi_2\}$, $\{\xi_{\min}, \xi_{\max}\}$, and $\{\xi_2, \xi_{\max}\}$ in (5.6) we get the desired result.

To prove property (iv), we compute

$$p_1^2 - p_2^2 = \frac{(\xi_{\max} - \xi_{\min})(\xi_{\min} + \xi_{\max})(y - 2\xi_{\max}\xi_{\min})(y + 2\xi_{\max}\xi_{\min})}{8\xi_{\max}^2\xi_{\min}^2},$$

which shows that $p_1 \leq p_2$ if $y < 2\xi_{\max}\xi_{\min}$ and $p_1 \geq p_2$ if $y \geq 2\xi_{\max}\xi_{\min}$.

To prove property (v), we compute

$$p_{12}^2 - p_{23}^2 = -\frac{\sqrt{y}\sqrt{2}(\xi_{\max} - \xi_{\min})(2\xi_{\max}\xi_{\min} - y)}{4\xi_{\max}\xi_{\min}}.$$

We can see that $p_{12} \leq p_{23}$ if $y < 2\xi_{\max}\xi_{\min}$ and $p_{12} \geq p_{23}$ if $y \geq 2\xi_{\max}\xi_{\min}$.

We now show (vi) by computing

$$\begin{aligned} p_{12}^2 - p_{13}^2 &= -\frac{(2y + 4\xi_{\min}^2)(\xi_{\max} - 1/2\sqrt{y}\sqrt{2})^2}{8\xi_{\max}\xi_{\min}} \leq 0, \\ p_{23}^2 - p_{13}^2 &= -\frac{(4\xi_{\max}^2 + 2y)(\xi_{\min} - 1/2\sqrt{y}\sqrt{2})^2}{8\xi_{\max}\xi_{\min}} \leq 0, \\ p_{12}^2 - \frac{3y}{2} &= \frac{\sqrt{y}\sqrt{2}(\xi_{\min} - 1/2\sqrt{y}\sqrt{2})^2}{2\xi_{\min}} \geq 0, \\ p_{23}^2 - \frac{3y}{2} &= \frac{\sqrt{y}\sqrt{2}(\xi_{\max} - 1/2\sqrt{y}\sqrt{2})^2}{2\xi_{\max}} \geq 0. \end{aligned}$$

This shows that $p_{i2}^2 - \frac{3y}{2} \geq 0$ and $p_{i2}^2 - p_{i3}^2 \leq 0$, $i = 1, 2$.

Finally we treat the last point. After some computations we obtain

$$\begin{aligned} E_2 &:= R(\xi_{\min}, y, p)^2 - R(\xi_{\max}, y, p)^2 \\ &= \frac{64p\xi_{\min}\xi_{\max}(\xi_{\max} - \xi_{\min})(2\xi_{\max}\xi_{\min} - y)(p^2 - p_{13}^2)}{\left(4\xi_{\min}^2(\xi_{\min} + p)^2 + (y + 2\xi_{\min}p)^2\right)\left(4\xi_{\max}^2(\xi_{\max} + p)^2 + (y + 2\xi_{\max}p)^2\right)}, \\ E_3 &:= R(\xi_{\max}, y, p)^2 - R(\xi_2, y, p)^2 \\ &= -\frac{32p\xi_{\max}(\xi_{\max} - 1/2\sqrt{y}\sqrt{2})^2(p^2 - p_{23}^2)}{\left(4\xi^2(\xi + p)^2 + (y + 2\xi p)^2\right)(y + 2\sqrt{y}\sqrt{2}p + 2p^2)}, \\ E_4 &:= R(\xi_{\min}, y, p)^2 - R(\xi_2, y, p)^2 \\ &= -\frac{32p\xi_{\min}(\xi_{\min} - 1/2\sqrt{y}\sqrt{2})^2(p^2 - p_{12}^2)}{\left(4\xi^2(\xi + p)^2 + (y + 2\xi p)^2\right)(y + 2\sqrt{y}\sqrt{2}p + 2p^2)}. \end{aligned}$$

From the expression above we can easily see that the following hold:

- If $y < 2\xi_{\max}\xi_{\min}$ and $p \in [\sqrt{\frac{y}{2}}, p_{23}]$, then $E_2 \leq 0$ because $p_{13} \geq p_{23}$ (see property (vi)) and $E_3 \geq 0$.
- If $y < 2\xi_{\max}\xi_{\min}$ and $p \in [p_{23}, +\infty)$, then $E_4 \leq 0$ since $p_{12} \leq p_{23}$ (see property (vi)) and $E_3 \leq 0$.
- If $y \geq 2\xi_{\max}\xi_{\min}$ and $p \in [\sqrt{\frac{y}{2}}, p_{12}]$, then $E_2 \geq 0$ and $E_4 \geq 0$.
- If $y \geq 2\xi_{\max}\xi_{\min}$ and $p \in [p_{12}, +\infty)$, then $E_3 \leq 0$ since $p_{12} \leq p_{23}$ (see property (vi)) and $E_4 \leq 0$. $\square$

Lemma 5.3 proves that the solution of the min-max problem (5.2) depends mainly on the sign of $y - 2\xi_{\max}\xi_{\min}$. We can see in Figure 5.1 the solution of the min-max problem for different values of y , $\xi_{\min}$, and $\xi_{\max}$. The pictures (a), (b), (c), and (d) show the four cases of Theorem 5.2 ($y < 2\xi_{\max}\xi_{\min}$, $p_2 \geq p_{23}$), ($y \geq 2\xi_{\max}\xi_{\min}$, $p_1 \geq p_{12}$), ($y < 2\xi_{\max}\xi_{\min}$, $p_2 \leq p_{23}$), and ($y \geq 2\xi_{\max}\xi_{\min}$, $p_1 \leq p_{12}$).

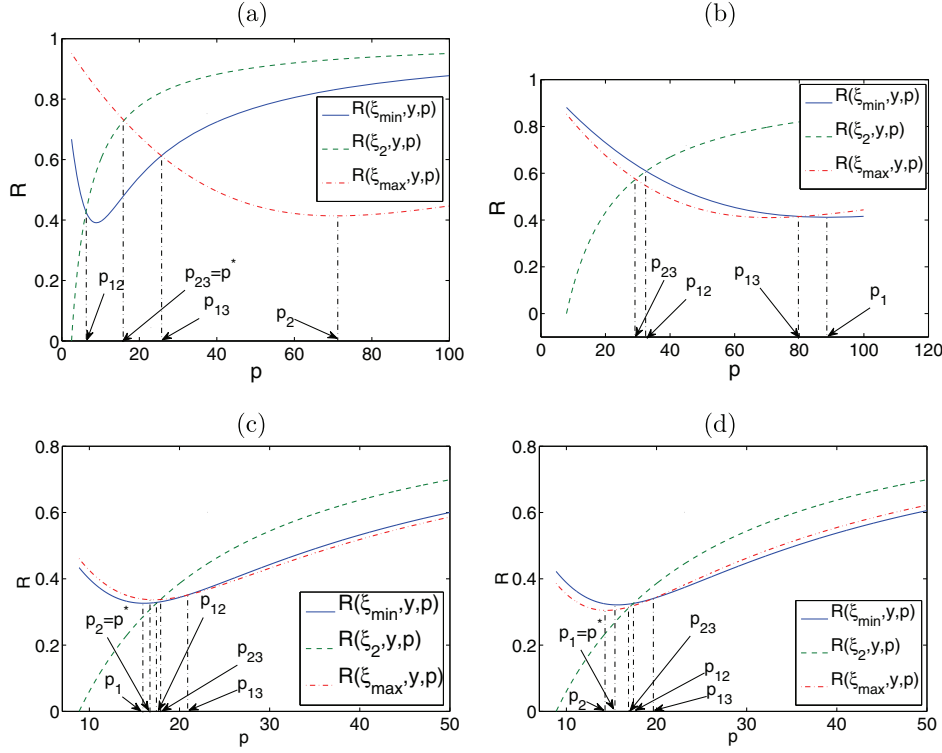


FIG. 5.1. Solutions of the min-max problem for different values of y , $\xi_{\min}$, and $\xi_{\max}$. (a) $y = 12.5$, $\xi_{\min} = 0.5$, $\xi_{\max} = 100$. (b) $y = 125.5$, $\xi_{\min} = 0.5$, $\xi_{\max} = 100$. (c) $y = 157$, $\xi_{\min} = 3.5$, $\xi_{\max} = 24$. (d) $y = 157$, $\xi_{\min} = 3.6$, $\xi_{\max} = 20$.

LEMMA 5.4. If $y < 2\xi_{\max}\xi_{\min}$, the solution of the min-max problem

$$\min_{\sqrt{\frac{y}{2}} \leq p \leq p_{23}} \left(\max_{\xi_{\min} \leq \xi \leq \xi_{\max}} R(\xi, y, p) \right)$$

is $p^* = \min(p_2, p_{23})$, and the solution of the min-max problem

$$\min_{p \geq p_{23}} \left(\max_{\xi_{\min} \leq \xi \leq \xi_{\max}} R(\xi, y, p) \right)$$

is $p^* = p_{23}$.

Proof. By using property (vii) of Lemma 5.3, for $p \in [\sqrt{\frac{y}{2}}, p_{23}]$, we have

$$R(\xi_{\max}, y, p) = \max_{\xi_{\min} \leq \xi \leq \xi_{\max}} R(\xi, y, p).$$

Hence the solution of the first min-max problem is the minimum of the mapping $p \mapsto R(\xi_{\max}, y, p)$ on $[\sqrt{\frac{y}{2}}, p_{23}]$. By property (c) in the proof of Theorem 5.2, the minimum of $R(\xi_{\max}, y, \cdot)$ on $[\sqrt{\frac{y}{2}}, p_{23}]$ is $\min(p_2, p_{23})$.

By using property (vii) of Lemma 5.3, for $p \in [p_{23}, +\infty)$, we have

$$R(\xi_2, y, p) = \max_{\xi_{\min} \leq \xi \leq \xi_{\max}} R(\xi, y, p).$$

Therefore the solution of the second min-max problem is the minimum of the mapping $R(\xi_2, y, \cdot)$ on $[p_{23}, +\infty)$. By property (b) in the proof of Theorem 5.2, the minimum of $R(\xi_2, y, \cdot)$ on $[p_{23}, +\infty)$ is p_{23} . $\square$

LEMMA 5.5. *If $y \geq 2\xi_{\max}\xi_{\min}$, the solution of the min-max problem*

$$\min_{\sqrt{\frac{y}{2}} \leq p \leq p_{12}} \left(\max_{\xi_{\min} \leq \xi \leq \xi_{\max}} R(\xi, y, p) \right)$$

is $p^ = \min(p_1, p_{12})$, and the solution of the min-max problem*

$$\min_{p \geq p_{12}} \left(\max_{\xi_{\min} \leq \xi \leq \xi_{\max}} R(\xi, y, p) \right)$$

is $p^ = p_{12}$.*

Proof. By using property (vii) of Lemma 5.3, for $p \in [\sqrt{\frac{y}{2}}, p_{12}]$, we have

$$R(\xi_{\min}, y, p) = \max_{\xi_{\min} \leq \xi \leq \xi_{\max}} R(\xi, y, p).$$

The solution of the first min-max problem is the minimum of the mapping $p \mapsto R(\xi_{\min}, y, p)$ on $[\sqrt{\frac{y}{2}}, p_{12}]$. Using now (a) in the proof of Theorem 5.2, the minimum of $R(\xi_{\min}, y, \cdot)$ on $[\sqrt{\frac{y}{2}}, p_{12}]$ is $\min(p_1, p_{12})$.

By proposition (vii) of Lemma 5.3, for $p \in [p_{12}, +\infty)$, we have

$$R(\xi_2, y, p) = \max_{\xi_{\min} \leq \xi \leq \xi_{\max}} R(\xi, y, p).$$

Therefore the solution of the second min-max problem is the minimum of the mapping $p \mapsto R(\xi_2, y, p)$ on $[p_{12}, +\infty)$. By property (a), the minimum of $R(\xi_2, y, \cdot)$ on $[p_{23}, +\infty)$ is p_{12} . $\square$

We can now return to the end of the theorem. The min-max problem (5.2) can be rewritten in the equivalent form

$$\begin{aligned} \min_{p \geq \sqrt{\frac{y}{2}}} \left(\max_{\xi_{\min} \leq \xi \leq \xi_{\max}} R(\xi, y, p) \right) &= \min \left(\min_{\sqrt{\frac{y}{2}} \leq p \leq p_{23}} \left(\max_{\xi_{\min} \leq \xi \leq \xi_{\max}} R(\xi, y, p) \right), \right. \\ &\quad \left. \min_{p \geq p_{23}} \left(\max_{\xi_{\min} \leq \xi \leq \xi_{\max}} R(\xi, y, p) \right) \right), \end{aligned}$$

which is useful for the case $y < 2\xi_{\max}\xi_{\min}$, because we then get, using Lemma 5.4,

$$\min_{p \geq \sqrt{\frac{y}{2}}} \left(\max_{\xi_{\min} \leq \xi \leq \xi_{\max}} R(\xi, y, p) \right) = R(\xi_{\max}, y, \min(p_2, p_{23})).$$

The min-max problem (5.2) can also be rewritten as

$$\min_{p \geq \sqrt{\frac{y}{2}}} \left(\max_{\xi_{\min} \leq \xi \leq \xi_{\max}} R(\xi, y, p) \right) = \min \left(\min_{\sqrt{\frac{y}{2}} \leq p \leq p_{12}} \left(\max_{\xi_{\min} \leq \xi \leq \xi_{\max}} R(\xi, y, p) \right), \right. \\ \left. \min_{p \geq p_{12}} \left(\max_{\xi_{\min} \leq \xi \leq \xi_{\max}} R(\xi, y, p) \right) \right),$$

which is now useful for the case $y \geq 2\xi_{\max}\xi_{\min}$, because we then get, using Lemma 5.5,

$$\min_{p \geq \sqrt{\frac{y}{2}}} \left(\max_{\xi_{\min} \leq \xi \leq \xi_{\max}} R(\xi, y, p) \right) = R(\xi_{\max}, y, \min(p_1, p_{12})). \quad \square$$

COROLLARY 5.6. *For h sufficiently small in the nonoverlapping case, $L = 0$, the solution of the min-max problem (5.2) is given by*

$$(5.7) \quad p^* = \frac{(\omega\sigma\mu)^{\frac{1}{4}}\sqrt{C}}{2^{\frac{1}{4}}\sqrt{h}} \quad \text{and} \quad \rho_2^* = 1 - \frac{2^{\frac{3}{4}}(\omega\sigma\mu)^{\frac{1}{4}}\sqrt{h}}{\sqrt{C}} + O(h).$$

Proof. If we assume h sufficiently small, then $\xi_{\max} = \xi(C/h)$ is large. By this assumption, the solution of the min-max problem (5.2) is given by the first equation of (5.3), i.e.,

$$p^* = \frac{\sqrt[4]{2y}\sqrt{\xi_{\max}(y + 2\xi_{\max}^2 + \xi_{\max}\sqrt{2}\sqrt{y})}}{2\xi_{\max}}.$$

By expanding p^* for h small, using that $\xi_{\max} = \xi(C/h)$, we get the desired result. $\square$

Corollary 5.6 gives the optimal parameter p^* and the corresponding convergence factor of Algorithm 2 in terms of the mesh size h . In some practical situations, where we consider high frequencies, it is interesting to give the optimized parameters in terms of ω . For the high frequencies, it has been shown in [2] that it is necessary, to avoid the pollution effect, to couple the frequency ω with h by the relation $h = C_h/\omega^\gamma$, where $\gamma \geq 1$ depends on the discretization method. In [2], it has been shown that $\gamma = \frac{3}{2}$ for a P_1 finite element method. We have the following corollary.

COROLLARY 5.7. *Let $h = C_h/\omega^\gamma$. If $\gamma = 1$, then for ω large, the solution of the min-max problem (5.2) is given by*

$$(5.8) \quad p^* = \frac{(\sigma\mu)^{\frac{1}{4}}(C^2 - \varepsilon\mu C_h^2)^{\frac{1}{4}}}{2^{\frac{1}{4}}\sqrt{C_h}}\omega^{\frac{3}{4}} \quad \text{and} \quad \rho_2^* = 1 - \frac{2^{\frac{3}{4}}(\sigma\mu)^{\frac{1}{2}}\sqrt{C_h}}{(C^2 - \varepsilon\mu C_h^2)^{\frac{1}{4}}}\omega^{-\frac{1}{4}}.$$

If $\gamma > 1$, for ω large, the solution is

$$(5.9) \quad p^* = \frac{(\sigma\mu)^{\frac{1}{4}}\sqrt{C}}{2^{\frac{1}{4}}\sqrt{C_h}}\omega^{\frac{2\gamma+1}{4}} \quad \text{and} \quad \rho_2^* = 1 - \frac{2^{\frac{3}{4}}(\sigma\mu)^{\frac{1}{4}}\sqrt{C_h}}{\sqrt{C}}\omega^{\frac{1-2\gamma}{4}}.$$

Proof. If we assume ω sufficiently large, then $\xi_{\max} = \xi(\omega C/C_h)$ is large. By this assumption, the solution of the min-max problem (5.2) is given by the first equation of (5.3), i.e.,

$$p^* = \frac{\sqrt[4]{2y} \sqrt{\xi_{\max} (y + 2\xi_{\max}^2 + \xi_{\max} \sqrt{2y})}}{2\xi_{\max}}.$$

By expanding p^* for ω large, using that $\xi_{\max} = \xi(\omega C/C_h)$, we get the desired result. $\square$

The overlapping case. After the complete analysis of the best approximation problem without overlap in the previous paragraph, we focus in the remainder of this paper on asymptotic analysis, in order to obtain tractable best approximation problems and to be able to present a complete set of optimized transmission conditions for all algorithms. We start with the case of the previous paragraph, but now consider an overlapping method. We set $L = h$, $k_{\min} = 0$, $k_{\max} = \frac{\pi}{h}$ and denote by p^* the solution of the min-max problem (5.2). Solving this min-max problem numerically for different parameter values and different mesh sizes h , we observe that the solution of (5.2) equioscillates once, i.e., p^* is the solution of

$$(5.10) \quad \rho_2(\bar{k}_1, \tilde{\omega}, \sigma, Z, L, p^*(1+i)) = \rho_2(\bar{k}_2, \tilde{\omega}, \sigma, Z, L, p^*(1+i)),$$

where $\bar{k}_1$ and $\bar{k}_2$ are two interior local maxima of ρ_2 . For small h , we see numerically that these maxima and the optimized parameter p^* behave like

$$\bar{k}_1 = C_{b_1}, \quad \bar{k}_2 = C_{b_2} h^{-\frac{2}{3}}, \quad \text{and} \quad p^* = C_p h^{-\frac{1}{3}} \quad \text{for some constants } C_p, C_{b_1}, C_{b_2}.$$

In order to determine the constants C_{b_1} , C_{b_2} , and C_p , we now solve the corresponding equations asymptotically: since $\bar{k}_1$ and $\bar{k}_2$ are local maxima of the convergence factor, the corresponding derivatives must vanish, and we find asymptotically

$$\begin{aligned} \frac{\partial \rho_2}{\partial k}(\bar{k}_1, \tilde{\omega}, \sigma, Z, L, p^*(1+i)) &= 0 \Rightarrow C_{b_1} = \tilde{\omega}, \\ \frac{\partial \rho_2}{\partial k}(\bar{k}_2, \tilde{\omega}, \sigma, Z, L, p^*(1+i)) &= 0 \Rightarrow C_{b_2} = \sqrt{2C_p}. \end{aligned}$$

The equioscillation equation (5.10) must also be satisfied, which gives asymptotically

$$\rho_2(\bar{k}_1, \tilde{\omega}, \sigma, Z, L, p^*(1+i)) = \rho_2(\bar{k}_2, \tilde{\omega}, \sigma, Z, L, p^*(1+i)) \Rightarrow \frac{\sqrt{2}\sqrt{\tilde{\omega}\sigma Z}}{C_p} = \frac{C_{b_2}^2 + 2C_p}{C_{b_2}}.$$

Solving the three equations obtained for the constants C_{b_1} , C_{b_2} , and C_p , and expressing the results in the original problem parameters ω , ϵ , μ , and σ , we find the asymptotic result

$$(5.11) \quad p^* = \frac{(2\omega\sigma\mu)^{\frac{1}{3}}}{2h^{\frac{1}{3}}}, \quad \rho_2^* = 1 - 2^{\frac{7}{6}} (\omega\sigma\mu)^{\frac{1}{6}} h^{\frac{1}{3}}.$$

LEMMA 5.8. *Asymptotically, the parameter p^* defined in (5.11) is a local minimum of the min-max problem (5.2).*

Proof. The parameter p^* is a local minimum if there exists no variation δp such that $\rho_2(k, \tilde{\omega}, \sigma, Z, 0, (p^* + \delta p)(1+i)) \leq \rho_2(k, \tilde{\omega}, \sigma, Z, 0, p^*(1+i))$ for $k = \bar{k}_1, \bar{k}_2$. By the Taylor formula, it suffices to prove that there is no variation δp such that $\delta p \frac{\partial \rho_2}{\partial p}(k, \tilde{\omega}, \sigma, Z, 0, p^*(1+i)) < 0$ for $k = \bar{k}_1, \bar{k}_2$. To prove this, it is necessary to obtain the next higher order terms in the expansions of p^* , $\bar{k}_1$, and $\bar{k}_2$. After a lengthy computation, we find that

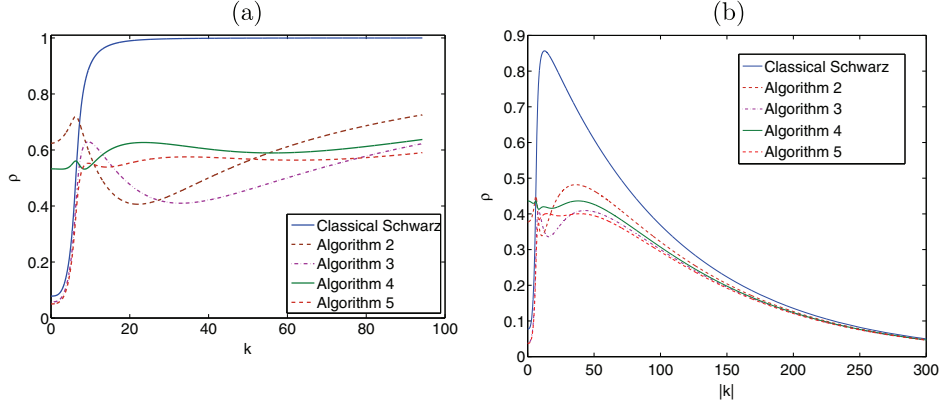


FIG. 5.2. Convergence factor comparisons of different algorithms for $\omega = 2\pi$, $\sigma = 2$, and $\mu = \varepsilon = 1$ for both nonoverlapping ($L = 0$) (left) and overlapping ($L = h = \frac{1}{30}$) (right) cases

$$\bar{k}_1 \sim \tilde{\omega} + \tilde{C}_{b_1} h^{\frac{2}{3}}, \quad \bar{k}_2 \sim C_{b_2} h^{-\frac{2}{3}} + \tilde{C}_{b_2}, \quad p^* \sim C_p h^{-\frac{1}{3}} + \tilde{C}_p h^{\frac{1}{3}}.$$

With these three new constants determined, we obtain for the derivative of ρ_3

$$\frac{\partial \rho_2}{\partial p}(k_1, \tilde{\omega}, \sigma, Z, h, p^*(1+i)) \sim 4 \frac{2^{\frac{5}{6}} h^{\frac{2}{3}}}{(\omega \sigma \mu)^{\frac{1}{6}}}, \quad \frac{\partial \rho_2}{\partial p}(\bar{k}_2, \tilde{\omega}, \sigma, Z, h, p^*(1+i)) \sim -2 \frac{2^{\frac{5}{6}} h^{\frac{2}{3}}}{(\omega \sigma \mu)^{\frac{1}{6}}}.$$

Since these leading terms differ in sign, there cannot be a variation δp which diminishes the contraction factor obtained for p^* , and we have asymptotically a local minimum. $\square$

In Figure 5.2 we can see the convergence factor in the nonoverlapping and overlapping cases for Algorithm 2 compared to the classical Algorithm 1.

5.2. Optimization of Algorithm 3. Here, we look for s of the form $s = p(1+i)$, such that p is the solution of the min-max problem

$$(5.12) \quad \min_{p \geq 0} \left(\max_{k \in K} \rho_3(k, \tilde{\omega}, \sigma, Z, L, p(1+i)) \right).$$

Again we are only searching for an asymptotic solution of this best approximation problem.

The nonoverlapping case. Setting $L = 0$, $k_{\min} = 0$, $k_{\max} = \frac{C}{h}$ and denoting by p^* the solution of (5.12), we obtain from numerical experiments that the solution of (5.12) equioscillates once, i.e., p^* is the solution of

$$(5.13) \quad \rho_3(k_{\max}, \tilde{\omega}, \sigma, Z, 0, p^*(1+i)) = \rho_3(\bar{k}, \tilde{\omega}, \sigma, Z, 0, p^*(1+i)),$$

where k_b is an interior local maximum of ρ_3 . For small h we find the asymptotic behavior $\bar{k} = C_b h^{-\frac{1}{7}}$, $p^* = C_p h^{-\frac{4}{7}}$, and to determine C_b and C_p , we solve asymptotically

$$\frac{\partial \rho_3}{\partial k}(\bar{k}, \tilde{\omega}, \sigma, Z, 0, p^*(1+i)) = 0 \Rightarrow C_p = \frac{C_b^4}{3 \tilde{\omega}^2 \sigma Z}$$

and the equioscillation equation (5.13), which leads to $C_b = (6C\tilde{\omega}^4\sigma^2Z^2)^{\frac{1}{7}}$. Solving the two equations for the two constants, and using again the original parameters ω , ϵ , μ , and σ , we find

$$(5.14) \quad p^* = \frac{1}{3} 6^{\frac{4}{7}} C^{\frac{4}{7}} \omega^{\frac{2}{7}} \mu^{\frac{3}{14}} \epsilon^{\frac{1}{14}} \sigma^{\frac{1}{7}} h^{-\frac{4}{7}}, \quad \rho_3^* = 1 - \frac{4}{3} \frac{6^{\frac{4}{7}} \omega^{\frac{2}{7}} \mu^{\frac{3}{14}} \epsilon^{\frac{1}{14}} \sigma^{\frac{1}{7}} h^{\frac{3}{7}}}{C^{\frac{3}{7}}}.$$

LEMMA 5.9. *Asymptotically, the parameter p^* defined in (5.14) is a local minimum of the min-max problem (5.12).*

Proof. The proof follows the same argument as the proof of Lemma 5.8. We obtain after a lengthy computation that

$$\bar{k} \sim C_b h^{-\frac{1}{7}} + C_{b_2} h^{\frac{1}{7}}, \quad p^* \sim C_p h^{-\frac{4}{7}} + C_{p_2} h^{-\frac{2}{7}}.$$

Determining the constants in the higher order terms as well, we can show that

$$\frac{\partial \rho_3}{\partial p}(k_{\max}, \tilde{\omega}, \sigma, Z, 0, p^*(1+i)) \sim -\frac{4h}{C}, \quad \frac{\partial \rho_3}{\partial p}(\bar{k}, \tilde{\omega}, \sigma, Z, 0, p^*(1+i)) \sim \frac{1}{4} \frac{2^{\frac{5}{7}} 3^{\frac{5}{7}} \sigma^{\frac{3}{7}} Z^{\frac{3}{7}} h^{\frac{9}{7}}}{\tilde{\omega}^{\frac{1}{7}} C^{\frac{9}{7}}},$$

which shows by the opposite sign that p^* is indeed a local minimum. $\square$

The overlapping case. Also with $L = h$ the solution of the min-max problem (5.12) equioscillates once, i.e., p^* is the solution of

$$\rho_3(\bar{k}_1, \tilde{\omega}, \sigma, Z, L, p^*(1+i)) = \rho_3(\bar{k}_2, \tilde{\omega}, \sigma, Z, L, p^*(1+i)),$$

where $\bar{k}_1$ and $\bar{k}_2$ are two interior local maxima of ρ_3 . For small h , we get

$$\bar{k}_1 = C_{b_1} h^{-\frac{1}{10}}, \quad \bar{k}_2 = C_{b_2} h^{-\frac{7}{10}}, \quad p^* = C_p h^{-\frac{2}{5}},$$

and solving the corresponding equations asymptotically, we find in the original parameters ω , ϵ , μ , and σ

$$(5.15) \quad p^* = \frac{1}{3} \frac{3^{\frac{2}{5}} \epsilon^{\frac{1}{10}} \mu^{\frac{3}{10}} \omega^{\frac{2}{5}} \sigma^{\frac{1}{5}} 2^{\frac{2}{5}}}{h^{\frac{2}{5}}}, \quad \rho_3^* = 1 - \frac{2}{3} 3^{\frac{7}{10}} 2^{\frac{7}{10}} \omega^{\frac{1}{5}} \epsilon^{\frac{1}{20}} \mu^{\frac{3}{20}} \sigma^{\frac{1}{10}} h^{\frac{3}{10}}.$$

In Figure 5.2 we can see the convergence factor in the nonoverlapping and overlapping cases for Algorithm 3 compared to the classical Algorithm 1 and Algorithm 2.

5.3. Optimization of Algorithm 4. Here, we look for s_1 and s_2 of the form $s_l = p_l(1+i)$, $l = 1, 2$, such that the p_l are solutions of the min-max problem

$$(5.16) \quad \min_{p_1, p_2 \geq 0} \left(\max_{k \in K} \rho_4(k, \tilde{\omega}, \sigma, Z, L, p_1(1+i), p_2(1+i)) \right).$$

As for Algorithm 3, we proceed directly with the asymptotic analysis.

The nonoverlapping case. Letting $L = 0$, $k_{\min} = 0$, $k_{\max} = \frac{C}{h}$ and denoting by (p_1^*, p_2^*) the solution of the min-max problem (5.16), we find numerically that the solution of (5.16) equioscillates twice, i.e., (p_1^*, p_2^*) is the solution of

$$\begin{aligned} \rho_4(\bar{k}_1, \tilde{\omega}, \sigma, Z, L, p_1^*(1+i), p_2^*(1+i)) &= \rho_4(\bar{k}_2, \tilde{\omega}, \sigma, Z, L, p_1^*(1+i), p_2^*(1+i)) \\ &= \rho_4(k_{\max}, \tilde{\omega}, \sigma, Z, L, p_1^*(1+i), p_2^*(1+i)), \end{aligned}$$

where $\bar{k}_1$ and $\bar{k}_2$ are two interior local maxima of ρ_4 . One can show that $\bar{k}_1 = \tilde{\omega}$, and for small h we get

$$\bar{k}_2 = C_{b_2} h^{-\frac{1}{2}}, \quad p_1^* = C_p h^{-\frac{3}{4}}, \quad p_2^* = C_q h^{-\frac{3}{4}}.$$

To determine the constants C_{b_2} , C_p , and C_q , we solve asymptotically the equioscillation conditions and the equation stating that $\bar{k}_2$ is a maximum, which leads to the system

$$C_{b_2} = \sqrt{2 C_p C_q}, \quad C_q = \frac{1}{2} \frac{C_p^3}{C^2}, \quad C_p = (2 \tilde{\omega} \sigma Z)^{\frac{1}{8}} C^{\frac{3}{4}}.$$

Solving this system, we obtain in the original parameters ω , ϵ , μ , and σ the asymptotic result

$$(5.17) \quad p_1^* = \frac{(2 \omega \sigma \mu)^{\frac{1}{8}} C^{\frac{3}{4}}}{h^{\frac{3}{4}}}, \quad p_2^* = \frac{(2 \omega \sigma \mu)^{\frac{3}{8}} C^{\frac{1}{4}}}{2 h^{\frac{1}{4}}}, \quad \rho_4^* = 1 - \frac{(2 \omega \sigma \mu)^{\frac{1}{8}} h^{\frac{1}{4}}}{C^{\frac{1}{4}}}.$$

The overlapping case. Also with overlap, $L = h$, the solution of of the min-max problem (5.16) equioscillates twice, and we find that (p_1^*, p_2^*) is the solution of

$$\begin{aligned} \rho_4(\bar{k}_1, \tilde{\omega}, \sigma, Z, L, p_1^*(1+i), p_2^*(1+i)) &= \rho_4(\bar{k}_2, \tilde{\omega}, \sigma, Z, L, p_1^*(1+i), p_2^*(1+i)) \\ &= \rho_4(\bar{k}_3, \tilde{\omega}, \sigma, Z, L, p_1^*(1+i), p_2^*(1+i)), \end{aligned}$$

where $\bar{k}_1$, $\bar{k}_2$, and $\bar{k}_3$ are three interior local maxima of ρ_4 . For small h , we get

$$\bar{k}_1 = C_{b_1}, \quad \bar{k}_2 = C_{b_2} h^{-\frac{2}{5}}, \quad \bar{k}_3 = C_{b_2} h^{-\frac{4}{5}}, \quad p_1^* = C_p h^{-\frac{3}{5}},$$

and solving the corresponding equations asymptotically, we find after a lengthy calculation for the constants the system

$$C_{b_1} = \tilde{\omega}, \quad C_{b_2} = \sqrt{2 C_p C_q}, \quad C_{b_3} = \sqrt{C_p}, \quad C_p = \frac{4 C_q^3}{\tilde{\omega} \sigma Z}, \quad C_q = 2 C_p^2.$$

After its solution, we obtain in the original parameters ω , ϵ , μ , and σ

$$p_1^* = \frac{(\omega \sigma \mu)^{\frac{1}{5}}}{2 h^{\frac{3}{5}}}, \quad p_2^* = \frac{(\omega \sigma \mu)^{\frac{2}{5}}}{2 h^{\frac{1}{5}}}, \quad \rho_4^* = 1 - 4 \sqrt{2} (\omega \sigma \mu)^{\frac{1}{10}} h^{\frac{1}{5}}.$$

In Figure 5.2 we show again a comparison of the convergence factors.

5.4. Optimization of Algorithm 5. We finally look for s_1 and s_2 of the form $s_l = p_l(1+i)$, $l = 1, 2$, such that p_l are solutions of the min-max problem

$$(5.18) \quad \min_{p_1, p_2 \geq 0} \left(\max_{k \in K} \rho_5(k, \tilde{\omega}, \sigma, Z, L, p_1(1+i), p_2(1+i)) \right).$$

The nonoverlapping case. We set $L = 0$, $k_{\min} = 0$, $k_{\max} = \frac{C}{h}$ and let (p_1^*, p_2^*) be the solution of the min-max problem (5.16). We observe numerically that this solution equioscillates twice, where (p_1^*, p_2^*) solves

$$\begin{aligned} \rho_5(\bar{k}_1, \tilde{\omega}, \sigma, Z, L, p_1^*(1+i), p_2^*(1+i)) &= \rho_5(\bar{k}_2, \tilde{\omega}, \sigma, Z, L, p_1^*(1+i), p_2^*(1+i)) \\ &= \rho_5(k_{\max}, \tilde{\omega}, \sigma, Z, L, p_1^*(1+i), p_2^*(1+i)), \end{aligned}$$

and $\bar{k}_1$ and $\bar{k}_2$ are two interior local maxima of ρ_5 . Asymptotically, we obtain the behavior

$$\bar{k}_1 = C_{b_1} h^{-\frac{1}{13}}, \quad \bar{k}_2 = C_{b_2} h^{-\frac{7}{13}}, \quad p_1^* = C_p h^{-\frac{10}{13}}, \quad p_2^* = C_q h^{-\frac{4}{13}}.$$

Proceeding as in the other cases, we find for the constants the system

$$C_{b_1}^4 = 6C_q \tilde{\omega} \sigma Z, \quad C_{b_2} = \sqrt{2C_p C_q}, \quad C_q = \frac{1}{2} \frac{C_p^3}{C^2}, \quad C_p = \frac{(2^8 \tilde{\omega}^2 \sigma Z C^{10})^{\frac{1}{13}}}{3^{\frac{3}{13}}},$$

which allows us to find the asymptotic result in the original parameters to be

$$p_1^* = \frac{2^{\frac{8}{13}} \omega^{\frac{2}{13}} \varepsilon^{\frac{1}{26}} \mu^{\frac{3}{26}} \sigma^{\frac{1}{13}} C^{\frac{10}{13}}}{3^{\frac{3}{13}} h^{\frac{10}{13}}}, \quad p_2^* = \frac{2^{\frac{11}{13}} \omega^{\frac{6}{13}} \sigma^{\frac{3}{13}} \varepsilon^{\frac{3}{26}} \mu^{\frac{9}{26}} C^{\frac{4}{13}}}{3^{\frac{9}{13}} h^{\frac{4}{13}}},$$

$$\rho_5^* = 1 - \frac{2^{\frac{34}{13}} \omega^{\frac{2}{13}} \sigma^{\frac{1}{13}} \varepsilon^{\frac{1}{26}} \mu^{\frac{3}{26}} h^{\frac{3}{13}}}{3^{\frac{3}{13}} C^{\frac{3}{13}}}.$$

The overlapping case. We finally treat the overlapping case $L = h$, where the solution of (5.18) equioscillates twice,

$$\begin{aligned} \rho_5(\bar{k}_1, \tilde{\omega}, \sigma, Z, L, p_1^*(1+i), p_2^*(1+i)) &= \rho_5(\bar{k}_2, \tilde{\omega}, \sigma, Z, L, p_1^*(1+i), p_2^*(1+i)) \\ &= \rho_5(\bar{k}_3, \tilde{\omega}, \sigma, Z, L, p_1^*(1+i), p_2^*(1+i)), \end{aligned}$$

TABLE 5.1

Asymptotic convergence factor and optimal choice of the parameters in the transmission conditions.

Algorithm	With overlap, $L = h$	
	ρ	Parameters
1	$1 - \frac{4}{3} (9\omega^4 \sigma^2 \mu^3 \varepsilon)^{\frac{1}{8}} h^{\frac{3}{4}}$	none
2	$1 - 2^{\frac{7}{6}} (\omega \sigma \mu)^{1/6} h^{\frac{1}{3}}$	$p = \frac{(2\omega \sigma \mu)^{1/3}}{2h^{\frac{1}{3}}}$
3	$1 - \frac{2^{\frac{17}{10}} (\omega^4 \varepsilon \mu^3 \sigma^2)^{\frac{1}{20}} h^{\frac{3}{10}}}{3^{\frac{3}{10}}}$	$p = \frac{2^{\frac{2}{5}} (\omega^4 \varepsilon \mu^3 \sigma^2)^{\frac{1}{10}}}{3^{\frac{3}{5}} h^{\frac{2}{5}}}$
4	$1 - 4\sqrt{2} (\omega \sigma \mu)^{\frac{1}{10}} h^{\frac{1}{5}}$	$p_1 = \frac{(\omega \sigma \mu)^{\frac{1}{5}}}{2h^{\frac{1}{5}}}, p_2 = \frac{(\omega \sigma \mu)^{\frac{2}{5}}}{2h^{\frac{1}{5}}}$
5	$1 - \frac{2^{\frac{23}{8}} (\omega^4 \varepsilon \mu^3 \sigma^2)^{\frac{1}{32}} h^{\frac{3}{16}}}{3^{\frac{3}{16}}}$	$p_1 = \frac{(\omega^4 \varepsilon \mu^3 \sigma^2)^{\frac{1}{16}}}{2^{\frac{1}{4}} 3^{\frac{3}{8}} h^{\frac{5}{8}}}, p_2 = \frac{\sqrt{2} (\omega^4 \varepsilon \mu^3 \sigma^2)^{\frac{1}{8}}}{3^{\frac{3}{4}} h^{\frac{1}{4}}}$
Without overlap, $L = 0$		
1	$1 - \frac{\omega^2 \sigma \sqrt{\mu^3 \varepsilon}}{C^3} h^3$	none
2	$1 - \frac{2^{\frac{3}{4}} (\omega \sigma \mu)^{\frac{1}{4}} \sqrt{h}}{\sqrt{C}}$	$p = \frac{(\omega \sigma \mu)^{\frac{1}{4}} \sqrt{C}}{2^{\frac{1}{4}} \sqrt{h}}$
3	$1 - \frac{2^{\frac{11}{7}} (\omega^4 \varepsilon \mu^3 \sigma^2)^{\frac{1}{14}} h^{\frac{3}{7}}}{3^{\frac{3}{7}} C^{\frac{2}{7}}}$	$p = \frac{2^{\frac{4}{7}} (\omega^4 \varepsilon \mu^3 \sigma^2)^{\frac{1}{14}} C^{\frac{4}{7}}}{3^{\frac{3}{7}} h^{\frac{4}{7}}}$
4	$1 - \frac{(2\omega \sigma \mu)^{\frac{1}{8}} h^{\frac{1}{4}}}{C^{\frac{1}{4}}}$	$p_1 = \frac{(2\omega \sigma \mu)^{\frac{1}{8}} C^{\frac{3}{4}}}{h^{\frac{1}{4}}}, p_2 = \frac{(2\omega \sigma \mu)^{3/8} C^{1/4}}{2h^{\frac{1}{4}}}$
5	$1 - \frac{2^{\frac{34}{13}} (\omega^4 \varepsilon \mu^3 \sigma^2)^{\frac{1}{26}} h^{\frac{3}{13}}}{3^{\frac{3}{13}} C^{\frac{3}{13}}}$	$p_1 = \frac{2^{\frac{8}{13}} (\omega^4 \varepsilon \mu^3 \sigma^2)^{\frac{1}{26}} C^{\frac{10}{13}}}{3^{\frac{3}{13}} h^{\frac{10}{13}}}, p_2 = \frac{2^{\frac{11}{13}} (\omega^4 \varepsilon \mu^3 \sigma^2)^{\frac{3}{26}} C^{\frac{4}{13}}}{3^{\frac{9}{13}} h^{\frac{4}{13}}}$

and $\bar{k}_1$, $\bar{k}_2$, and $\bar{k}_3$ are three interior local maxima of ρ_5 . Proceeding as before, we arrive after a long asymptotic analysis at

$$p_1^* = \frac{\omega^{\frac{1}{4}} \varepsilon^{\frac{1}{16}} \mu^{\frac{3}{16}} \sigma^{\frac{1}{8}}}{2^{\frac{1}{4}} 3^{\frac{3}{8}} h^{\frac{5}{8}}}, \quad p_2^* = \frac{\sqrt{2} \sqrt{\omega} \varepsilon^{\frac{1}{8}} \mu^{\frac{3}{8}} \sigma^{\frac{1}{4}}}{3^{\frac{3}{4}} h^{\frac{1}{4}}}, \quad \rho_5^* = 1 - \frac{2^{\frac{23}{8}} \omega^{\frac{1}{8}} \varepsilon^{\frac{1}{32}} \mu^{\frac{3}{32}} \sigma^{\frac{1}{16}} h^{\frac{3}{16}}}{3^{\frac{3}{16}}}.$$

In Figure 5.2 we show the convergence factor in the nonoverlapping and overlapping cases for Algorithm 5 compared to the classical Algorithm 1, Algorithm 2, Algorithm 3, and Algorithm 4. Table 5.1 summarizes the asymptotic results that have been obtained for all algorithms.

6. Numerical results. We present here numerical experiments in order to illustrate the performance of the optimized algorithms developed in the previous sections. The domain Ω is partitioned into several subdomains Ω_j . The Maxwell equations are approximated by a discontinuous Galerkin method ($DG - \mathbb{P}_p$); see [7] for further details. In short, given a mesh $\mathcal{T}_h$ of Ω_j such that $\bar{\Omega}_j = \bigcup_{\tau_k \in \mathcal{T}_h}$, the ($DG - \mathbb{P}_p$) method consists in searching an approximate solution of each field component in the set

$$V_h = \{V \in L^2(\Omega) / \forall \tau_k \in \mathcal{T}_h, V|_{\tau_k} \in \mathbb{P}_p[\tau_k]\},$$

$$\mathbb{P}_p[\tau_k] = \{\text{polynomial function on } \tau_k \text{ of degree } \leq p\}.$$

In all that follows we note that the computational cost for one iteration of all algorithms is very similar since optimized algorithms introduce only a few modified entries in the discretized problems at the nodes of the interface, and these modified entries have no influence on the cost of the direct solvers we use in the interior.

6.1. Performance for a two subdomain decomposition. We first test the propagation of a plane wave in a homogeneous and conductor medium using a transverse magnetic formulation. To perform these tests, which confirm the theoretical asymptotic results, we used the optimized Schwarz methods as solvers without a Krylov method. The domain is $\Omega = (0, 1)^2$, and the parameters are constant in Ω , with $\varepsilon = \mu = 1$, $\sigma = 5$, and $\omega = 2\pi$. We impose on the boundary an incident field $\mathbf{W}^{\text{inc}} = (H_x^{\text{inc}}, H_y^{\text{inc}}, E_z^{\text{inc}}) = (\frac{k_y}{\mu\omega}, \frac{-k_x}{\mu\omega}, 1)e^{-i\mathbf{k}\cdot\mathbf{x}}$ with $\mathbf{k} = (k_x, k_y) = (\omega\sqrt{\varepsilon - i\frac{\sigma}{\omega}}, 0)$, $\mathbf{x} = (x, y)$. The domain Ω is decomposed into two subdomains $\Omega_1 =$

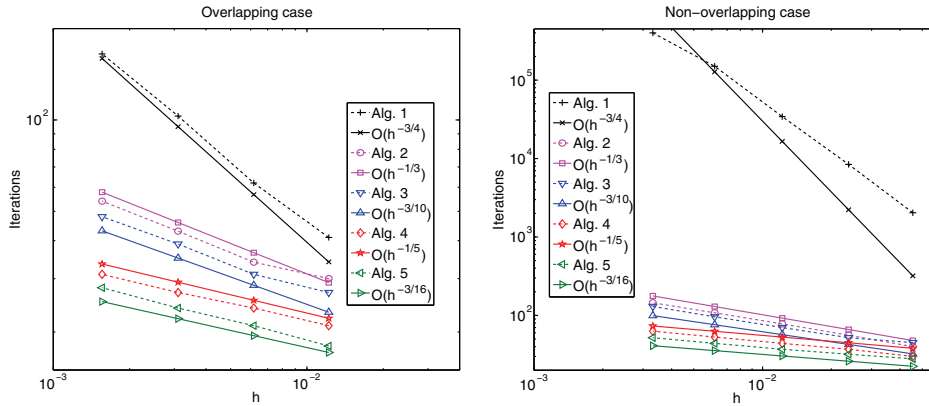


FIG. 6.1. Number of iterations against the mesh size h , to attain a relative residual reduction of 10^{-8} .

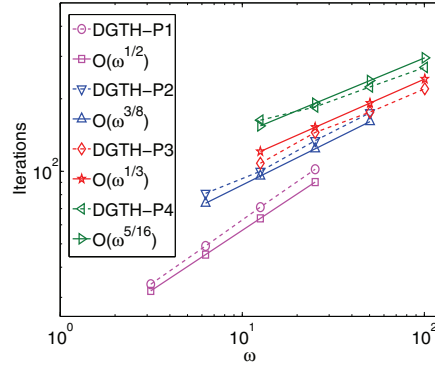


FIG. 6.2. Number of iterations against the wave number ω for four different DG-element orders, and theoretically predicted asymptotics.

$(0, 1/2 + L) \times (0, 1)$ and $\Omega_2 = (1/2, 1) \times (0, 1)$; L is the size of the overlap and is equal to the mesh size h for the methods with overlap and is equal to zero for the methods without overlap. For this test, we use a $DG - \mathbb{P}_2$ method, i.e., a uniform polynomial approximation of order two. The performance of the algorithms is shown in Figure 6.1. In this figure, we show the number of iterations as a function of the mesh size h for both the nonoverlapping and overlapping cases. We have also represented in this figure the theoretical asymptotic results denoted by $O(h^\alpha)$. The numerical results are in good agreement with the theoretical asymptotic results which are summarized in Table 5.1, except for Algorithm 1 (the classical algorithm) without overlap, where we get better results. The fact that the discretization can improve the performance of Schwarz algorithms has also been observed and explained in [8] for Cauchy–Riemann equations.

We next show a numerical experiment in order to illustrate the weak dependence of our optimized Schwarz method on the wave number ω , and to illustrate Corollary 5.7. We choose the same setting as in the previous experiment, but now set $h := C_h/\omega^\gamma$, with $\gamma = (2p + 1)/(2p)$, where p is the polynomial degree of the DG-element we are using. This choice of the mesh size avoids the pollution effect; see [2]. We show in Figure 6.2 for $\sigma = 0.1$ the iteration number our optimized Schwarz method from Corollary 5.7 needs when we increase the wave number ω . These numerical experiments illustrate well the weak dependence of the convergence behavior on the wave number ω , as predicted by our analysis.

6.2. A more realistic application. We now present a second, more realistic, test which consists in the simulation of electromagnetic wave propagation in a heterogeneous subsurface medium. For this test case, the optimized Schwarz methods are used as preconditioners for a Krylov method (here BiCGStab). This kind of simulation is very important in imaging; see [13] for details. The configuration of the subsurface is shown in Figure 6.3 and is composed of media which are characterized by various parameter values ε and σ : $\varepsilon = 1.5, 2.25, 3.25, 4.25$ and $\sigma = 10^{-6}, 10^{-5}, 10^{-4}, 10^{-3}$. The computational domain is decomposed into several subdomains without overlap; an example with eight subdomains for a mesh with 49430 elements is shown in Figure 6.4. We test here the performance of the algorithms for a decomposition into two, four, eight, and 16 subdomains, for two mesh resolutions: 49430 and 197720 elements. In Table 6.2, we show the number of iterations needed for convergence, i.e.,

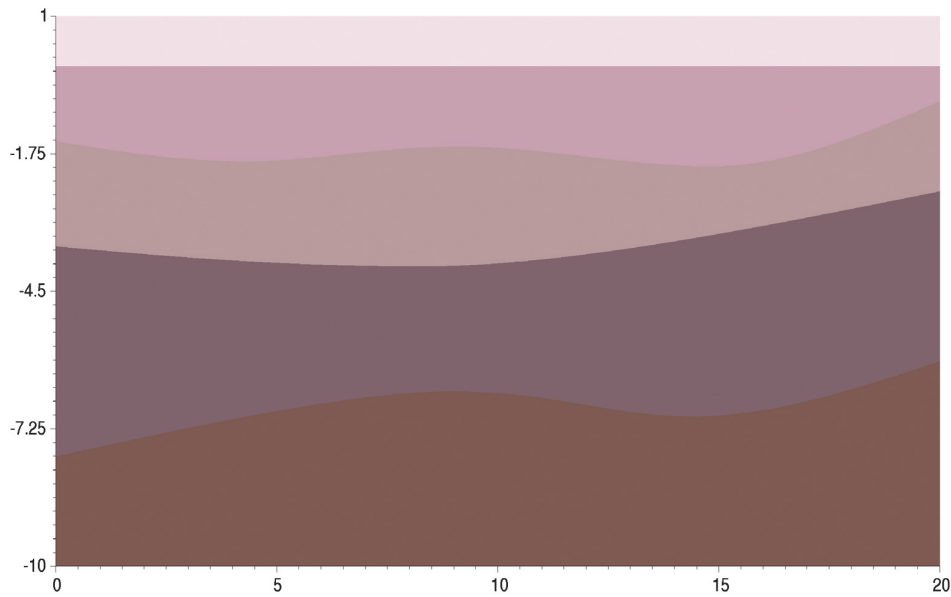
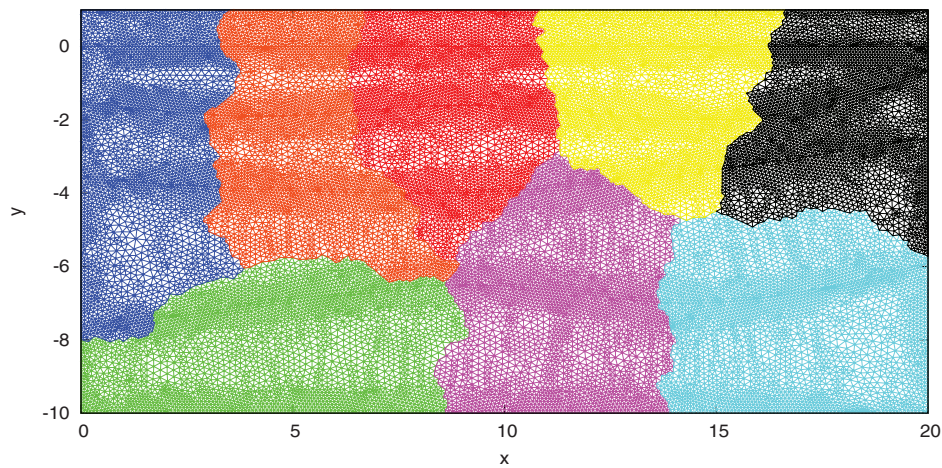
FIG. 6.3. *Configuration of a subsurface.*FIG. 6.4. *Mesh and decomposition in eight subdomains.*

TABLE 6.2

Iteration number comparison for the wave propagation problem in a subsurface with two meshes, the original one shown in Figure 6.4 and a refined one where each triangle was cut into four.

No. of subdomains/processors	49430 elements				197720 elements			
	2	4	8	16	2	4	8	16
Algorithm 1	59	72	80	85	88	100	109	199
Algorithm 2	39	47	53	61	47	57	62	75
Algorithm 3	37	48	51	58	46	55	60	69
Algorithm 4	35	41	49	57	46	53	58	69
Algorithm 5	33	40	47	54	45	51	56	67

to attain a relative residual of 10^{-8} , depending on the number of subdomains and the resolution. These results show that the optimized algorithms converge much faster than the classical algorithm (i.e., Algorithm 1). We can also see that the hierarchy of the optimized algorithms, in terms of number of iterations, is respected, as predicted by the theoretical results in Corollary 4.1, and that refining the mesh leads only to a very moderate increase in the iteration numbers required for the solution when using optimized algorithms. Even increasing the number of subdomains only leads to a moderate increase in the iteration numbers in these experiments.

7. Conclusions. We have developed several domain decomposition algorithms based on optimized transmission conditions for Maxwell's equations with nonzero electric conductivity. Two algorithms are obtained by approximating the transparent operator by zeroth order transmission conditions, while two further algorithms are based on second order transmission conditions. We have shown that the convergence factor of each algorithm can be written as $1 - O(h^{\alpha_i})$, $i = 1, \dots, 5$. Our results are well confirmed by the first numerical test with a decomposition into two subdomains, where we also obtained numerical convergence factors in the form $1 - O(h^{\beta_i})$, $i = 1, \dots, 4$, with $\alpha_i \approx \beta_i$. The first test also shows that optimized Schwarz algorithms converge much faster than the classical one. This is also confirmed by the second more realistic test which also shows that these optimized algorithms can be effective for more complex problems and for arbitrary decompositions.

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