An Optimized Schwarz Waveform Relaxation Algorithm for Micro-Magnetics

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Summary. We present an optimized Schwarz waveform relaxation algorithm for the parallel solution in space-time of the equations of ferro-magnetics in the micro-magnetic model. We use Robin transmission conditions, and observe fast convergence of the discretized algorithm. We show numerically the existence of an optimal parameter in the Robin condition, and study its dependence on the various physical and numerical parameters.

1 Introduction

Over the last decades, ferro-magnetics has been the subject of renewed interest due to its omnipresence in industrial applications, and the need for correctly predicting the behavior of ferro-magnets, which is best achieved by numerical simulations, see the historical introduction in Trémolet de Lacheisserie [2000] and Aharoni [1996]. Since micro-magnetic simulations are very costly, we present in this paper an optimized Schwarz waveform relaxation algorithm for the micro-magnetic equation. These algorithms have the advantage of independent adaptive discretizations per subdomain both in space and time, and they are naturally parallel, see Gander et al. [1999, 2003], Gander and Halpern [2004, 2007]. We present a numerical analysis of the algorithm with Robin transmission conditions applied to the equation of ferro-magnetics for a two subdomain decomposition, and study the dependence of the optimal parameter on the various physical and numerical parameters.

2 The micro-magnetic model

Let Ω be a bounded open set in $\mathbb{R}^3$ filled with a ferromagnetic material. The magnetic state of the material is given by its magnetization vector $\boldsymbol{m} \in \mathbb{R}^3$, vanishing outside Ω, with the non-convexity constraint

$$|\boldsymbol{m}| = 1 \text{ a.e. in } \Omega. \tag{1}$$

The behavior of $\boldsymbol{m}$ is modeled by the Landau-Lifshitz equation,

$$\frac{\partial \boldsymbol{m}}{\partial t} = \mathcal{L}(\boldsymbol{m}) := -\boldsymbol{m} \times \boldsymbol{h}(\boldsymbol{m}) - \alpha \boldsymbol{m} \times (\boldsymbol{m} \times \boldsymbol{h}(\boldsymbol{m})), \tag{2}$$

where $\alpha > 0$ is the dissipation parameter. As a first step toward real computations, we include only the exchange interaction, which is local, and produces a magnetic excitation $\boldsymbol{h}(\boldsymbol{m}) = A \triangle \boldsymbol{m}$, where $A > 0$ is the exchange constant. The equation in $\Omega \times (0,T)$ is subject to homogeneous Neumann boundary conditions on the boundary of Ω, i.e. $\frac{\partial \boldsymbol{m}}{\partial \boldsymbol{\nu}} = \boldsymbol{0}$, where $\boldsymbol{\nu}$ is the unit outward normal on the boundary. Equation (2) is often used to compute the steady states of the magnetization field; for more information, see Halpern and Labbé [2001].

3 Optimized Schwarz Waveform Relaxation Algorithm

We decompose the domain Ω into p non-overlapping subdomains $(\widetilde{\Omega}_i)_{i=1\dots p}$, $\bigcup_{i=1}^{p} \overline{\widetilde{\Omega}_i} = \overline{\Omega}$. We then derive from this non-overlapping decomposition an overlapping one by choosing $(\Omega_i)_{i=1\dots p}$ such that $\widetilde{\Omega}_i \subset \Omega_i$ and $\bigcup_{i=1}^{p} \Omega_i = \Omega$.

We define the interfaces and exterior boundary by

$$\Gamma_{ij} = \partial\Omega_i \cap \overline{\widetilde{\Omega}_j}, \qquad\qquad \Gamma_i^e = \partial\Omega_i \cap \partial\Omega.$$

An optimized Schwarz waveform relaxation algorithm computes for $n = 1, 2, \dots$ the iterates $(\boldsymbol{m}_i^n)_{1 \le i \le p}$ defined by

$$\begin{aligned}
\frac{\partial \boldsymbol{m}_i^n}{\partial t} &= \mathcal{L}(\boldsymbol{m}_1^n) && \text{in } \Omega_i \times (0,T),\\
\boldsymbol{m}_i^n(\cdot, 0) &= \boldsymbol{m}_0 && \text{on } \Omega_i,\\
\mathcal{B}_{ij}\boldsymbol{m}_i^n &= \mathcal{B}_{ij}\boldsymbol{m}_j^{n-1} && \text{on } \Gamma_{ij} \times (0,T),\\
\frac{\partial \boldsymbol{m}_i^n}{\partial \boldsymbol{\nu}} &= \boldsymbol{0} && \text{on } \Gamma_i^e \times (0,T),
\end{aligned} \tag{3}$$

where the $\mathcal{B}_{ij}$ are linear operators.

In Bennequin et al. [2004], Gander and Halpern [2007], several strategies for choosing these boundary operators are proposed both in the case with and without overlap, and a complete convergence analysis is provided for a linear advection-diffusion equation. Here, because of the non-linearity, such an analysis is not yet available, and we use for our numerical study for the non-overlapping case Robin transmission conditions, which are robust and easy to implement, *i.e.* $\mathcal{B}_{ij} = \frac{\partial}{\partial \boldsymbol{\nu}} + \beta_{ij}\mathrm{I}$, where β_{ij} is a positive real number to be chosen optimally for best convergence.

4 Discretization

We use in space a finite difference discretization on a regular rectangular grid, where the Laplace operator can be approximated by the standard five point finite difference stencil. Since we use cell-centered nodes, the boundaries and the interfaces are halfway in between two nodes, and hence values there can be approximated using the mean of the two adjacent nodes, denoted by A and B in Figure 1, whereas the normal derivative can be approximated by a finite difference between the same nodes A and B. The Robin condition $\partial_{\boldsymbol{\nu}}\boldsymbol{m}+\beta\boldsymbol{m} =$

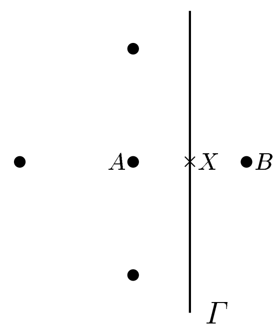


Fig. 1. Position of the interface in our cell-centered finite difference discretization.

g at point X in Figure 1 is thus discretized by $\frac{\boldsymbol{m}_B-\boldsymbol{m}_A}{\Delta x}+\beta\frac{\boldsymbol{m}_A+\boldsymbol{m}_B}{2} = g$, where Δx is the space step-size. This yields $\boldsymbol{m}_B = (2\Delta x g+(2-\beta\Delta x)\boldsymbol{m}_A)/(2+\beta\Delta x)$, which is then used to complete the missing value at the node B in the five-point finite difference stencil centered at A in Figure 1.

For the time discretization, we use the explicit second order scheme from Labbé [1998], Labbé and Bertin [1999],

$$\boldsymbol{m}_{i+1} = \boldsymbol{m}_i + \Delta t\boldsymbol{F}(\boldsymbol{m}_i) + \frac{\Delta t^2}{2}\mathrm{D}\boldsymbol{F}(\boldsymbol{m}_i)\cdot\boldsymbol{F}(\boldsymbol{m}_i), \tag{4}$$

where D is the differentiation operator and

$$\boldsymbol{F}(\boldsymbol{m}_i) = -\boldsymbol{m}_i \times \boldsymbol{h}(\boldsymbol{m}_i) - \alpha\boldsymbol{m}_i \times (\boldsymbol{m}_i \times \boldsymbol{h}(\boldsymbol{m}_i)).$$

To satisfy the non-convexity constraint (1), we renormalize the magnetization after each time step. Our implementation can use an optimized time step per subdomain in order to maximize energy dissipation and speed up convergence to the steady state, and thus the algorithm is truly non-conforming in time. To study however the convergence to the discrete solution on the entire domain Ω numerically, we use in the sequel fixed time steps in the subdomains.

5 Numerical Study of the Algorithm

We consider a squared ferromagnetic thin plate of dimension $3.68e{-6} \times 3.68e{-6} \times \Delta x$, with the parameter $A = 4.06e{-18}$. We also fix the parameter α to be $\frac{1}{2}$.

We divide the domain into two subdomains, as shown in Figure 2. In this non-overlapping case, $\Gamma_{12} = \Gamma_{21} = \Gamma$, and we consider the case $\beta_{12} = \beta_{21} = \beta$ only. We discretize the problem as shown in Section 4, and we first compute

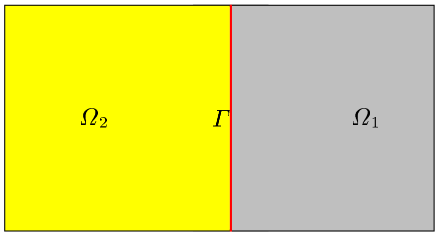


Fig. 2. The two subdomain decomposition used for our numerical experiments.

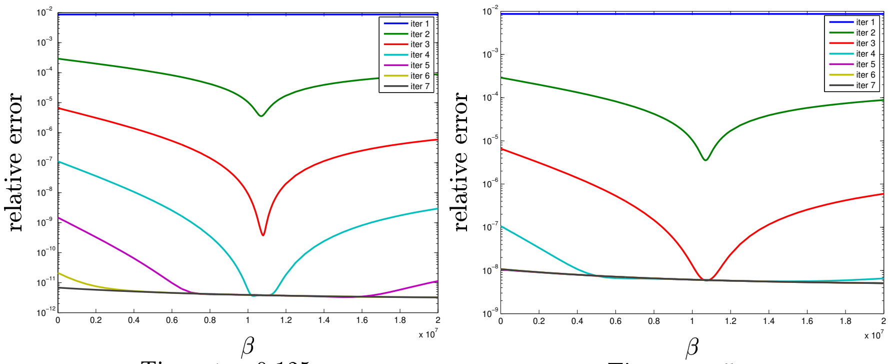
the discrete solution on the entire domain. We then measure the relative error between the mono-domain solution and each iteration of the optimized Schwarz waveform relaxation algorithm in the l_h^2 norm,

$$\|\boldsymbol{u}\|_{l_h^2}^2 = \sum_{i=1}^{N} |\omega_i| |\boldsymbol{u}_i|^2.$$

Since (ω_i) is a rectangular mesh, $|\omega_i| = |\Delta x|^3$.

We mesh the ferromagnetic domain with a space step of $\Delta x = 1.84e{-7}$, which yields a $20 \times 20 \times 1$ mesh. A numerical experiment over long time shows that for the physical parameters chosen above, the equilibrium state has not yet been reached at $T = 30000$.

We first study the convergence behavior of the optimized Schwarz waveform relaxation algorithm. The simulations presented here are done without overlap. We choose a final time of $T = 500$, and perform a series of computations for fixed Δt, for various values of the parameter β. In Figure 3, we show for the time discretization steps $\Delta t = 0.125$ and $\Delta t = 5$ the relative error curves for a sequence of iterates as a function of the parameter β in the Robin transmission condition. The algorithm is convergent, and there is a numerically optimal choice β_{opt} for β: in both cases, $\beta_{opt} \approx 1.05e{+}7$, which indicates that β_{opt} does not depend on the time step.



Time step 0.125 Time step 5

Fig. 3. Convergence curves as function of the optimization parameter β for two different time steps.

We now study the dependence of β_{opt} on the space discretization step Δx. As the step size increases, β_{opt} in the Robin transmission conditions decreases. The least squared best fit shown in Figure 4 gives $\ln \beta_{opt} \approx 1.03 - 0.97 \ln(\Delta x)$, which indicates that

$$\beta_{opt} \approx \frac{2.8}{\Delta x}. \tag{5}$$

In the next sequence of numerical experiments, we study the dependence of β_{opt} on the physical parameters of the problem. In Figure 5, we show for the final times $T = 100$ and $T = 4000$ the relative error curves for a sequence of iterations as a function of the parameter β in the Robin transmission condition. These results, and a large sample of final times between 1 and 4000 indicate that β_{opt} does not seem to depend on the final time. This is somewhat unexpected, since for the linear heat equation, β_{opt} depends on the final time, see Gander and Halpern [2003].

The fact that β_{opt} does not depend on the final time of the simulation implies that β_{opt} does not depend on the physical parameter A in our case, since dividing the entire equation by A shows that A can be interpreted as a scaling factor for the final time. If however other exchange interactions were present, such as the demagnetization field, this scaling argument would not hold any more.

It remains to study the behavior of β_{opt} when α varies. To this end, we compute the convergence curves with parameters $T = 200$ and $\Delta x = 1.84e{-}7$ for α ranging from 0.1 to 100. We present some of these results in Figure 6. For small α, the algorithm converges very well and the β_{opt} has an approximate

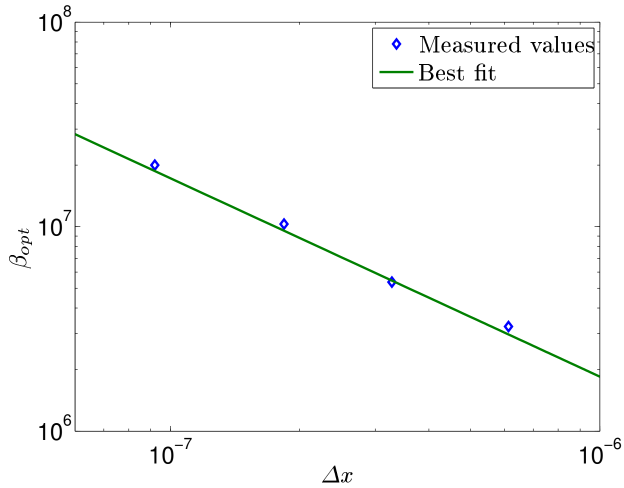


Fig. 4. β_{opt} as a function of the space discretization step

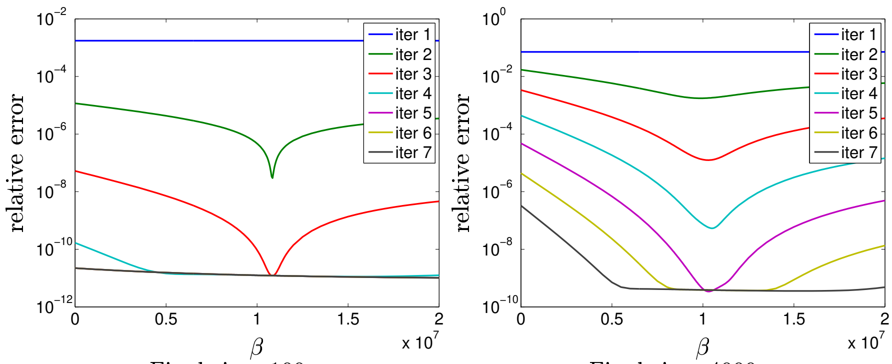


Fig. 5. Convergence curves as function of the optimization parameter for two final times.

value of 1.05e+7. However as α increases, the value of β_{opt} varies as shown in Figure 7 and the optimal error increases, see Figure 7.

6 Conclusion

We presented an optimized Schwarz waveform relaxation algorithm for the equations of ferro-magnetics in the micro-magnetic model. We studied numerically the convergence behavior of the algorithm with Robin transmission conditions. This study revealed that the algorithm converges very fast: after only few iterations, an error reduction by a factor of 10^{-9} is achieved. Using

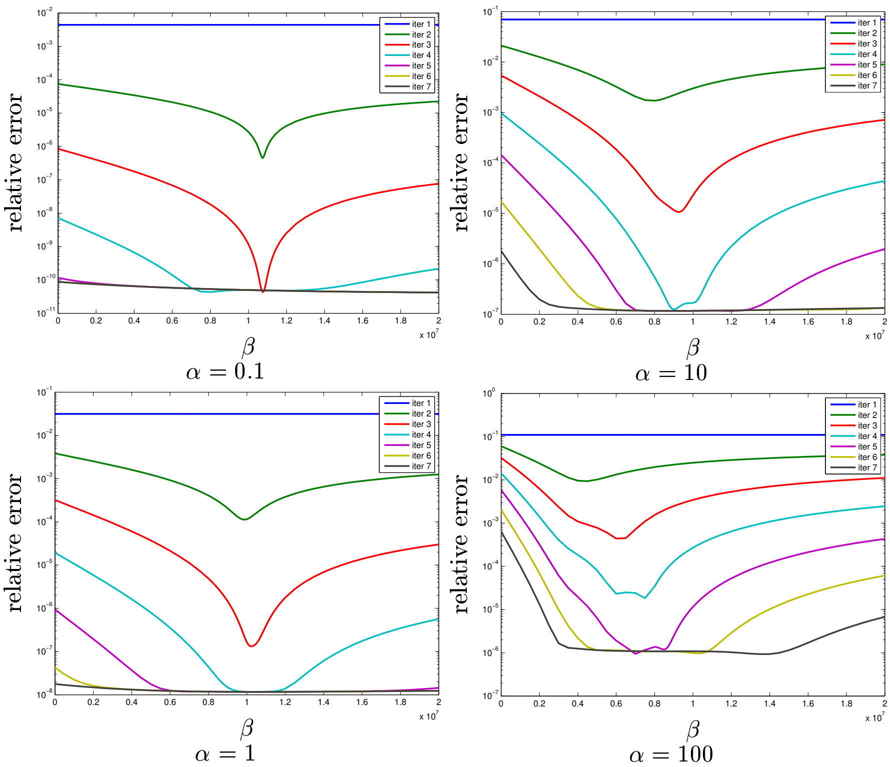


Fig. 6. Convergence curves for various choice of α

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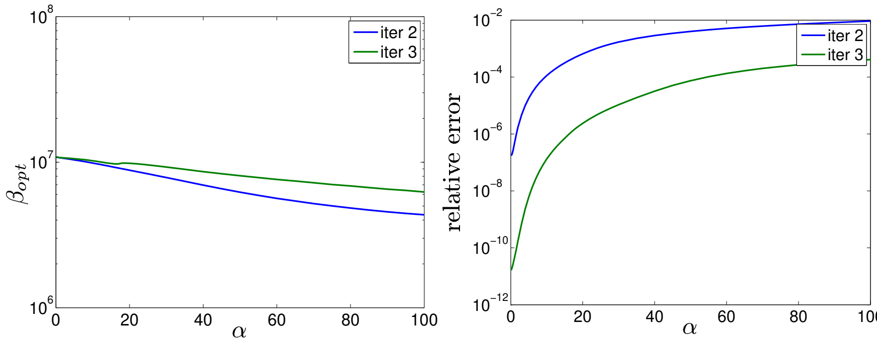

Fig. 7. β_{opt} and optimal error as functions of α

extensive numerical results, we determined a heuristic formula for the value of β_{opt}, in the non-overlapping case,

$$\beta_{opt} \approx \frac{g(\alpha)}{\Delta x},$$

where $g(\alpha)$ is represented in Figure 7 on the left.

We are currently working on a convergence analysis of the algorithm, and the extension to other interaction terms; in particular adding the demagnetization field interaction is challenging, since it represents a global operator.

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