Non shape regular domain decompositions: an analysis using a stable decomposition in H_0^1

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Abstract In this paper, we establish the existence of a stable decomposition in the Sobolev space H_0^1 for domain decompositions which are not shape regular in the usual sense. In particular, we consider domain decompositions where the largest subdomain is significantly larger than the smallest subdomain. We provide an explicit upper bound for the stable decomposition that is independant of the ratio between the diameter of the largest and the smallest subdomain.

1 Introduction

One of the great success stories in domain decomposition methods is the invention and analysis of the additive Schwarz method by Dryja and Widlund [1987]. Even before the series of international conferences on domain decomposition methods started, Dryja and Widlund [1987] presented a variant of the classical alternating Schwarz method (see Schwarz [1870]), which has the advantage of being symmetric for symmetric problems, and it also contains a coarse space component. In a fully discrete analysis, Dryja and Widlund [1987] proved, based on a stable decomposition result for shape regular decompositions, that the condition number of the preconditioned operator with a decomposition into many subdomains only grows as a function of $\frac{H}{\delta}$, where H is the subdomain diameter, and δ is the overlap between subdomains.

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This analysis inspired a generation of numerical analysts, who used these techniques in order to analyze many other domain decomposition methods, see the reference books Smith et al. [1996], Quarteroni and Valli [1999], Toselli and Widlund [2004], or the monographs Xu [1992], Chan and Mathew [1994], and references therein.

The key assumption that the decomposition is shape regular is however often not satisfied in practice: because of load balancing, highly refined subdomains are often physically much smaller than subdomains containing less refined elements, and it is therefore of interest to consider domain decompositions that are only locally shape regular, *i.e.* domain decompositions where the largest subdomain can be considerably larger than the smallest subdomain. In such a domain decomposition, the ratio $\frac{H}{\delta}$ can be given at least two different meanings: let H_i refer to the diameter of subdomain number i and δ_i refer to the width of the overlap around subdomain number i. Is the explicit upper bound of the stable decomposition linear in $\max_i(\frac{H_i}{\delta_i})$ or is it only linear in $\frac{\max_i(H_i)}{\min_i(\delta_i)}$? The latter estimate is much more pessimistic than the former when the subdomains are of wildly different size, and the general analysis based on a shape regular decomposition of the additive Schwarz method does not permit to answer this question.

In Gander et al. [2011], we established the existence of a stable decomposition in the continuous setting with an explicit upper bound and a quantitative definition of shape regularity. The explicit upper bound is also linear in $\frac{H}{\delta}$, and the result is limited to shape regular domain decompositions where all subdomains have similar size and where the overlap width is uniform over all subdomains. Having explicit upper bounds however allows us now, using similar techniques, to establish the existence of a stable decomposition in the continuous setting with explicit upper bounds when $\max_i(H_i) \gg \min_i(H_i)$. We provide an explicit upper bound which is linear in $\max_i(H_i/\delta_i)$. To get this result, only a few of the inequalities established in Gander et al. [2011] need to be reworked, and it would be very difficult to obtain such a result without the explicit upper bounds from the continuous analysis in Gander et al. [2011].

2 Geometric parameters and main theorem

In the remainder of this paper, we always consider a domain decomposition that has the following properties:

- Ω is a bounded domain of $\mathbb{R}^2$.
- The $(U_i)_{1\leq i\leq N}$ are a non overlapping domain decomposition of Ω, *i.e.* $\bigcup_{i=1}^N \overline{U}_i = \overline{\Omega}$. The U_i are bounded connected open sets of $\mathbb{R}^2$ and for all subdomains U_i the measure of $\overline{U}_i \setminus U_i$ is zero.
- We set $H_i := \operatorname{diam}(U_i)$.

- Two distinct subdomains U_i and U_j are said to be neighbors if $\overline{U}_i \cap \overline{U}_j \neq \emptyset$.
- For each subdomain U_i, let $\delta_i > 0$ be such that $2\delta_i \leq \min_{j, \overline{U}_i \cap \overline{U}_j = \emptyset}(\text{dist}(U_i, U_j))$. We set $\Omega_i := \{\boldsymbol{x} \in \Omega,\ \text{dist}(\boldsymbol{x}, U_i) < \delta_i\}$. The Ω_i form an overlapping domain decomposition of Ω. When subdomains U_i and U_j are neighbors, then the overlap between Ω_i and Ω_j is $\delta_i + \delta_j$ wide. The intersection $\Omega_i \cap \Omega_j$ is empty if and only if the distance between U_i and U_j is positive.
- We set $\delta_i^s = \min_{j \neq i, \overline{U}_i \cap \overline{U}_j \neq \emptyset} \delta_j$ and $\delta_i^l = \max_{j \neq i, \overline{U}_i \cap \overline{U}_j \neq \emptyset} \delta_j$.
- The domain decomposition has N_c colors: there exists a partition of $\mathbb{N} \cap [1, N]$ into N_c sets I_k such that $\Omega_i \cap \Omega_j$ is empty whenever $i \neq j$ and i and j belong to the same color I_k.
- $\mathcal{T}$ is a coarse triangular mesh of Ω: one node $\boldsymbol{x}_i$ per subdomain Ω_i (not counting the nodes located on $\partial\Omega$).
- Let θ_{min} be the minimum of all angles of mesh $\mathcal{T}$.
- No node (including the nodes located on $\partial\Omega$) of the coarse mesh has more than K neighbors.
- Let d_i be the length of the largest edge originating from node $\boldsymbol{x}_i$ in the mesh $\mathcal{T}$.
- Let $H_{h,i}$ be the length of the shortest height through $\boldsymbol{x}_i$ of any triangle in the coarse mesh $\mathcal{T}$ that connects to $\boldsymbol{x}_i$. We also set $H'_{h,i}$ as the minimum of $H_{h,j}$ over i and its direct neighbors in mesh $\mathcal{T}$.
- We suppose that for each subdomain U_i, there exists $r_i > 0$ such that U_i is star-shaped with respect to any point in the ball $B(\boldsymbol{x}_i, r_i)$. We also suppose $r_i \leq \frac{H_{h,i}}{4K+1}$ and $r_i \leq H'_{h,i}/2$.
- For all i, we set

$$\ell_i = \frac{1}{\pi r_i^2} \int_{B(\boldsymbol{x}_i, r_i)} u(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \frac{1}{\pi} \int_{B(\mathbf{0},1)} u(\boldsymbol{x}_i + r_i \boldsymbol{y}) \mathrm{d}\boldsymbol{y}.$$

- We suppose that for each U_i there exists an open layer L_i containing ∂U_i, a vector field $\boldsymbol{X}_i$ continuous on $L_i \cap \overline{U}_i$, $\mathcal{C}^\infty$ on $L_i \cap U_i$ such that $\mathrm{D}\boldsymbol{X}_i(\boldsymbol{x})(\boldsymbol{X}_i(\boldsymbol{x})) = 0$, $\|\boldsymbol{X}_i(\boldsymbol{x})\| = 1$, and $\varepsilon_0 > 0$ such that for all positive $\varepsilon < \varepsilon_0$ and for all $\hat{\boldsymbol{x}}$ in ∂U_i, $\hat{\boldsymbol{x}} + \varepsilon \boldsymbol{X}_i(\hat{\boldsymbol{x}}) \in U_i$ and $\hat{\boldsymbol{x}} - \varepsilon \boldsymbol{X}_i(\hat{\boldsymbol{x}}) \notin U_i$. We set, for all positive δ', $U_i^{\delta'} = \{\boldsymbol{x} \in U_i,\ \text{dist}(\boldsymbol{x}, \partial U_i) < \delta'\}$, and $V_i^{\delta'} = \{\hat{\boldsymbol{x}} + s\boldsymbol{X}_i(\hat{\boldsymbol{x}}), \hat{\boldsymbol{x}} \in \partial U_i, 0 < s < \delta'\}$. We assume there exist $\hat{R}_i > 0$, $\theta_{\boldsymbol{X}}$, $0 < \theta_{\boldsymbol{X}} \leq \pi/2$, and δ_{0i}, $0 < \delta_{0i} \leq \hat{R}_i \sin\theta_{\boldsymbol{X}}$ such that $V_i^{\hat{R}} \subset L_i \cap U_i$ and $U_i^{\delta'} \subset V^{\delta'/\sin\theta_{\boldsymbol{X}}}$ for all positive $\delta' \leq \delta_{0i}$. Set $\tilde{R}_i := 1/\|\text{div}\boldsymbol{X}_i\|_\infty$. We suppose $\delta_{0i} > \delta_i^l$.

We now state our main result, the existence of a stable decomposition of $H_0^1(\Omega)$ whose upper bound is independant of $\frac{\max_i(H_i)}{\min_i(H_i)}$.

Theorem 1. *For u in $H_0^1(\Omega)$, there exists a stable decomposition $(u_i)_{0 \leq i \leq N}$ of u, i.e. $u = \sum_{i=0}^N u_i$, u_0 in $P_1(\mathcal{T})$ and $u_i \in H_0^1(\Omega_i)$ such that*

$$\sum_{i=1}^{N}\|\nabla u_i\|^2_{L^2(\Omega_i)} \leq C\|\nabla u\|^2_{L^2(\Omega)},$$

where $C = 2C_1 + 2(1+C_1)C_2$ *and*

$$C_1 = \frac{1}{\tan\theta_{min}}\frac{\big(1+2\max_i(\frac{r_i}{H_{h,i}})\big)K(\frac{25}{6\pi}\max_i(\frac{d_i}{r_i})+2\pi\big)}{1-\big((2K+1)+(4K+1)\max_i(\frac{r_i}{H_{h,i}})\big)\max_i(\frac{r_i}{H_{h,i}})},$$

$$C_2 = 2 + 8\lambda_2^2(N_c-1)^2(1+\max_i\frac{\hat{R}_i}{\tilde{R}_i})\max_i\frac{\delta_i^l}{\delta_i^s}\max_i\frac{\hat{R}_i}{\delta_i^s\sin\theta_{\boldsymbol{X}}}$$
$$+\frac{8}{3}\lambda_2^2(N_c-1)^2(1+\max_i\frac{\hat{R}_i}{\tilde{R}_i})\max_i\frac{\delta_i^l}{\delta_i^s}\max_i\frac{r_i^2}{\delta_i^s\hat{R}_i\sin\theta_{\boldsymbol{X}}}\times$$
$$\times\max_i\left(\left(\left(\frac{H_i^2}{r_i^2}+\frac{1}{2}\right)^{\frac{1}{4}}+\frac{H_i}{\sqrt[4]{2}r_i}\right)^4-\frac{1}{2}-\frac{H_i^2}{r_i^2}-\frac{H_i^4}{2r_i^4}\right),$$

with λ_2 *being a universal constant.*

Note that the condition $r_i \leq \frac{H_{h,i}}{4K+1}$ ensures that $1-\big((2K+1)+(4K+1)\max_i(r_i/H_{h,i})\big)\max_i(r_i/H_{h,i})$ remains positive.

3 Proof of Theorem 1

3.1 Constructing the fine component

We begin by establishing a stable decomposition when there is no coarse mesh.

Lemma 1. *Let* u *be in* $H_0^1(\Omega)$*. Then, there exist* $(u_i)_{1\leq i\leq N}$*,* u_i *in* $H_0^1(\Omega_i)$ *such that* $u=\sum_{i=1}^N u_i$*, and*

$$\sum_{i=1}^{N}\|\nabla u_i\|^2_{L^2(\Omega)} \leq 2\|\nabla u\|^2_{L^2(\Omega)} + 8\lambda_2^2(N_c-1)^2\left(\sum_{i=1}^{N}(1+\frac{\hat{R}_i}{\tilde{R}_i})\frac{\delta_i^l}{\delta_i^s}\frac{\hat{R}_i}{\delta_i^s\sin\theta_{\boldsymbol{X}}}\|\nabla u\|^2_{L^2(U_i)}\right)$$
$$+8\lambda_2^2(N_c-1)^2\left(\sum_{i=1}^{N}(1+\frac{\hat{R}_i}{\tilde{R}_i})\frac{\delta_i^l}{\delta_i^s}\frac{1}{\delta_i^s\hat{R}_i\sin\theta_{\boldsymbol{X}}}\|u\|^2_{L^2(U_i)}\right), \tag{1}$$

where λ_2 *is a universal constant that depends only on the dimension. We further have, for all* $\eta > 0$,

$$\sum_{i=1}^{N}\|\nabla u_i\|^2_{L^2(\Omega)} \le 2\|\nabla u\|^2_{L^2(\Omega)} + 8\lambda_2^2(N_c-1)^2\sum_{i=1}^{N}(1+\frac{\hat{R}_i}{\tilde{R}_i})\frac{\delta_i^l}{\delta_i^s}\frac{\hat{R}_i}{\delta_i^s\sin\theta_{\boldsymbol{X}}}\|\nabla u\|^2_{L^2(U_i)}$$
$$+\frac{8(1+\eta)}{3}\lambda_2^2(N_c-1)^2\sum_{i=1}^{N}(1+\frac{\hat{R}_i}{\tilde{R}_i})\frac{\delta_i^l}{\delta_i^s}\frac{r_i^2}{\delta_i^s\hat{R}_i\sin\theta_{\boldsymbol{X}}}\times$$
$$\times\left(\left(\left(\frac{H_i^2}{r_i^2}+\frac{1}{2}\right)^{\frac{1}{4}}+\frac{H_i}{\sqrt[4]{2}r_i}\right)^4-\frac{1}{2}-\frac{H_i^2}{r_i^2}-\frac{H_i^4}{2r_i^4}\right)\|\nabla u\|^2_{L^2(U_i)}$$
$$+8(1+\frac{1}{\eta})\pi\lambda_2^2(N_c-1)^2\sum_{i=1}^{N}(1+\frac{\hat{R}_i}{\tilde{R}_i})\frac{\delta_i^l}{\delta_i^s}\frac{H_i^2}{\delta_i^s\hat{R}_i\sin\theta_{\boldsymbol{X}}}|\ell_i(u)|^2. \quad (2)$$

Proof. We follow the proof of [Gander et al., 2011, Th. 4.6]. Let ρ be a $\mathcal{C}^\infty$ non negative function whose support is included in the closed unit ball of $\mathbb{R}^2$ and whose L^1 norm is 1. Let $\rho_\varepsilon(\boldsymbol{x}) = \rho(\boldsymbol{x}/\varepsilon)/\varepsilon^2$ for all $\varepsilon > 0$. Let h_i be the characteristic function of the set $\{\boldsymbol{x}\in\mathbb{R}^2, \mathrm{dist}(\boldsymbol{x},U_i) < \delta_i/2\}$. Let $\phi_i = \rho_{\delta_i/2} * h_i$. The function ϕ_i is equal to 1 inside U_i, vanishes outside of $\{\boldsymbol{x}\in\mathbb{R}^2, \mathrm{dist}(\boldsymbol{x},U_i) < \delta_i\}$, and $\|\phi_i\|_\infty \le 2\|\nabla\rho\|_{L^1(\mathbb{R}^2)}/\delta_i$. For i in $\mathbb{N}\cap[1,N]$, let $\psi_i = \phi_i\prod_{k=1}^{i-1}(1-\phi_k)$. We have $0\le\psi_i\le 1$, ψ_i zero in $\Omega\backslash\Omega_i$ and $\sum_i\psi_i = 1$ in Ω. Set $u_i = \psi_i u$. The function u_i is in $H_0^1(\Omega_i)$ and $u=\sum_i u_i$. Following the proof of [Gander et al., 2011, Lemma 4.3], we get $\sum_{i=1}^{N}\|\nabla\psi_i(\boldsymbol{x})\|_2^2 \le 2(N_C-1)\sum_{i=1}^{N}\|\nabla\phi_i(\boldsymbol{x})\|_2^2$. Therefore, for all $\boldsymbol{x}$ in Ω,

$$\sum_{i=1}^{N}\|\nabla\psi_i(\boldsymbol{x})\|_2^2 \le 8(N_c-1)\|\nabla\rho\|^2_{L^1(\mathbb{R}^2)}\sum_{i=1}^{N}\frac{1_{\Omega_i\backslash U_i}(\boldsymbol{x})}{\delta_i^2}.$$

Since $\sum_i\|\nabla u_i\|^2_{L^2(\Omega)} \le 2\|\nabla u\|^2_{L^2(\Omega)}\mathrm{d}\boldsymbol{x} + 2\int_\Omega|u(\boldsymbol{x})|^2\sum_i|\nabla\psi_i(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x}$, we get

$$\sum_{i=1}^{N}\|\nabla u_i\|^2_{L^2(\Omega)} \le 2\|\nabla u\|^2_{L^2(\Omega)}\mathrm{d}\boldsymbol{x}+4\lambda_2^2(N_c-1)^2\sum_{i=1}^{N}\int_{U_i}1\{\mathrm{dist}(\boldsymbol{x},\partial U_i)<\delta_i^l\}\frac{|u(\boldsymbol{x})|^2}{(\delta_i^s)^2}\mathrm{d}\boldsymbol{x},$$

with $\lambda_2 := 2\|\nabla\rho\|_{L^1(\mathbb{R}^2)}$. To get (1), we apply Lemma 4.5 in Gander et al. [2011] to each U_i, and to obtain (2), we apply Lemma 5.10 from the same reference.

To obtain a stable decomposition with a coarse component, we want to construct u_0 in $P_1(\mathcal{T})$ such that for all i, $\ell_i(u_0) = \ell_i(u)$.

3.2 Constructing the coarse component

To construct u_0, we follow the ideas of [Gander et al., 2011, §5.2]. First, we define a special norm.

Definition 1. Let $\mathcal{T}$ be the coarse mesh of domain Ω. Let $\mathcal{V}$ be the set of pairs of neighboring nodes in $\mathcal{T}$, and $\mathcal{B}$ be the set of boundary nodes[1] of $\mathcal{T}$. We define

$$\begin{aligned}\|\cdot\|_{\mathcal{V},\mathcal{B}} : \mathbb{R}^N &\to \mathbb{R}^+,\\ \boldsymbol{y} &\mapsto \sqrt{\sum_{(i,j)\in\mathcal{V}} |y_i - y_j|^2 + \sum_{i\in\mathcal{B}} |y_i|^2}.\end{aligned}$$

When u is in $P_1(\mathcal{T}) \cap H_0^1(\Omega)$, set $\|u\|_{\mathcal{V},\mathcal{B}} := \|(u(\boldsymbol{x}_i))_{1\le i\le N}\|_{\mathcal{V},\mathcal{B}}$, where the $\boldsymbol{x}_i$ are the interior nodes of the mesh $\mathcal{T}$.

Lemma 2. *For u in $H_0^1(\Omega)$, there exists u_0 in $P_1(\mathcal{T}) \cap H_0^1(\Omega)$ such that, for all i in $\{1,\dots,N\}$, $\ell_i(u_0) = \ell_i(u)$ and*

$$\|\nabla u_0\|^2_{L^2(\Omega)} \le \frac{1}{\tan\theta_{min}} \frac{\left(1 + 2\max_i(\frac{r_i}{H_{h,i}})\right) K\left(\frac{25}{6\pi}\max_i(\frac{d_i}{r_i}) + 2\pi\right)}{1 - \left((2K+1) + (4K+1)\max_i(\frac{r_i}{H_{h,i}})\right)\max_i(\frac{r_i}{H_{h,i}})}.$$

Proof. The results of [Gander et al., 2011, Lemmas 5.6,and 5.8] stand without modifications. Therefore u_0 exists, and we have

$$\|\nabla u_0\|^2_{L^2(\Omega)} \le \frac{1}{\tan\theta_{min}} \frac{1 + 2\max_i(\frac{r_i}{H_{h,i}})}{1 - \left((2K+1) + (4K+1)\max_i(\frac{r_i}{H_{h,i}})\right)\max_i(\frac{r_i}{H_{h,i}})} \|u\|^2_{\mathcal{V},\mathcal{B}}.$$

Note that the condition $r_i \le \frac{H_{h,i}}{4K+1}$ ensures that $1 - \left((2K+1) + (4K+1)\max_i(r_i/H_{h,i})\right)\max_i(r_i/H_{h,i})$ remains positive. It remains to compare $\|u\|^2_{\mathcal{V},\mathcal{B}}$ and $\|\nabla u\|^2_{L^2(\Omega)}$. We need to adapt the proof of [Gander et al., 2011, Lemma 5.7]. We can suppose without any loss of generality that u is in $\mathcal{C}^\infty(\overline{\Omega})$. Let i, j in $\{1,\dots,N\}$ be indices of neighboring nodes of $\mathcal{T}$. Let $\boldsymbol{d}_{ij} = \boldsymbol{x}_i - \boldsymbol{x}_j$, and $d_{ij} = \|\boldsymbol{d}_{ij}\|$. We have for all $(i,j) \in \mathcal{V}$

[1] The nodes that are located on $\partial\Omega$ are not numbered among $\{1,\dots,N\}$, and $\mathcal{B}$ contains only the nodes which are neighbor to a node located on $\partial\Omega$.

$$
\begin{aligned}
|\ell_i(u)-\ell_j(u)|^2 &= \frac{1}{\pi^2}\left(\int_{B(\mathbf{0},1)}(u(\boldsymbol{x}_i+r_i\boldsymbol{y})-u(\boldsymbol{x}_j+r_j\boldsymbol{y}))\mathrm{d}\boldsymbol{y}\right)^2\\
&\le \frac{1}{\pi}\int_{B(\mathbf{0},1)}\int_0^1\|\nabla u\big(t(\boldsymbol{x}_i+r_i\boldsymbol{y})+(1-t)(\boldsymbol{x}_j+r_j\boldsymbol{y})\big)\|_2^2\|\boldsymbol{x}_i-\boldsymbol{x_j}+(r_i-r_j)\boldsymbol{y}\|_2^2\mathrm{d}t\mathrm{d}\boldsymbol{y}\\
&\le \frac{(d_{ij}+|r_i-r_j|)^2}{\pi}\int_{B(\mathbf{0},1)}\int_0^1\|\nabla u\big(t(\boldsymbol{x}_i+r_i\boldsymbol{y})+(1-t)(\boldsymbol{x}_j+r_j\boldsymbol{y})\big)\|_2^2\mathrm{d}t\mathrm{d}\boldsymbol{y}\\
&\le \frac{(d_{ij}+|r_i-r_j|)^2}{\pi}\int_{T_{i,j}}\|\nabla u(\boldsymbol{y'})\|_2^2\int_0^1\frac{1\{\|\boldsymbol{y'}-t\boldsymbol{x}_i-(1-t)\boldsymbol{x}_j\|\le tr_i+(1-t)r_j\}}{(tr_i+(1-t)r_j)^2}\mathrm{d}t\mathrm{d}\boldsymbol{y'},
\end{aligned}
$$

where the tube $T_{i,j}$ is the convex hull of $B(\boldsymbol{x}_i,r_i)\cup B(\boldsymbol{x}_j,r_j)$. We get

$$
\begin{aligned}
&\max_{\boldsymbol{y}\in\mathbb{R}^2}\int_0^1\frac{1\{\|\boldsymbol{y'}-t\boldsymbol{x}_i-(1-t)\boldsymbol{x}_j\|\le tr_i+(1-t)r_j\}}{(tr_i+(1-t)r_j)^2}\mathrm{d}t\\
&\quad=\max_{(s,\sigma)\in\mathbb{R}^2}\int_0^1\frac{1\{\sqrt{(s-td_{ij})^2+\sigma^2}\le tr_i+(1-t)r_j\}}{(tr_i+(1-t)r_j)^2}\mathrm{d}t\\
&\quad=\max_{s\in[-r_j,d_{ij}+r_i]}\int_0^1\frac{1\{|s-td_{ij}|\le tr_i+(1-t)r_j\}}{(tr_i+(1-t)r_j)^2}\mathrm{d}t\\
&\quad\le\max_{s\in[-r_j,d_{ij}+r_i]}\left(\frac{2}{d_{ij}r_j+s(r_i-r_j)}\right)\\
&\quad=\frac{2}{\min(r_i,r_j)(d_{ij}-|r_i-r_j|)}.
\end{aligned}
$$

Since $d_{ij}\ge H_{h,i}\ge 4\max(r_i,r_j)$, $|\ell_i(u)-\ell_j(u)|^2\le 25d_{ij}/(6\pi\min(r_i,r_j))\|\nabla u\|^2_{L^2(T_{ij})}$.

If i is in the boundary set of the coarse mesh, then the node $\boldsymbol{x}_i$ is neighbor to a node $\boldsymbol{x}_{i'}$ located on $\partial\Omega$. Note that i' lies outside of the range $\{1,\ldots,N\}$. Using [Gander et al., 2011, Eqs (5.7) and (5.9)], we get

$$
\sum_{i\in\mathcal{B}}|\ell_i(u)|\le\left(\sum_{i\in\mathcal{B}}\frac{4\|\boldsymbol{x}_i-\boldsymbol{x}_{i'}\|}{\pi r_i}\int_{T_i'}\|\nabla u(\boldsymbol{x})\|^2\mathrm{d}\boldsymbol{x}\right)+2K\pi\|\nabla u\|^2_{L^2(\Omega)},\qquad(3)
$$

where T_i' is the convex hull of $B(\boldsymbol{x}_i,r_i)\cup B(\boldsymbol{x}_{i'},r_i)$. We sum $|\ell_i(u)-\ell_j(u)|^2\le 25d_{ij}/(6\pi\min(r_i,r_j))\|\nabla u\|^2_{L^2(T_{ij})}$ over all i,j in the neighbor set and combine it with equation (3). Since $\max(r_i,r_j)\le H'_{h,i}/2\le\min(H_{h,i},H_{h,j}))/2$, no point can belong to more than K tubes $T_{i,j}$ or T_i'. Therefore $\|u\|^2_{\mathcal{V},\mathcal{B}}\le K\big(25\max_i(d_i/r_i)/(6\pi)+2\pi\big)\|\nabla u\|^2_{L^2(\Omega)}$. This concludes the proof.

To prove Theorem 1, we use Lemma 2 to construct the coarse component u_0. We then apply Lemma 1 to $u-u_0$ to get the fine components u_i. The terms in $\ell_i(u)$ vanish.

Conclusion

We have proven the existence of a stable decomposition of the Sobolev space $H_0^1(\Omega)$ in the presence of a coarse mesh when the domain decomposition is only guaranteed to be locally shape regular. We provided an explicit upper bound for the stable decomposition that depends neither on $\max_i(H_i)/\min_i(H_i)$, nor on the number of subdomains.

Establishing the existence of a stable decomposition with a uniform upper bound that does not explode when $\max_i(H_i)/\min_i(H_i)$ does would not have been possible without the explicit upper bounds provided in Gander et al. [2011]. This shows that deriving such explicit upper bounds can be important for problems arising naturally in applications, i.e. load balanced domain decompositions with local refinement.

References

Tony F. Chan and Tarek P. Mathew. Domain decomposition algorithms. In *Acta Numerica 1994*, pages 61–143. Cambridge University Press, 1994.

Maksymilian Dryja and Olof B. Widlund. An additive variant of the Schwarz alternating method for the case of many subregions. Technical Report 339, also Ultracomputer Note 131, Department of Computer Science, Courant Institute, 1987.

Martin Gander, Laurence Halpern, and Kévin Santugini-Repiquet. Continuous Analysis of the Additive Schwarz Method: a Stable Decomposition in H^1. *Submitted*, 2011. URL `http://hal.archives-ouvertes.fr/hal-00462006/fr/`.

Alfio Quarteroni and Alberto Valli. *Domain Decomposition Methods for Partial Differential Equations*. Oxford Science Publications, 1999.

H. A. Schwarz. Über einen Grenzübergang durch alternierendes Verfahren. *Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich*, 15:272–286, May 1870.

Barry F. Smith, Petter E. Bjørstad, and William Gropp. *Domain Decomposition: Parallel Multilevel Methods for Elliptic Partial Differential Equations*. Cambridge University Press, 1996.

Andrea Toselli and Olof Widlund. *Domain Decomposition Methods - Algorithms and Theory*, volume 34 of *Springer Series in Computational Mathematics*. Springer, 2004.

Jinchao Xu. Iterative methods by space decomposition and subspace correction. *SIAM Review*, 34(4):581–613, December 1992.