Optimized RAS Methods

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Summary. A small modification of additive, multiplicative and restricted additive Schwarz preconditioners at the algebraic level, motivated by continuous optimized Schwarz methods, leads to a greatly improved convergence rate of the iterative solver. These modifications imply specific compatibility conditions at subdomain boundaries. Numerical experiments using finite difference and spectral element discretizations of the modified Helmholtz problem $u - \Delta u = f$ illustrate the effectiveness of the new approach.

1 Introduction

Optimized Schwarz methods were originally derived from Fourier analysis of the continuous elliptic partial differential equation [5, 4]. Until now it was not clear how to introduce optimized transmission conditions in the additive (AS), multiplicative (MS) and restricted additive (RAS) forms of Schwarz preconditioners. We show that small modifications of the subdomain matrices in these preconditioners leads to optimized Schwarz methods. These also lead to specific compatibility conditions at subdomain boundaries. For optimized RAS (ORAS), an overlap condition must be satisfied. In the case of optimized MS (OMS), there is no such condition on the overlap. Whereas optimized AS (OAS) requires an enhanced system, permitting the use of non-overlapping regions, where the common unknowns at subdomain boundaries are duplicated. Because our results are algebraic, they apply to any space discretization of the continuous PDE. We show how these methods can be applied to discretisations of the modified Helmholtz problem $u - \Delta u = f$, discretized with finite differences and spectral finite elements, by a simple modification of already existing classical implementations of RAS, MS and AS, with and without overlap.

2 Schwarz Methods at the Algebraic Level

The discretization of an elliptic partial differential equation leads to a linear system of equations

$$A\mathbf{u} = \mathbf{f}. \tag{1}$$

A stationary iterative method for (1) is given by

$$\mathbf{u}^{n+1} = \mathbf{u}^n + M^{-1}(\mathbf{f} - A\mathbf{u}^n). \tag{2}$$

An initial guess $\mathbf{u}^0$ is required to start the iteration. Algebraic domain decomposition methods group the unknowns into subsets, $\mathbf{u}_j = R_j\mathbf{u}$, $j = 1, \dots, J$, where R_j are rectangular matrices. Coefficient matrices for subdomain problems are defined by $A_j = R_jAR_j^T$. The classical multiplicative Schwarz method (MS) is a stationary iteration (2) where the preconditioner M is

$$M_{MS}^{-1} = A^{-1} - \prod_{j=1}^{J} \left(I - R_j^T A_j^{-1} R_j\right) A^{-1}. \tag{3}$$

Multiplicative Schwarz is equivalent to a Gauss-Seidel iteration on the subdomains [10] and thus corresponds to the original alternating method introduced by Schwarz at the continuous level [9]. A more parallel variant is the additive Schwarz (AS) preconditioner introduced in [2],

$$M_{AS}^{-1} = \sum_{j=1}^{J} R_j^T A_j^{-1} R_j. \tag{4}$$

In general, the additive Schwarz method does not converge, nor does it correspond to a classical iteration per subdomain, [3]. The Restricted Additive Schwarz method (RAS) [1] corrects this problem by introducing prolongation matrices $\tilde{R}_j$, corresponding to a non-overlapping decomposition. Each entry u_l of the vector $\mathbf{u}$ occurs in $\tilde{R}_j\mathbf{u}$ for exactly one j. The RAS preconditioner is

$$M_{RAS}^{-1} = \sum_{j=1}^{J} \tilde{R}_j^T A_j^{-1} R_j. \tag{5}$$

Another way to alleviate the convergence problems of AS is to apply M_{AS} as a preconditioner in a Krylov iteration. In general, MS and RAS will be accelerated by Krylov iteration.

The algebraic formulation of Schwarz methods has an important feature: a subdomain matrix A_j is not necessarily the restriction of A to a subdomain j. For example, if A represents a spectral element discretization of a differential operator, then A_j can be obtained from a finite element discretization at the collocation points. Furthermore, subdomain matrices A_j can be chosen to accelerate convergence and this is the focus of the next section.

3 Optimized Schwarz Methods

It was shown in [3] that the original alternating Schwarz algorithm [9] is equivalent to MS in (3), and that the discrete form of the Jacobi variant of the algorithm introduced in [6] is equivalent to RAS in (5). In optimized algorithms, the subdomain matrices A_j are replaced by $\tilde{A}_j$ and the transmission matrices B_j are replaced by $\tilde{B}_j$, corresponding to optimal transmission conditions. The discrete form of the optimized alternating Schwarz algorithm for two subdomains is given by

$$\tilde{A}_1 \mathbf{u}_1^{n+1} = \mathbf{f}_1 + \tilde{B}_1 \mathbf{u}_2^n, \quad \tilde{A}_2 \mathbf{u}_2^{n+1} = \mathbf{f}_2 + \tilde{B}_2 \mathbf{u}_1^{n+1}, \tag{6}$$

and the optimized form of the discrete Jacobi algorithm is

$$\tilde{A}_1 \mathbf{u}_1^{n+1} = \mathbf{f}_1 + \tilde{B}_1 \mathbf{u}_2^n, \quad \tilde{A}_2 \mathbf{u}_2^{n+1} = \mathbf{f}_2 + \tilde{B}_2 \mathbf{u}_1^{n}. \tag{7}$$

3.1 Optimized Restricted Additive Schwarz Methods

Here it is shown that for sufficient overlap, the subdomain matrices A_j in the RAS algorithm (5) can be replaced by the matrices $\tilde{A}_j$,

$$\mathbf{u}^{n+1} = \mathbf{u}^n + (\sum_{j=1}^{2} \tilde{R}_j^T \tilde{A}_j^{-1} R_j)(\mathbf{f} - A\mathbf{u}^n), \tag{8}$$

to obtain an optimized RAS method (ORAS) and thus the matrices $\tilde{B}_j$ in (7) are not needed in (8). The iteration (7) is consistent if it converges to the solution $\mathbf{u}$ of (1) for all $\mathbf{f}$.

Lemma 1. *Let A in (1) have full rank. For a consistent matrix splitting R_j, $\tilde{A}_j$, $\tilde{B}_j$, the following matrix identities hold*

$$\tilde{A}_1 R_1 - \tilde{B}_1 R_2 = R_1 A, \quad \tilde{A}_2 R_2 - \tilde{B}_2 R_1 = R_2 A. \tag{9}$$

Proof. Apply R_1 to (1) and by consistency, $R_1 A\mathbf{u} = R_1 \mathbf{f} = \mathbf{f}_1 = \tilde{A}_1 \mathbf{u}_1 - \tilde{B}_1 \mathbf{u}_2$. Applying $\mathbf{u}_1 = R_1 \mathbf{u}$ and $\mathbf{u}_2 = R_2 \mathbf{u}$ on the rhs, $(\tilde{A}_1 R_1 - \tilde{B}_1 R_2 - R_1 A)\mathbf{u} = 0$. $\mathbf{f}$ is arbitrary and both identities hold for all $\mathbf{u}$.

Lemma 2. *Let R_j cover the entire discrete domain and let $\tilde{R}_j$ be the corresponding RAS matrices. If $\tilde{B}_1 R_2 \tilde{R}_1^T = 0$, then $\tilde{B}_1 R_2 \tilde{R}_2^T = \tilde{B}_1$, and if $\tilde{B}_2 R_1 \tilde{R}_2^T = 0$, then $\tilde{B}_2 R_1 \tilde{R}_1^T = \tilde{B}_2$.*

Proof. From the definition of non-overlapping $\tilde{R}_j$,

$$I = \tilde{R}_1^T \tilde{R}_1 + \tilde{R}_2^T \tilde{R}_2. \tag{10}$$

Multiplying $\tilde{B}_1 R_2 \tilde{R}_1^T = 0$ on the right by R_1, and using (10)

$$(\tilde{B}_1 - \tilde{B}_1 R_2 \tilde{R}_2^T) R_2 = 0,$$

which completes the proof, because R_2 has full rank.

Theorem 1. *Let R_j, $\tilde{A}_j$, $\tilde{B}_j$ be a consistent matrix splitting, and let $\tilde{R}_j$ be the corresponding RAS versions of R_j. If the initial iterates $\mathbf{u}_j^0$ of (7) and the initial iterate $\mathbf{u}^0$ of (8) satisfy*

$$\mathbf{u}^0 = \tilde{R}_1^T \mathbf{u}_1^0 + \tilde{R}_2^T \mathbf{u}_2^0, \tag{11}$$

and if the overlap condition

$$\tilde{B}_1 R_2 \tilde{R}_1^T = 0, \quad \tilde{B}_2 R_1 \tilde{R}_2^T = 0 \tag{12}$$

is satisfied, then (7) and (8) generate an identical sequence of iterates,

$$\mathbf{u}^n = \tilde{R}_1^T \mathbf{u}_1^n + \tilde{R}_2^T \mathbf{u}_2^n. \tag{13}$$

Proof. The proof is by induction. For $n = 0$, we have (13) by assumption (11) on the initial iterates. Assume that $\mathbf{u}^n = \tilde{R}_1^T \mathbf{u}_1^n + \tilde{R}_2^T \mathbf{u}_2^n$ and show that the identity (13) holds for $n+1$. Applying Lemma 1 to the first term of (8),

$$\begin{aligned} \tilde{R}_1^T \tilde{A}_1^{-1} R_1 (\mathbf{f} - A\mathbf{u}^n) &= \tilde{R}_1^T \tilde{A}_1^{-1} (\mathbf{f}_1 - R_1 A \mathbf{u}^n) \\ &= \tilde{R}_1^T \tilde{A}_1^{-1} (\mathbf{f}_1 - (\tilde{A}_1 R_1 - \tilde{B}_1 R_2)\mathbf{u}^n) \\ &= \tilde{R}_1^T (\tilde{A}_1^{-1} \mathbf{f}_1 - R_1 \mathbf{u}^n + \tilde{A}_1^{-1} \tilde{B}_1 R_2 \mathbf{u}^n), \end{aligned} \tag{14}$$

and similarly for the second term of the sum,

$$\tilde{R}_2^T \tilde{A}_2^{-1} R_2 (\mathbf{f} - A\mathbf{u}^n) = \tilde{R}_2^T (\tilde{A}_2^{-1} \mathbf{f}_2 - R_2 \mathbf{u}^n + \tilde{A}_2^{-1} \tilde{B}_2 R_1 \mathbf{u}^n). \tag{15}$$

Substituting these into (8) and invoking (10)

$$\mathbf{u}^{n+1} = \tilde{R}_1^T (\tilde{A}_1^{-1} (\mathbf{f}_1 + \tilde{B}_1 R_2 \mathbf{u}^n)) + \tilde{R}_2^T (\tilde{A}_2^{-1} (\mathbf{f}_2 + \tilde{B}_2 R_1 \mathbf{u}^n)).$$

By induction hypothesis, replacing $\mathbf{u}^n$ by $\tilde{R}_1^T \mathbf{u}_1^n + \tilde{R}_2^T \mathbf{u}_2^n$ on the rhs and applying Lemma 2 with (12)

$$\begin{aligned} \mathbf{u}^{n+1} &= \tilde{R}_1^T (\tilde{A}_1^{-1} (\mathbf{f}_1 + \tilde{B}_1 R_2 (\tilde{R}_1^T \mathbf{u}_1^n + \tilde{R}_2^T \mathbf{u}_2^n))) \\ &+ \tilde{R}_2^T (\tilde{A}_2^{-1} (\mathbf{f}_2 + \tilde{B}_2 R_1 (\tilde{R}_1^T \mathbf{u}_1^n + \tilde{R}_2^T \mathbf{u}_2^n))) \\ &= \tilde{R}_1^T (\tilde{A}_1^{-1} (\mathbf{f}_1 + \tilde{B}_1 \mathbf{u}_2^n)) + \tilde{R}_2^T (\tilde{A}_2^{-1} (\mathbf{f}_2 + \tilde{B}_2 \mathbf{u}_1^n)), \end{aligned}$$

together with (7) implies $\mathbf{u}^{n+1} = \tilde{R}_1^T \mathbf{u}_1^{n+1} + \tilde{R}_2^T \mathbf{u}_2^{n+1}$.

Similar results can be derived in the case of optimized multiplicative (OMS) Schwarz methods and these will be presented in a future paper.

3.2 Optimized Additive Schwarz Methods for Enhanced Systems

To develop parallel optimized Schwarz methods, an enhanced system must be introduced, where overlap and interface unknowns are duplicated. In contrast with ORAS, these can be applied in a non-overlapping fashion. At convergence, (7) results in the enhanced system [11]

$$\begin{bmatrix} \tilde{A}_1 & -\tilde{B}_1 \\ -\tilde{B}_2 & \tilde{A}_2 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix} \tag{16}$$

or $\tilde{A}\mathbf{u} = \tilde{\mathbf{f}}$. Applying a standard additive Schwarz method with non-overlapping matrices $\hat{R}_j$,

$$\hat{R}_1 \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} = \mathbf{u}_1 \quad \text{and} \quad \hat{R}_2 \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} = \mathbf{u}_2, \tag{17}$$

to the enhanced system (16), results in the iterative method

$$\mathbf{u}^{n+1} = \mathbf{u}^n + (\hat{R}_1^T \tilde{A}_1^{-1} \hat{R}_1 + \hat{R}_2^T \tilde{A}_2^{-1} \hat{R}_2)(\mathbf{f} - \tilde{A} u^n). \tag{18}$$

The non-overlapping matrices $\hat{R}_j$ are related to the restriction matrices for the consistent splitting R_j, $\tilde{A}_j$, $\tilde{B}_j$. Their range is identical and subdomains are enlarged to account for the matrix $\tilde{A}$. The advantage of this formulation is that non-overlapping preconditioners can be employed. A disadvantage is that the $\tilde{B}_j$ must be explicitly constructed as they are needed in (18).

3.3 Schur Complement Matrices

To determine the optimal form of the subdomain matrices $\tilde{A}_j$ and $\tilde{B}_j$, consider a partition of (1) into two blocks with a common interface

$$\begin{bmatrix} A_{1i} & C_1 & \\ B_2 & A_\Gamma & B_1 \\ & C_2 & A_{2i} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{1i} \\ \mathbf{u}_\Gamma \\ \mathbf{u}_{2i} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{1i} \\ \mathbf{f}_\Gamma \\ \mathbf{f}_{2i} \end{bmatrix},$$

where A_{1i} and A_{2i} correspond to the interior unknowns and A_Γ to the interface unknowns. The classical Schwarz iteration (7) thus becomes

$$\begin{bmatrix} A_{1i} & C_1 \\ B_2 & A_\Gamma \end{bmatrix} \begin{bmatrix} \mathbf{u}_{1i}^{n+1} \\ \mathbf{u}_{1\Gamma}^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{1i} \\ \mathbf{f}_\Gamma - B_1 \mathbf{u}_{2i}^n \end{bmatrix}, \quad \begin{bmatrix} A_\Gamma & B_1 \\ C_2 & A_{2i} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{2\Gamma}^{n+1} \\ \mathbf{u}_{2i}^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_\Gamma - B_2 \mathbf{u}_{1i}^n \\ \mathbf{f}_{2i} \end{bmatrix}.$$

At convergence, eliminating the unknowns $\mathbf{u}_{2i}$ on the first subdomain gives

$$\begin{bmatrix} A_{1i} & C_1 \\ B_2 & A_\Gamma - B_1 A_{2i}^{-1} C_2 \end{bmatrix} \begin{bmatrix} \mathbf{u}_{1i} \\ \mathbf{u}_{1\Gamma} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{1i} \\ \mathbf{f}_\Gamma - B_1 A_{2i}^{-1} \mathbf{f}_{2i} \end{bmatrix},$$

and $\mathbf{f}_{2i}$ can be expressed in terms of the unknowns on the second subdomain

$$\mathbf{f}_{2i} = C_2 \mathbf{u}_{2\Gamma} + A_{2i} \mathbf{u}_{2i}.$$

Similarly, eliminating the unknowns $\mathbf{u}_{1i}$ on the second subdomain, yields

$$\begin{aligned} \begin{bmatrix} A_{1i} & C_1 \\ B_2 & A_\Gamma - B_1 A_{2i}^{-1} C_2 \end{bmatrix} \begin{bmatrix} \mathbf{u}_{1i}^{n+1} \\ \mathbf{u}_{1\Gamma}^{n+1} \end{bmatrix} &= \begin{bmatrix} \mathbf{f}_{1i} \\ \mathbf{f}_\Gamma - B_1 \mathbf{u}_{2i}^n - B_1 A_{2i}^{-1} C_2 \mathbf{u}_{2\Gamma}^n \end{bmatrix}, \\ \begin{bmatrix} A_\Gamma - B_2 A_{1i}^{-1} C_1 & B_1 \\ C_2 & A_{2i} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{2\Gamma}^{n+1} \\ \mathbf{u}_{2i}^{n+1} \end{bmatrix} &= \begin{bmatrix} \mathbf{f}_\Gamma - B_2 \mathbf{u}_{1i}^n - B_2 A_{1i}^{-1} C_1 \mathbf{u}_{1\Gamma}^n \\ \mathbf{f}_{2i} \end{bmatrix}. \end{aligned} \tag{19}$$

It is straightforward to show that the above method converges in two iterations. $\tilde{A}_1$ is the matrix obtained by subtracting $B_1 A_{2i}^{-1} C_2$ from the interface block and similarly for $\tilde{A}_2$. These matrices are dense and expensive to compute. However, sparse approximations based on the underlying PDE can be computed using discretized local operators acting at the interface [4].

4 Numerical Results

The optimized Schwarz algorithms are now applied to finite difference and spectral element discretizations of the modified Helmholtz problem on the domain Ω,

$$\mathcal{L}u = (\eta - \Delta)u = f, \quad \text{in } \Omega, \tag{20}$$

where Ω is an open set in two dimensions with the appropriate boundary conditions. Discretization of (20) using a standard five point finite difference stencil on an equidistant grid on the domain $\Omega = (0,1) \times (0,1)$ with homogeneous Dirichlet boundary conditions leads to the matrix problem

$$A^{FD}\mathbf{u} = \mathbf{f}, \quad A^{FD} = \frac{1}{h^2}\begin{bmatrix} T_\eta & -I & \\ -I & T_\eta & \ddots \\ & \ddots & \ddots \end{bmatrix}, \quad T_\eta = \begin{bmatrix} \eta h^2 + 4 & -1 & \\ -1 & \eta h^2 + 4 & \ddots \\ & \ddots & \ddots \end{bmatrix}.$$

Figure 1 illustrates the effect of replacing the interface blocks on the performance of the RAS iteration for the model problem on the unit square with $\eta = 1$ and $h = 1/30$. The asymptotic formulas from [4] were employed for the various choices of the parameters appearing in the discretized local operators acting at the interface. Clearly, the convergence of RAS is greatly accelerated and the number of operations per iteration is identical. Figure 2, shows the impact of the optimized subdomain matrices on the performance of multiplicative Schwarz (MS). As expected, convergence is about two times faster than RAS. In the case of no overlap, Taylor transmission conditions do not significantly improve the convergence rate.

In a nodal spectral element discretization, the computational domain Ω is partitioned into K elements Ω_k in which u is expanded in terms of the N–th degree Lagrangian interpolants h_i defined in Ronquist [7]. A weak variational problem is obtained by integrating the equation with respect to test functions and directly evaluating inner products using Gaussian quadrature. The model problem (20) is discretized on the domain $\Omega = (0,2) \times (0,4)$ with periodic boundary conditions. The rhs is constructed to be C^0 along element boundaries as displayed in Figure 3. Non-overlapping Schwarz methods are well-suited to spectral element discretizations. Here, a zero-th order optimized transmission condition is employed in the enhanced system. The resulting optimized Schwarz iteration is accelerated by a generalized minimal residual (GMRES) Krylov method [8]. Figure 3 also contains a plot of the residual

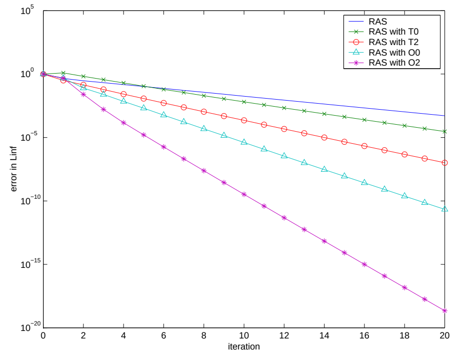

Fig. 1. Convergence curves of classical RAS with overlap $3h$, compared to ORAS methods with overlap h. Taylor optimized zero-th order (T0) and second order (T2). RAS optimized zero-th and second order.

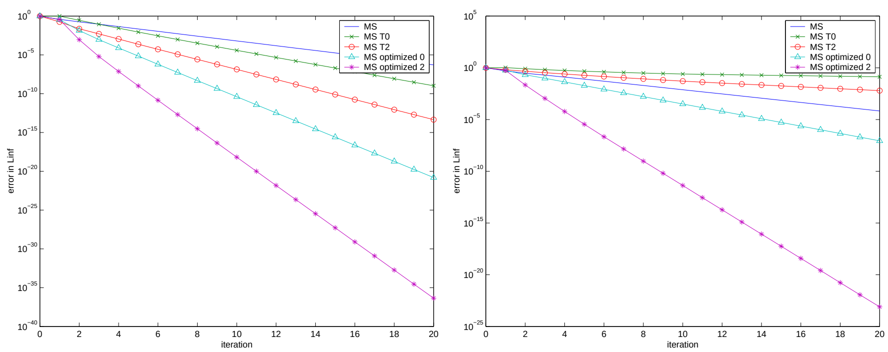

Fig. 2. Convergence curves of classical MS compared to OMS methods. Left: overlap h ($3h$ for classical MS). Right: no overlap ($2h$ for classical MS).

error versus the number of GMRES iterations for diagonal (the inverse mass matrix) and optimized Schwarz preconditioning.

5 Conclusion

We demonstrated that a simple algebraic modification of the subdomain matrices arising in the restrictive additive (RAS), and multiplicative (MS) Schwarz methods leads to greatly accelerated convergence rates. Both overlapping and non-overlapping variants were examined. In the case of two subdomains, the optimal algebraic modification is obtained directly from the Schur complement at the interface. Application of these results to finite difference and spectral element discretizations of the modified Helmholtz problem confirms our theoretical results.

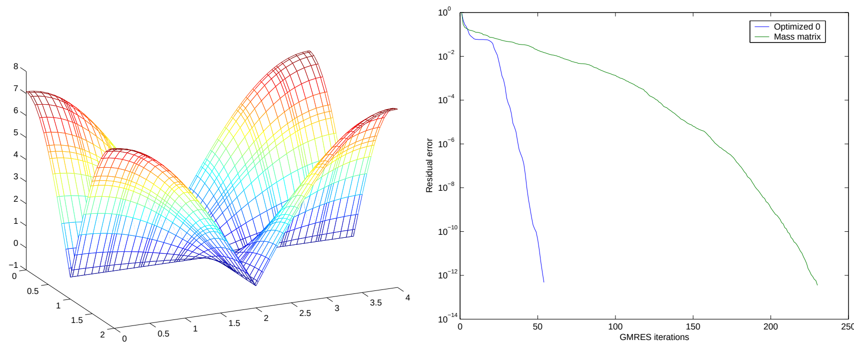

Fig. 3. Left panel: Right-hand side of modified Helmholtz problem. Right panel: Residual error versus GMRES iterations.

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