Techniques for Locally Adaptive Timestepping Developped over the Last Two Decades

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Joint work with Laurence Halpern

Overview

- Methods from the ODE community:
  - Split Runge-Kutta methods
  - Multirate Methods
  - Multirate Extrapolation Methods
- Methods from the PDE community:
  - Hyperbolic Problems
    - Interpolation based
    - Energy conservation
    - Symplectic methods
  - Parabolic Problems
    - Explicit-Implicit
    - Fully Implicit
  - Space-Time Finite Elements
  - One-Way and Two-Way methods
  - Schwarz Waveform Relaxation

Multirate Methods

Gear, Wells (1984): Multirate linear multistep methods
Partitioning of the system of ODEs into slow and fast components:

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y' &= b(y, z, t)\\
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"Hence the values of y will have to be approximated by interpolation from mesh values of y. This process will cost approximately the same amount of computer time as the prediction step in a predictor-corrector process, because the prediction process is a linear combination of past values."

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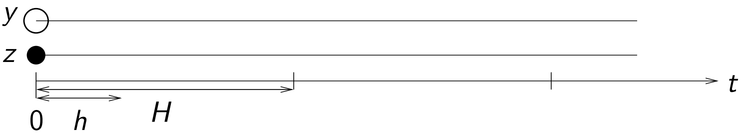

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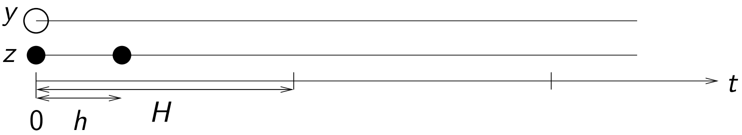

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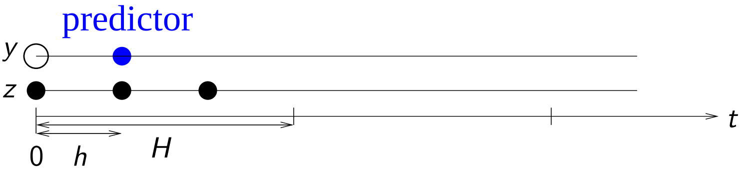

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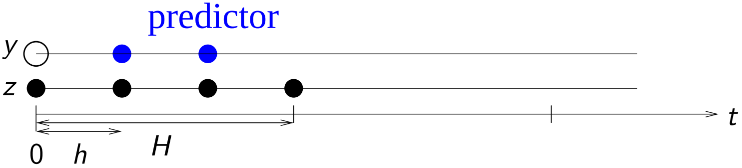

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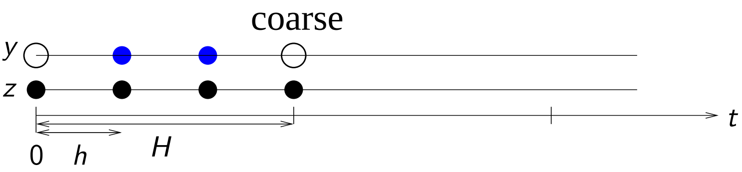

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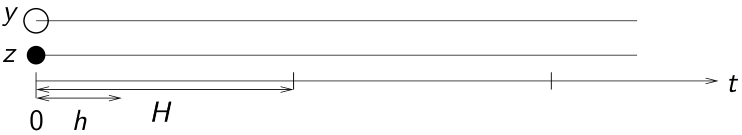

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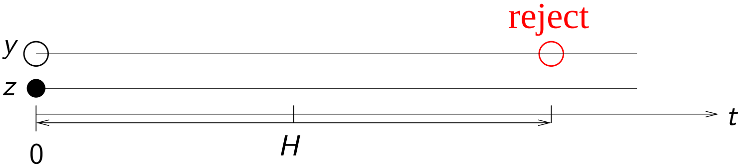

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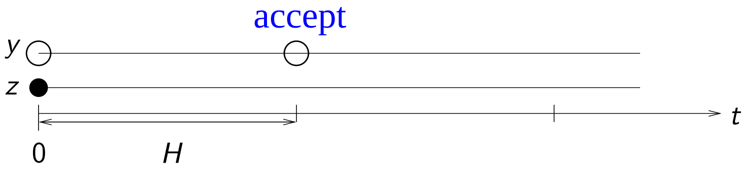

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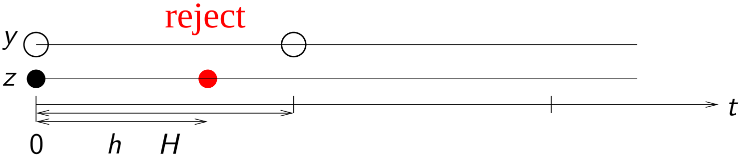

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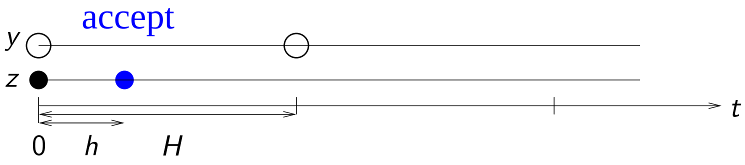

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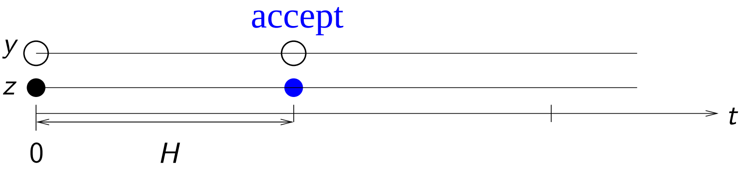

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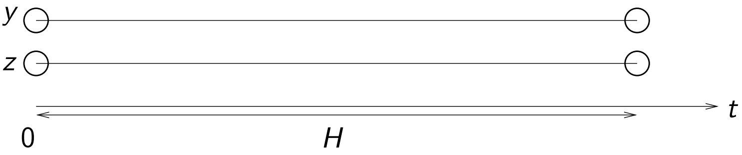
"There are several possible ways to control the fast extrapolation error, none of which is entirely satisfactory"

Multirate Extrapolation Methods

Engstler, Lubich (1996): Multirate Extrapolation Methods for Differential Equations with Different Time Scales

Based on Richardson extrapolation:



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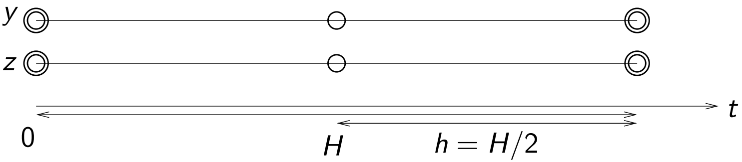

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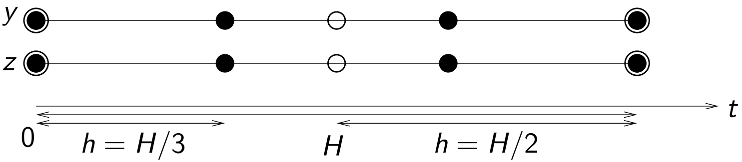

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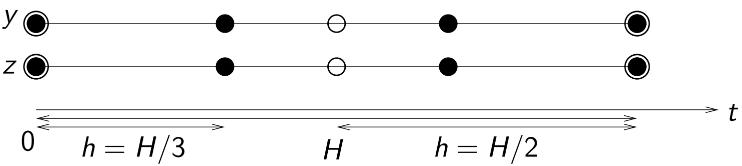
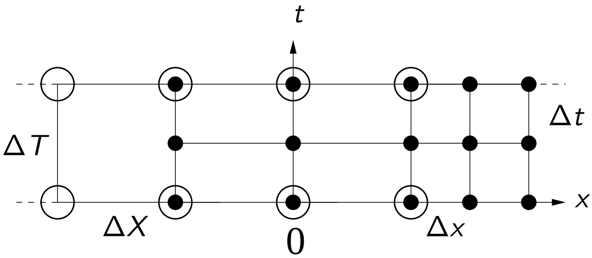


- As soon as a component has reached the desired accuracy at H, extrapolation for this component is marked inactive
- Inactive components must then be either
  - interpolated from the continuous approximation obtained from the extrapolation method (can destroy extrapolation!)
  - approximated from the asymptotic expansion assumption
- Inactivation can fail, need defect control

Hyperbolic Problems: Many Experimental Papers

Löhner, Morgan, Zienkiewicz (1984): The Use of Domain Splitting with an Explicit Hyperbolic Solver

For explicit schemes and general systems of hyperbolic problems



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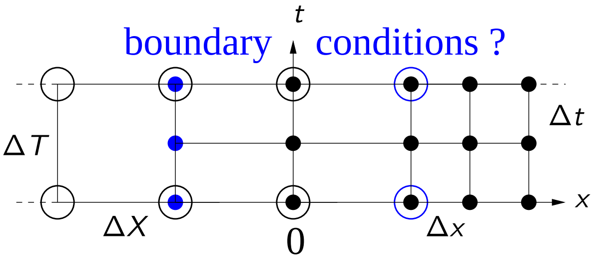

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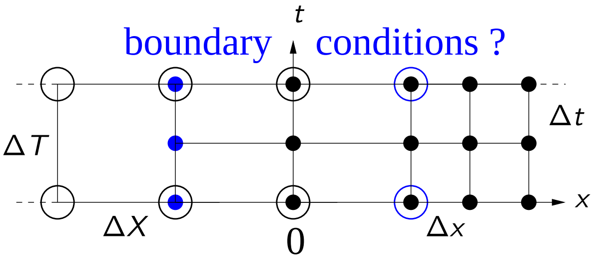


Can keep u fixed or free at subdomain boundaries

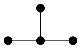
Numerical Results:

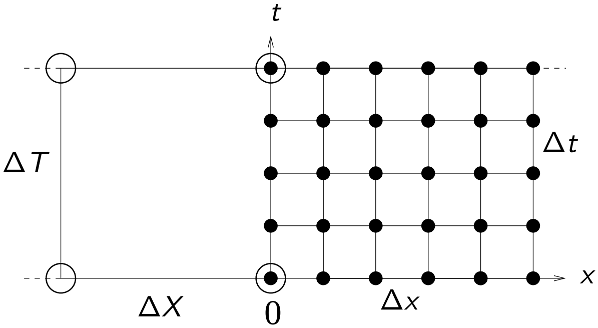
For simple linear advection in two dimensions:

> *"As one can see, the best solution is obtained when u is left free at soubdomain boundaries".*

First Analytical Results for Local Time Stepping

Marsha J. Berger (1985) Stability of Interfaces with Mesh Refinement

For explicit 3 point schemes 



Using interpolation

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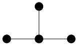
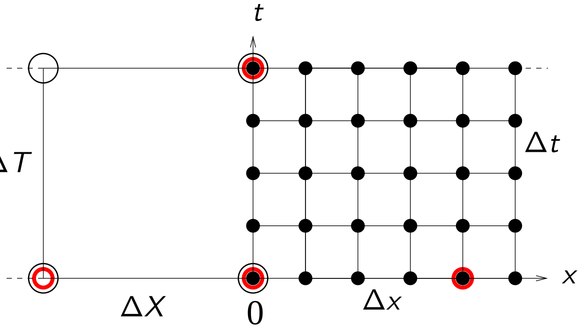
For explicit 3 point schemes 



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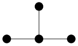
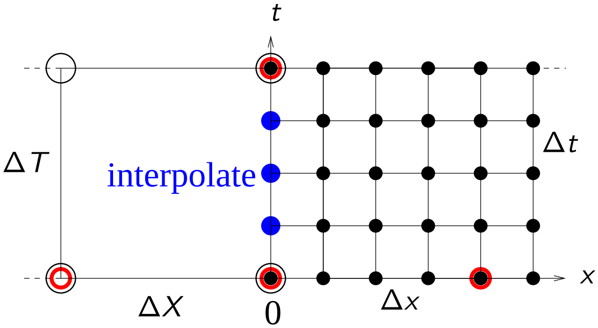
For explicit 3 point schemes 



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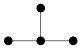
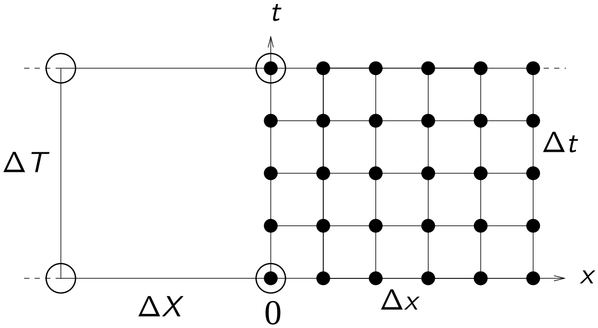
For explicit 3 point schemes 



Coarse Mesh Approximation Method

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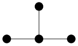
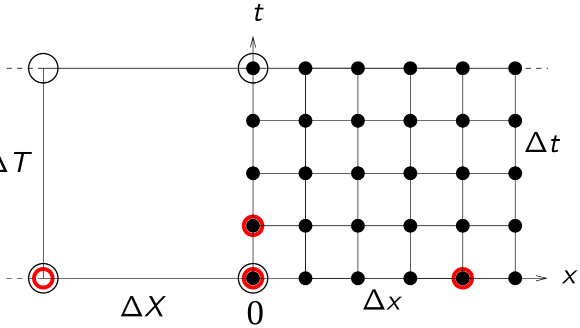
For explicit 3 point schemes 



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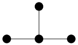

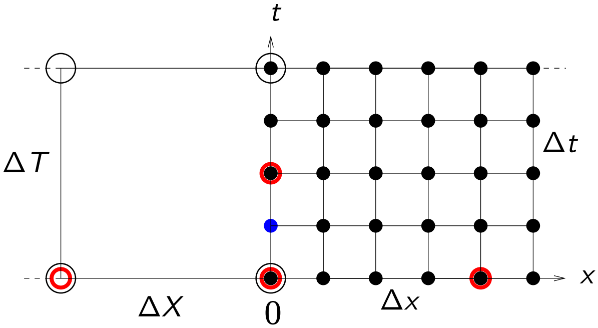

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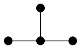

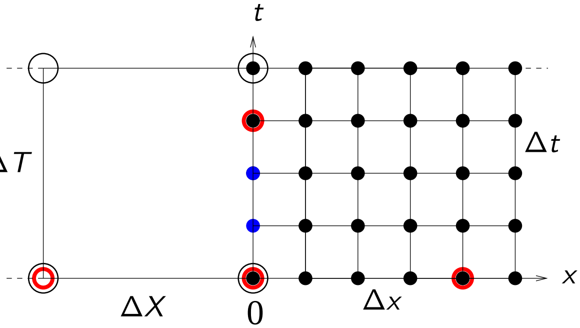

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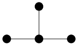

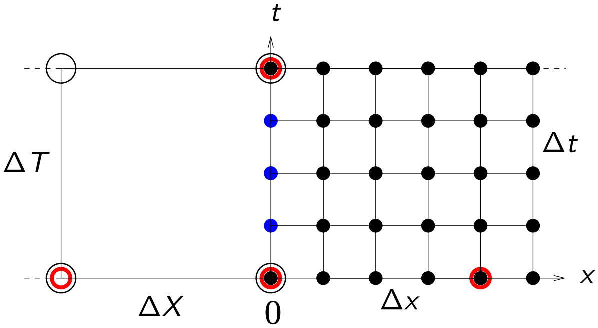

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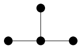
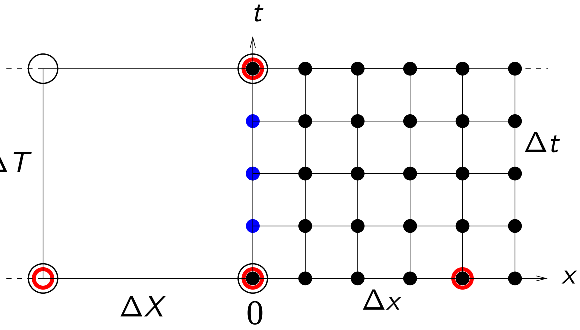
For explicit 3 point schemes 



Results:

For the hyperbolic model problem $u_t = u_x$:

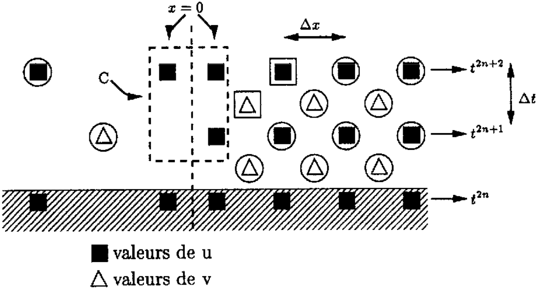
- Stability for both approaches for Lax-Wendroff
- Stability for Leapfrog *only with overlap*!

Energy Conservation

Collino, Fouquet, Joly (1998): Une méthode de raffinement de maillage espace-temps pour le système de Maxwell en dimension un

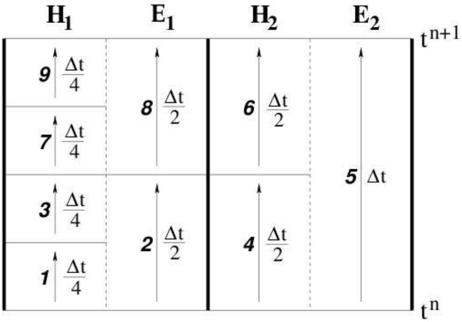
$$u_t + v_x = 0, \qquad v_t + u_x = 0$$

Centered finite difference discretization in time and space



Further Techniques

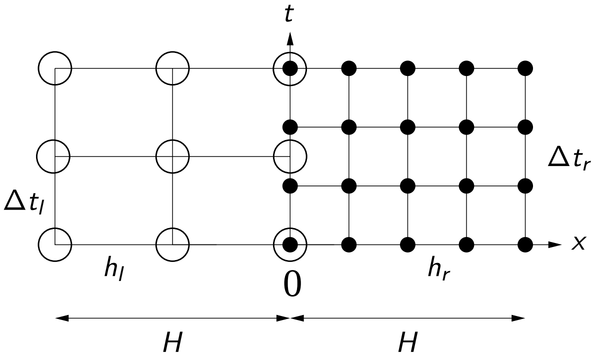
- **Piperno (2006):** Symplectic Local Time-Stepping in Non-Dissipative DGTD Methods Applied to Wave Propagation Problems



- **Diaz, Grote (2007):** Energy Conserving Explicit Local Time-Stepping for Second-Order Wave Equations
complicated CFL

Parabolic Problems: Explicit-Implicit Schemes

Dawson, Du, Dupont (1991) A Finite Difference Domain Decomposition Algorithm for Numerical Solution of the Heat Equation



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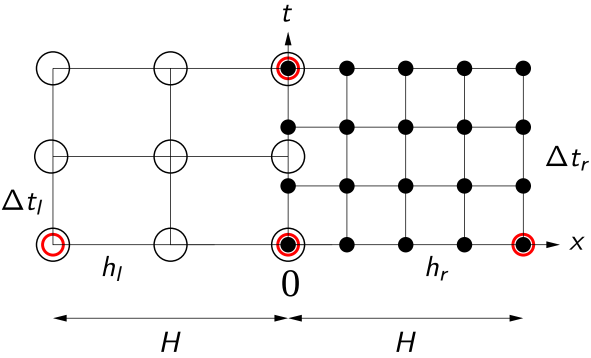

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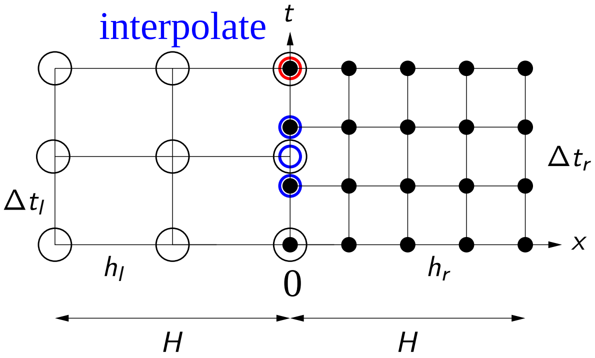

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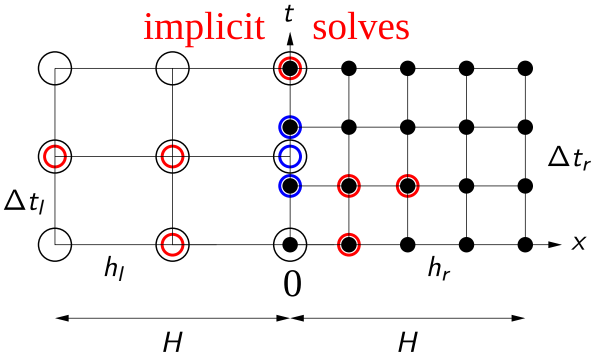

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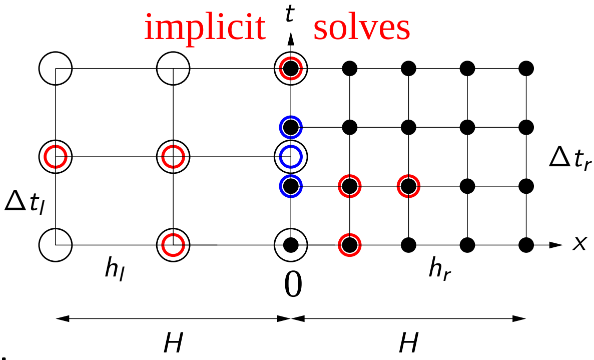

Results:

- Without local refinement, $h_l = h_r = h$, $\Delta t_l = \Delta t_r = \Delta t$ stable if $\Delta t \leq \frac{1}{2}H^2$

$$\max|\text{err}| \leq C(h^2 + H^3 + \Delta t)$$

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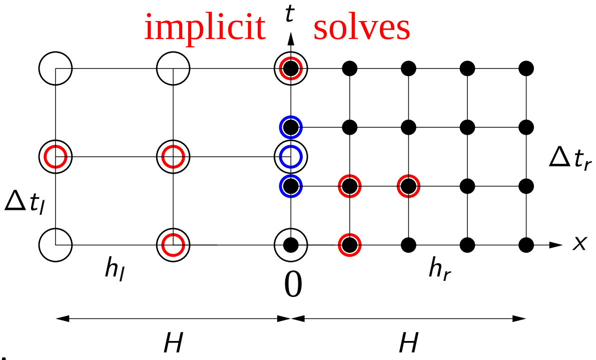


Results:

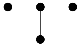
- With local refinement, stable if $\Delta t \leq \frac{1}{2}H^2$

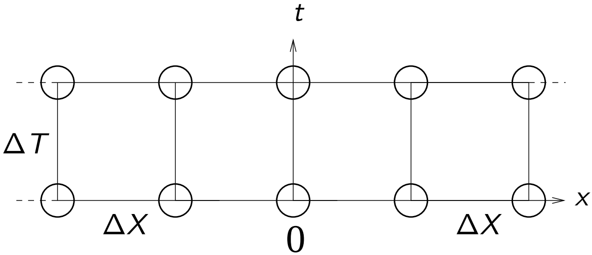
$$\max|\text{err}| \leq C(h_l^2 + h_r^2 + H^3 + \Delta t_l + \Delta t_r + H\Delta t)$$

Parabolic Problems: Explicit-Implicit Schemes

Blum, Lisky and Rannacher (1992) A Domain Splitting Algorithm for Parabolic Problems

Want to use implicit time integration





Without local refinement

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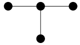

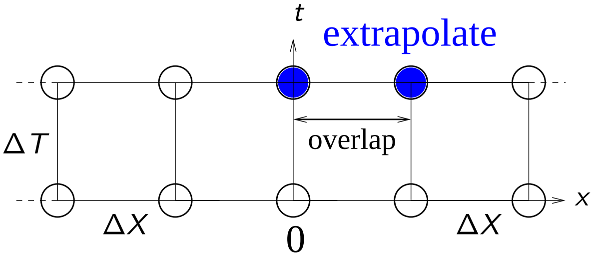

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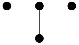

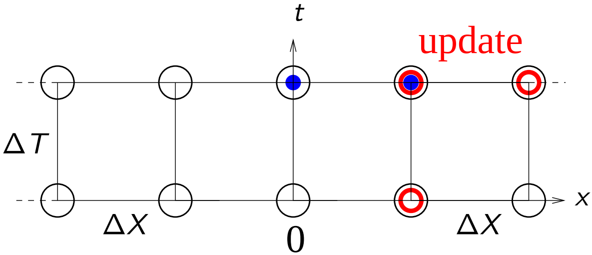

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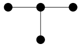
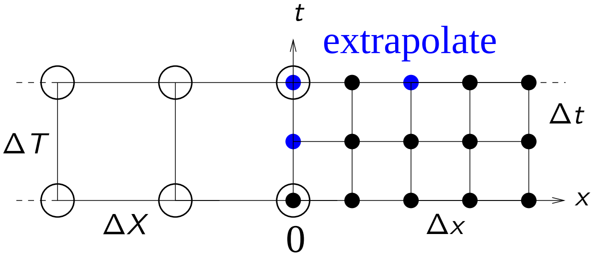
Want to use implicit time integration 



Could also do local refinement

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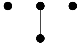

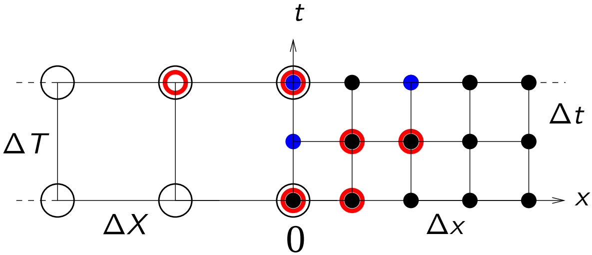

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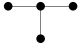
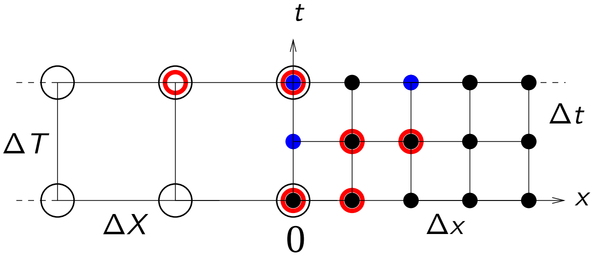




Could also do local refinement

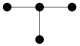
Results:

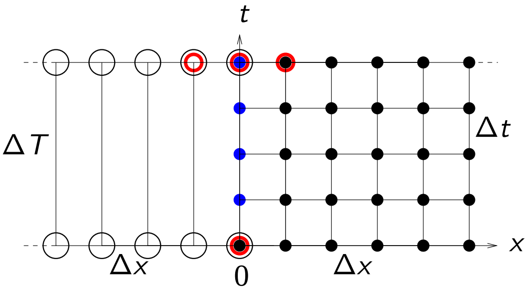
For the heat equation, without local refinement:

- Stable under the CFL condition $\Delta t \leq C\left(\frac{L}{\log L}\right)^2 \Delta x^2$, Lh overlap
- With Crank-Nicolson truncation error $O(\Delta t^2 + \Delta x^2)$.

Schemes with Unconditional Stability

Ewing, Lazarov, Vassilev (1994): Finite Difference Scheme for Parabolic Problems on Composite Grids with Refinement in Time and Space

For implicit 3 point schemes 



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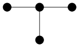
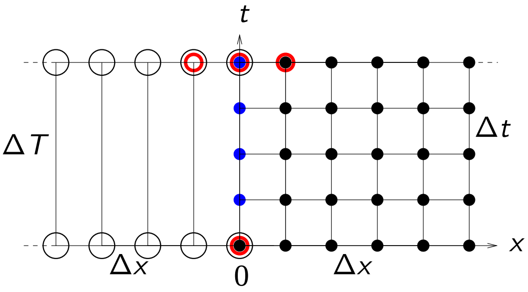
For implicit 3 point schemes 



Results: For a linear advection reaction diffusion equation:

- Unconditional stability
- For $\Delta t = O(\Delta x)$, error$= O(\Delta t + \Delta x^2)$ in 1d (2d loss of $|\log \Delta x|^{\frac{1}{2}}$, 3d loss of $\frac{1}{\sqrt{\Delta x}}$)

Conservative Schemes

Faille, Nataf, Willien Wolf (2009): Two Local Time stepping Schemes for Parabolic Problems
For the heat equation $u_t = u_{xx}$, decomposition of the domain $\Omega = (-1, 1)$ into two subdomains $\Omega_1 = (-1, 0)$ and $\Omega_2 = (0, 1)$, and the coupling conditions

$$u_1(0) = u_2(0), \quad \partial_x u_1(0) = \partial_x u_2(0)$$

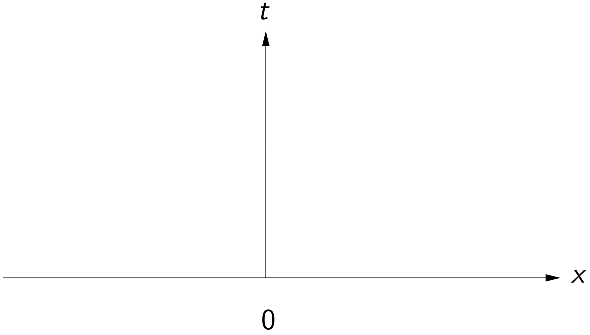
In a finite volume discretization, one obtains similar implicitly coupled schemes as before

Results:

- Schemes are conservative
- Error estimate $O(\Delta t + h)$ under certain conditions
- Iterative algorithm like Dirichlet-Neumann
- One of the choices corresponds to Ewing et al.

Space-Time Finite Elements

Johnson (1987): Space-Time Finite Element Method for $u_t + u_x = f$ and $u_t + u_x = u_{xx} + f$ using DG
Hulbert, Hughes (1990): Space-Time Finite Elements for Second Order Hyperbolic Equations



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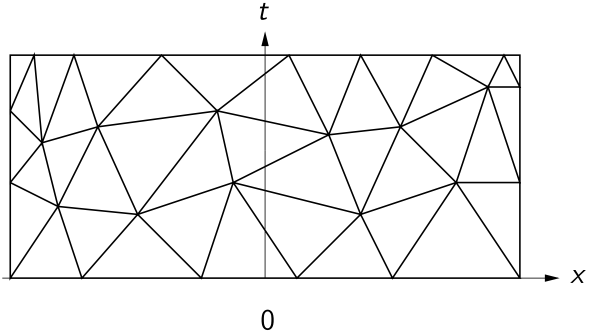

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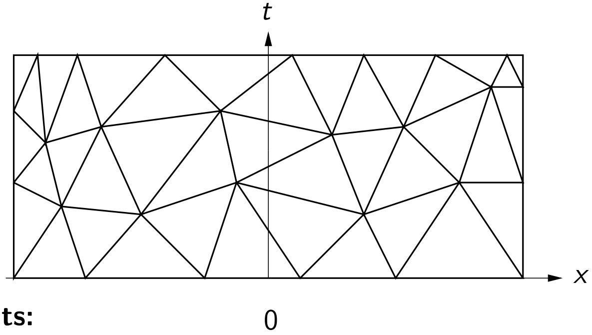
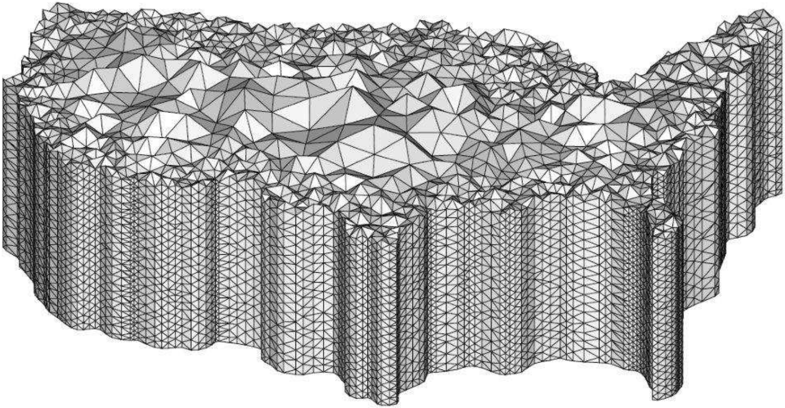


Results:

- Meshes need to satisfy a cone constraint
- **Üngör, Sheffer (2000):** Tent-Pitcher: a Meshing Algorithm for Space-Time Discontinuous Galerkin Methods

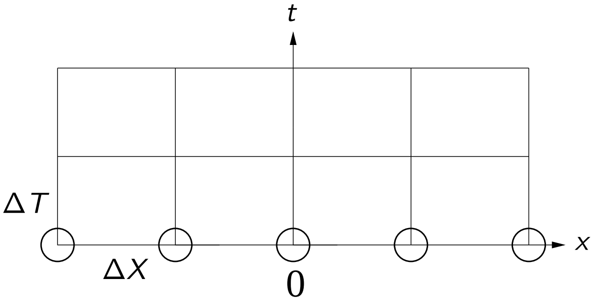
A Real Space Time Mesh

Erickson, Guoy, Sullivan, Üngör (2005): Building Spacetime Meshes over Arbitrary Spatial Domains



One-Way and Two-Way Methods

Cailleau, Fedorenko, Barnier, Blayo, Debreu (2008): Comparison of different numerical methods used to handle the open boundary of a regional ocean circulation model of the Bay of Biscay



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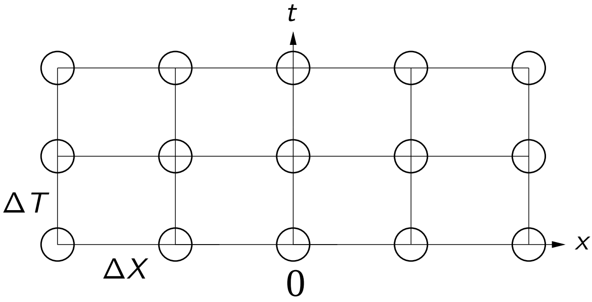

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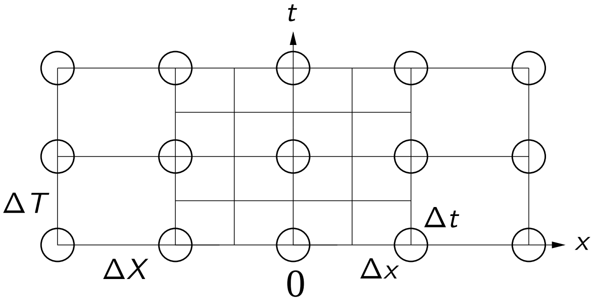

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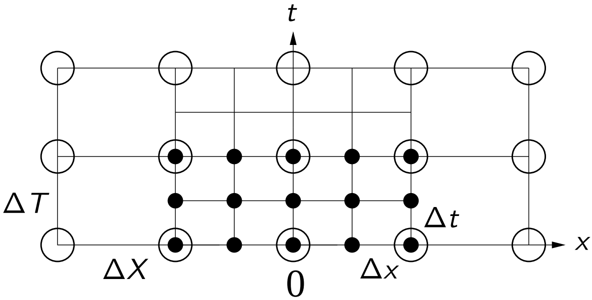

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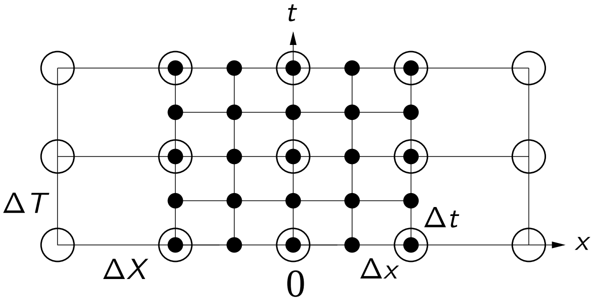

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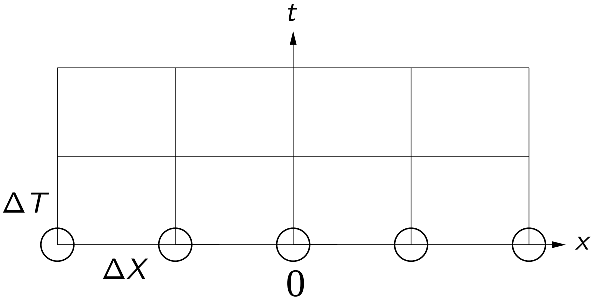

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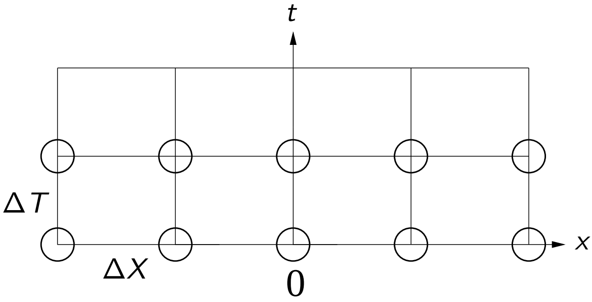

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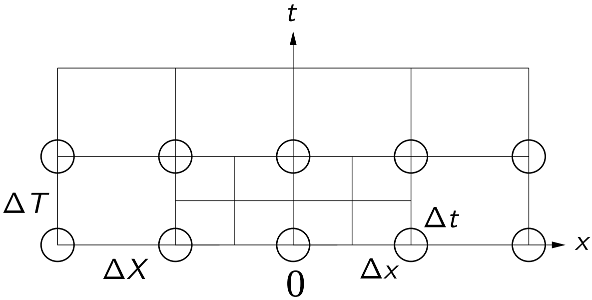

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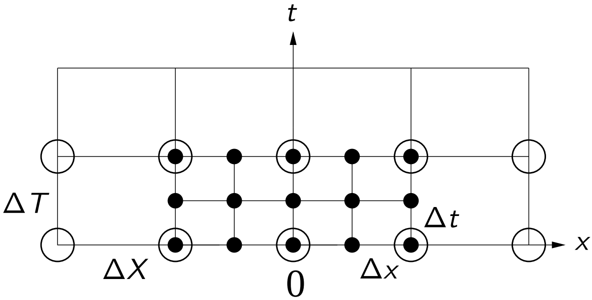

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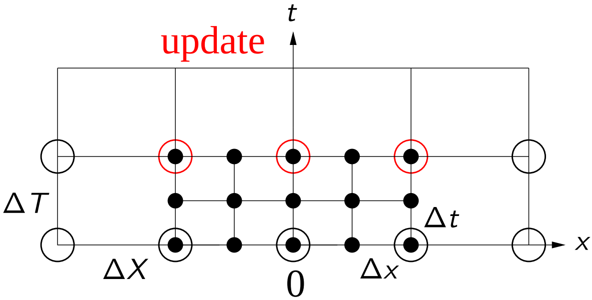

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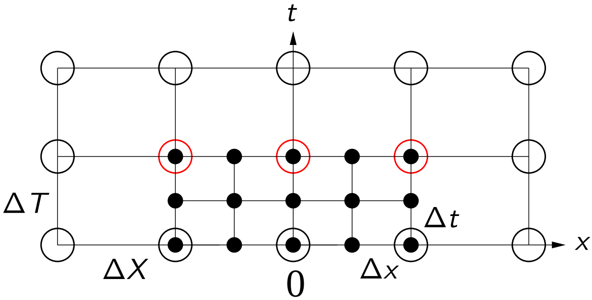

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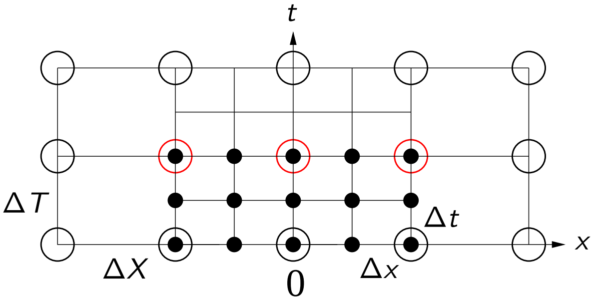

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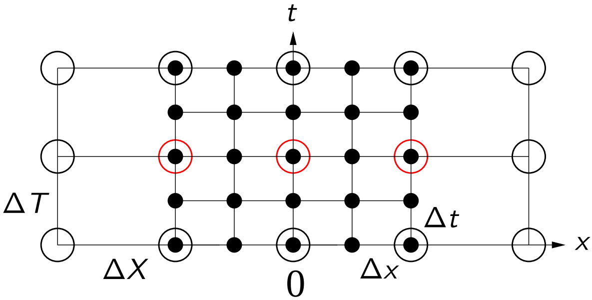

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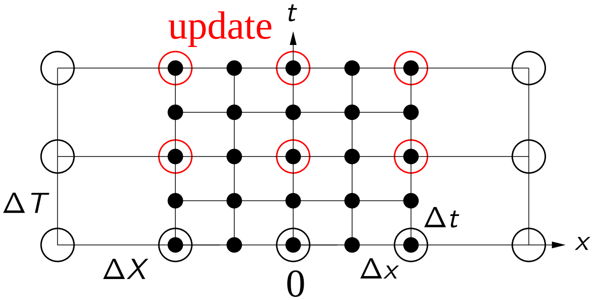


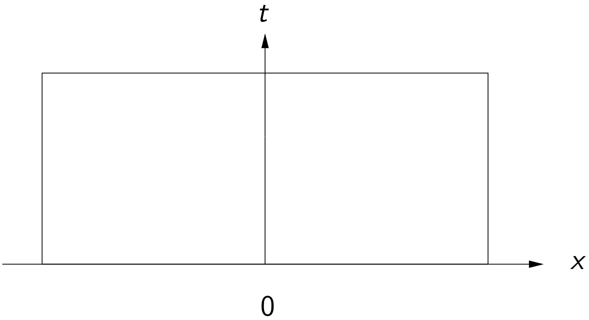
Two-Way Approach

Results:

- Experimental results, on real application, show that the two way method leads to substantial more accuracy
- No theoretical results

Schwarz Waveform Relaxation Methods

G, Halpern, Nataf (1999): Optimal Convergence of Overlapping and Non-Overlapping Schwarz Waveform Relaxation



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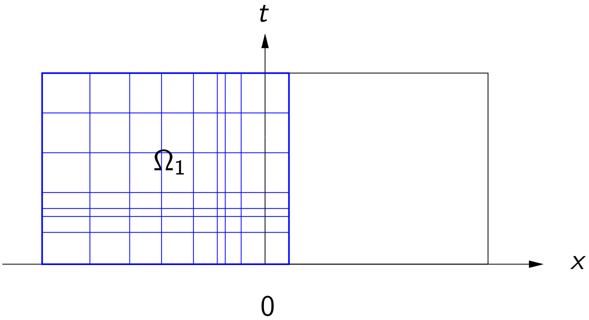

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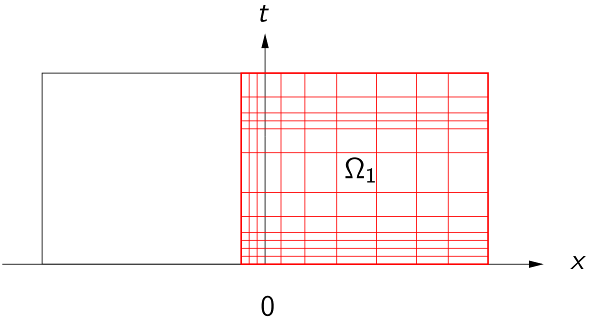

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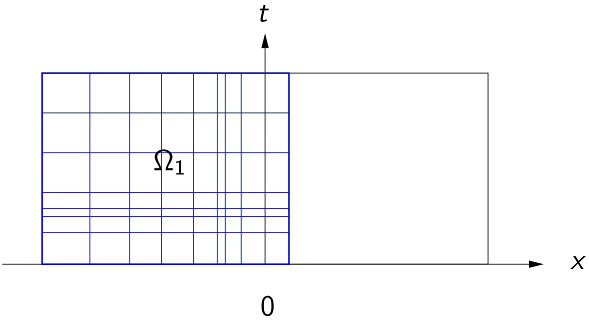

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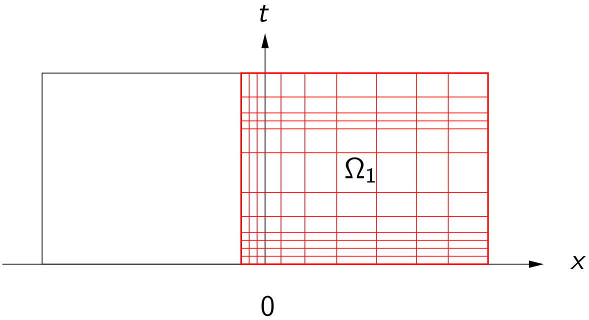

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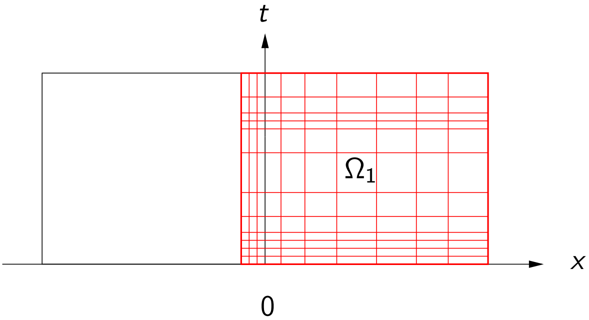

$$\begin{aligned} \mathcal{L}u_1^n &= f && \text{in } \Omega_1 \qquad & \mathcal{L}u_2^n &= f && \text{in } \Omega_2 \\ \mathcal{B}_1 u_1^n &= \mathcal{B}_1 u_2^{n-1} && & \mathcal{B}_2 u_2^n &= \mathcal{B}_2 u_1^n && \end{aligned}$$

- Very general method, can even use different models in different subdomains
- Iteration is expensive, need to use optimized $\mathcal{B}_i$

Outlook for the Minisymposium

- ODE based techniques
  - **Adrian Sandu:** Recent Developments in Multirate Time Integration
- PDE based techniques
  - Local Timestepping:
    - **Julien Diaz:** Explicit hp-Adaptive Time Scheme for the Wave Equation
    - **Jerónimo Rogríguez:** Coupling Discontinuous Galerkin Methods and Retarded Potentials for Transient Wave Propagation on Unbounded Domains
    - **Vadim Lisitsa:** Local Low-Reflection Space-Time Mesh Refinement for Finite-Difference Simulation of Seismic Waves
  - Domain Decomposition Based Techniques:
    - **Ron Haynes:** Equidistributing Grids via Domain Decomposition
    - **Florian Haeberlein:** Krylov Subspace Accelerators for Non-Overlapping Schwarz Waveform Relaxation Methods
    - **Laurence Halpern:** Space-time Refinement for the 1-D Wave Equation