How Close to the Fully Viscous Solution can one get When Inviscid Approximations are Used in Subregions ?

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1 Introduction

The coupling of different types of partial differential equations is an active field of research, since the need for such coupling arises in various applications. A first main area is the simulation of complex objects, composed of different materials, which are naturally modeled by different equations; fluid-structure interaction is a typical example. A second main area is when homogeneous objects are simulated, but the partial differential equation modeling the object is too expensive to solve over the entire object, and a simpler, less expensive model would suffice in most of the object to reach the desired accuracy; fluid flow around an airplane could serve as an example, where viscous effects are important close to the airplane, but can be neglected further away. A third emerging area is the coupling of equations across dimensions, for example the blood flow in the artery can be modeled by a one dimensional model, but in the heart, it needs to be three dimensional.

We are interested in this paper in the second situation, where the motivation for using different equations comes from the fact that we would like to use simpler, less expensive equations in areas of the domain where the full model is not needed, and we use as our guiding example the advection reaction diffusion equation. We are in principle interested in the fully viscous solution, but we would like to solve only an advection reaction equation for computational savings in part of the domain. Coupling conditions for this type of problem have been developed in the seminal paper [6], but with the first situation described above in mind, i.e. there is indeed a viscous and an inviscid physical domain, and the coupling conditions are obtained by a limiting process as the viscosity goes to zero, see also [7], and [1] for an innovative correction layer.

In his PhD thesis [2], Dubach developed coupling conditions based on absorbing boundary conditions, and such conditions have been used in order to define heterogeneous domain decomposition methods in [4]. A fundamental question however in the second situation described above is how far the solution obtained from the coupled problem is from the solution of the original, more expensive one on the entire domain. A first comparison of different transmission conditions focusing on this aspect appeared in [5]. In [3], coupling conditions were developed for stationary advection reaction diffusion equations in one spatial dimensions, which lead to solutions of the coupled problem that can be exponentially close to the fully viscous solution, and rigorous error estimates are provided. The coupling conditions are based on the factorization of the differential operator, and the exact factorization can be used in this one dimensional steady case. We study in this paper time dependent advection reaction diffusion problems, where the exact factorization of the differential operator can not be used any more, due to the non-local nature of the factors, and new ideas are needed in order to obtain better coupling conditions than the classical ones developed for situation one, where the domains are really physically different.

2 Model Problem

We consider the time dependent advection reaction diffusion equation

$$\begin{aligned} \mathcal{L}_{ad}u := \frac{\partial u}{\partial t} - \nu\frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x} + cu = f \quad & \text{in } (-L_1, L_2)\times(0,T), \\ \mathcal{B}_1 u(-L_1,\cdot) = g_1 \quad & \text{on } (0,T), \\ \mathcal{B}_2 u(L_2,\cdot) = g_2 \quad & \text{on } (0,T), \\ u(x,0) = 0 \quad & \text{on } (-L_1, L_2), \end{aligned} \tag{1}$$

where a is the velocity field, $\nu > 0$ is the viscosity, $c > 0$ is the reaction and $\mathcal{B}_j$, $j = 1, 2$ are suitable boundary operators: if $a > 0$ (resp. $a < 0$) a Dirichlet condition is imposed at $x = -1$ (resp. $x = 1$) and an absorbing boundary condition of order 1 is imposed at $x = 1$ (resp. $x = -1$). We present for convenience our results using a homogeneous initial condition; in the case of an inhomogeneous initial condition u_0, the change of variables $\tilde{u}(x,t) = u(x,t) - e^{-t}u_0(x)$ leads to a problem of the form (1).

We suppose now that the viscosity term is only important in part of the domain, say in $(-L_1, 0)$, and we are thus willing to solve the full advection reaction diffusion equation there,

$$\begin{aligned} \mathcal{L}_{ad}u_{ad} = f \quad & \text{in } (-L_1, 0)\times(0,T), \\ \mathcal{B}_1 u_{ad}(-L_1,\cdot) = g_1 \quad & \text{on } (0,T), \\ \mathcal{B}_{ad} u_{ad}(0,\cdot) = g_a \quad & \text{on } (0,T), \\ u_{ad}(x,0) = 0 \quad & \text{on } (-L_1, L_2). \end{aligned} \tag{2}$$

We want to determine a boundary operator $\mathcal{B}_{ad}$ and a function g_a, which can only use information from solutions of advection reaction equations on the remaining domain $(0, L_2)$, such that u_{ad} is as close as possible to the fully viscous solution u on $(-L_1, 0)$. Because the viscosity is small, a first idea is to solve on the remaining domain $(0, L_2)$ the advection equation

$$\mathcal{L}_a u_a := \frac{\partial u_a}{\partial t} + a\frac{\partial u_a}{\partial x} + cu_a \quad \text{on } (0, L_2) \times (0, T). \tag{3}$$

This choice was made in [6] for the 2D stationary advection diffusion equation, and a variational coupling condition was introduced, which in our time dependent case is

$$\begin{aligned} (-\nu u'_{ad} + a u_{ad})(0, \cdot) &= a u_a(0, \cdot) \quad \text{if } a > 0 \text{ or } a < 0, \\ u_{ad}(0, \cdot) &= u_a(0, \cdot) \quad \text{if } a > 0. \end{aligned} \tag{4}$$

We have shown in [3] that for the stationary case of (2) there exist coupling conditions which lead to coupled solutions that are much closer to the fully viscous solution on the entire domain than with the coupling conditions (4). The purpose of the present paper is to investigate if similar coupling conditions exist for (2).

3 Factorization of the Differential Operator

Let $\hat{u}(s) = \int_0^{+\infty} u(t)e^{-st}dt$, $\mathcal{R}(s) > \alpha$ be the Laplace transform of the continuous function u with $|u(t)| \leq e^{\alpha t}$, $t > 0$. Performing the Laplace transform of equation (1), we obtain

$$-\nu\frac{\partial^2 \hat{u}}{\partial x^2} + a\frac{\partial \hat{u}}{\partial x} + (c + s)\hat{u} = \hat{f}.$$

The characteristic roots of this equation are

$$\lambda^+ = \frac{1}{2\nu}(a + \sqrt{a^2 + 4\nu(c + s)}) \quad \text{and} \quad \lambda^- = \frac{1}{2\nu}(a - \sqrt{a^2 + 4\nu(c + s)}), \tag{5}$$

and we obtain a factorization of the Laplace transformed operator,

$$\widehat{\mathcal{L}}_{ad} = (a\partial_x - a\lambda^+)(-\frac{\nu}{a}\partial_x + \frac{\nu}{a}\lambda^-).$$

The two factors represent evolution operators in the x direction, one into the positive, and the other into the negative x direction, due to the square root with principal branch having positive real part. The evolution operators are however non-local, so we propose to expand $\lambda^\pm$ for small viscosity ν,

$$\lambda^+ = \frac{a + |a|}{2\nu} + \frac{c + s}{|a|} + \mathcal{O}(\nu) \quad \text{and} \quad \lambda^- = \frac{a - |a|}{2\nu} - \frac{c + s}{|a|} + \mathcal{O}(\nu). \tag{6}$$

If we truncate these expansions to obtain approximations $\lambda^{\pm}_{app}$, the factorization is not exact anymore, a rest appears on the right hand side, see for example [8], and we obtain

$$\begin{aligned}
&(a\partial_x - a\lambda^+_{app})(-\frac{\nu}{a}\partial_x + \frac{\nu}{a}\lambda^-_{app})\hat{u}(x,s)\\
&= -\nu\frac{\partial^2\hat{u}}{\partial x^2}(x,s) + a\frac{\partial\hat{u}}{\partial x}(x,s) + (c+s)\hat{u}(x,s) - (\nu\lambda^+_{app}\lambda^-_{app} + (c+s))\hat{u}(x,s)\\
&= \hat{f} - (\nu\lambda^+_{app}\lambda^-_{app} + (c+s))\hat{u}(x,s).
\end{aligned} \tag{7}$$

4 Optimal Coupling Conditions and Approximations

We start by deriving an optimal coupling condition: integrating (7) on $(0, L_2)$ once yields

$$\begin{aligned}
(-\nu\frac{\partial\hat{u}}{\partial x} + \nu\lambda^-_{app}\hat{u})(0,s) &= (-\nu\frac{\partial\hat{u}}{\partial x} + \nu\lambda^-_{app}\hat{u})(L_2,s)e^{-\lambda^+_{app}L_2}\\
&- \int_0^{L_2}(\hat{f}(x,s) - (\nu\lambda^+_{app}\lambda^-_{app} + (c+s))\hat{u}(x,s))e^{-\lambda^+_{app}x}dx.
\end{aligned} \tag{8}$$

The integral term suggests introducing the modified advection operator and associated equation

$$\widehat{\widetilde{\mathcal{L}}}_a\hat{w} := \left(a\partial_x - a\lambda^+_{app}\right)\hat{w} = \hat{\tilde{f}}, \tag{9}$$

since integrating this equation on $(0, L_2)$ gives $\hat{w}(0) = -\frac{1}{a}\int_0^{L_2}\hat{\tilde{f}}(x)e^{-\lambda^+_{app}x}\,dx + \hat{w}(L_2)e^{-\lambda^+_{app}L_2}$. Using this idea to replace the integral term, we find that the solution of (1) satisfies at $x = 0$ the coupling relation

$$(-\nu\frac{\partial\hat{u}}{\partial x} + \nu\lambda^-_{app}\hat{u})(0,s) = (-\nu\frac{\partial\hat{u}}{\partial x} + \nu\lambda^-_{app}\hat{u} - a\hat{\tilde{u}}_a)(L_2,s)e^{-\lambda^+_{app}L_2} + a\hat{\tilde{u}}_a(0,s), \tag{10}$$

where $\hat{\tilde{u}}_a$ is the solution of $\widehat{\widetilde{L}}_a\hat{\tilde{u}}_a = \hat{f} - (\nu\lambda^+_{app}\lambda^-_{app} + (c+s))\hat{u}$ on $(0, L_2)$. The coupling relation (10) gives an optimal coupling condition, since solving the advection diffusion equation on $(-L_1, 0)$ with the coupling condition

$$(-\nu\frac{\partial\hat{u}_{ad}}{\partial x} + \nu\lambda^-_{app}\hat{u}_{ad})(0,s) = (-\nu\frac{\partial\hat{u}}{\partial x} + \nu\lambda^-_{app}\hat{u} - a\hat{\tilde{u}}_a)(L_2,s)e^{-\lambda^+_{app}L_2} + a\hat{\tilde{u}}_a(0,s), \tag{11}$$

implies that u_{ad} is the restriction of u on $(-L_1, 0)$. But the right hand side of the coupling condition (11) depends on the fully viscous solution $\hat{u}$ on $(0, L_2)$, which we obviously do not know. We can however solve the advection equation $L_a u_a = f$ on $(0, L_2) \times (0, T)$, so that we obtain an approximation of u. This leads for $a > 0$ to the iterative procedure

$g^0 = 0$, $u_a^0 = 0$
for $k = 0, 1, 2, \ldots$

$$\begin{cases} \widetilde{\mathcal{L}}_a \tilde{u}_a^{k+1} = f - g^k & \text{on } (0, L_2) \times (0, T) \\ \tilde{u}_a^{k+1}(L_2, \cdot) = \frac{\nu}{a}(-\partial_x + \Lambda_{app}^-) u_a^k(L_2, \cdot) & \text{on } (0, T) \end{cases}$$

$$\begin{cases} \mathcal{L}_{ad} u_{ad}^{k+1} = f & \text{on } (-L_1, 0) \times (0, T) \\ \mathcal{B}_1 u_{ad}^{k+1}(-L_1, \cdot) = g_1 & \text{on } (0, T) \\ \mathcal{B}_{ad} u_{ad}^{k+1}(0, \cdot) = a \tilde{u}_a^{k+1}(0, \cdot) & \text{on } (0, T) \end{cases}$$

$$\begin{cases} \mathcal{L}_a u_a^{k+1} = f & \text{on } (0, L_2) \times (0, T) \\ u_a^{k+1}(0, \cdot) = u_{ad}^{k+1}(0, \cdot) & \text{on } (0, T) \end{cases}$$

$$g^{k+1} = \mathcal{G}(u_a^{k+1})$$

end;

where $\mathcal{G} := \nu \Lambda_{app}^+ \Lambda_{app}^- + c + \partial_t$ and $\Lambda_{app}^{\pm}$ are the differential operators corresponding to the symbols $\lambda_{app}^{\pm}$. The differential operators in this algorithm depend on the order of approximation of $\lambda^{\pm}$, and are given in Table 1 on the left. The initial conditions are all homogeneous, except in the case of

	$a > 0$		$a < 0$	
	Order 0	Order 1	Order 0	Order 1
$\widetilde{\mathcal{L}}_a$	$a\frac{\partial}{\partial x} - \frac{a^2}{\nu}$	$a\frac{\partial}{\partial x} - \frac{\partial}{\partial t} - (\frac{a^2}{\nu} + c)$	$a\frac{\partial}{\partial x}$	$a\frac{\partial}{\partial x} + \frac{\partial}{\partial t} + c$
$\mathcal{B}_{ad}$	$-\nu\frac{\partial}{\partial x}$	$-\nu\frac{\partial}{\partial x} - \frac{\nu}{a}\frac{\partial}{\partial t} - \frac{c\nu}{a}$	$-\nu\frac{\partial}{\partial x} + a$	$-\nu\frac{\partial}{\partial x} + \frac{\nu}{a}\frac{\partial}{\partial t} + (a + \frac{c\nu}{a})$
$\mathcal{G}$	$\frac{\partial}{\partial t} + c$	$-\frac{\nu}{a^2}(\frac{\partial^2}{\partial t^2} + 2c\frac{\partial}{\partial t} + c^2)$	$\frac{\partial}{\partial t} + c$	$-\frac{\nu}{a^2}(\frac{\partial^2}{\partial t^2} + 2c\frac{\partial}{\partial t} + c^2)$

Table 1. Local approximation of the operators in the coupling algorithm

approximation of order 1 in the modified advection problem, where the initial condition is $\tilde{u}_a^{k+1}(\cdot, 0) = -\frac{\nu}{a^2}\partial_t u_a^k(\cdot, 0)$.

If $a < 0$, the algorithm is in principle not iterative, since the advection problem (the third in the algorithm) has now as boundary condition

$$u_a^{k+1}(L_2, \cdot) = g_2 \text{ on } (0, T),$$

and can only improve the situation once. One could thus start directly with this step, but in order to compare the situation with and without this step, we leave the algorithm sequence as stated for the case of $a > 0$. The differential operators in this algorithm depend on the order of approximation of $\lambda^{\pm}$, and are given in Table 1 on the right. To investigate how small the error becomes in ν, and how this depends on the iteration of the algorithm, we present in the next section a numerical asymptotic study when the viscosity goes to zero.

5 Numerical Asymptotic Study

We chose for the data $f = 0$ and $u_0(x) = e^{-100x^2}$, $a = \pm 10$ and $c = 1$, and will consider several values for ν. The domain is $(-1, 1) \times (0, 0.1)$. Note that the support of the initial condition contains the interface between the two subdomains. We discretize the equations by centered finite differences, and the Crank-Nicolson scheme in time, with $\Delta x = \frac{1}{12800}$ and $\Delta t = \frac{1}{128000}$.

We show in Figure 1 both for the zeroth and first order approximation the L^2 error in space and time between the coupled solution and the fully

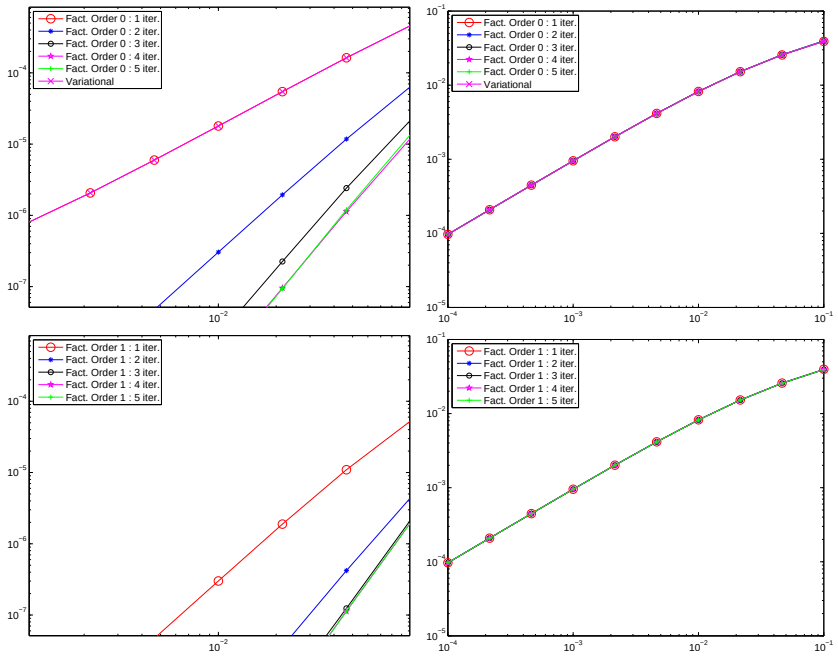

Fig. 1. Positive advection. Factorization method of order 0 and variational method (4) in the top row, and first order method in the bottom row. Error versus viscosity on $(-L_1, 0)$ on the left, and on $(0, L_2)$ on the right

viscous mono-domain solution versus the viscosity when $a = 10$. The error in the advection region is always $\|u_a - u\|_{L^2(0,T;(0,1))} = \mathcal{O}(\nu)$, and the variational method (4) and the factorization method of order 0 with one iteration give similar results, but as soon as one adds iterations, which seem to converge, or uses the first order approximation, significantly better results are obtained.

In Figure 2, we show the L^2 error in space and time between the coupled solution and the fully viscous mono-domain solution versus the viscosity when $a = -10$. We see that the error $\|u_a - u\|_{L^2(0,T;(0,1))}$ is always $\mathcal{O}(\nu)$. It seems that using the factorization method of order 0 without iteration is not a

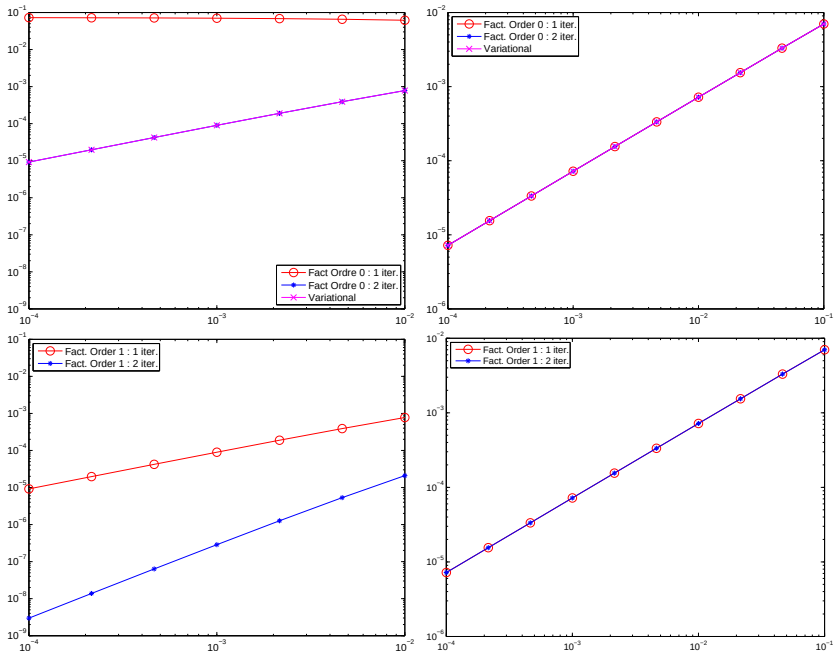

Fig. 2. Negative advection. Factorization method of order 0 and variational method in the top row, and first order method in the bottom row. Error versus the viscosity on $(-L_1, 0)$ on the left, and on $(0, L_2)$ on the right

useful method: in that case the neglected term $\nu\lambda^+_{app}\lambda^-_{app} + c + s = c + s$ is not small in ν, we need to iterate at least once, which is equivalent to changing the order in the algorithm, see the comment in Section 4. With the factorization method of order 1 and one iteration we obtain an error of $\mathcal{O}(\nu^2)$ which is a substantial improvement over the variational method, since in that case, because the advection is negative, the information comes from the right where the error is $\mathcal{O}(\nu)$. Our modified advection operator clearly carries more relevant information in this case.

In Table 2 we show in summary the numerically estimated dependence on the viscosity ν, both for the case of positive and negative advection.

6 Conclusions

We have derived coupling conditions for the time dependent advection reaction diffusion equation, which lead to coupled solutions that are closer to the fully viscous solution than when using classical variational coupling conditions. Our numerical experiments allowed us to estimate the asymptotic dependence on the viscosity of the new approach, and we are currently working on rigorous error estimates for the new coupling mechanism.

$a > 0$					$a < 0$			
Order 0			Order 1		Order 0		Order 1	
1 iter.	2 iter.	5 iter.	1 iter.	5 iter.	1 iter.	2 iter.	1 iter.	2 iter.
$\mathcal{O}(\nu^{1.4})$	$\mathcal{O}(\nu^{2.4})$	$\mathcal{O}(\nu^{3.3})$	$\mathcal{O}(\nu^{2.4})$	$\mathcal{O}(\nu^{3.3})$	$--$	$\mathcal{O}(\nu)$	$\mathcal{O}(\nu)$	$\mathcal{O}(\nu^2)$

Table 2. Numerically measured error estimates for $\|u_{ad} - u\|_{L^2(0,T;(0,1))}$

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