Optimized Schwarz methods for domain decompositions with parabolic interfaces

Martin J. Gander[1] and Yingxiang Xu[2]

1 Introduction

Optimizing parameters involved in the transmission conditions of subdomain iterations leads to the well-known optimized Schwarz methods, see [2, 3] and references therein, where for analysis usually a model problem is considered on $\mathbb{R}^2$, decomposed into two half planes with a straight interface. In applications the interface is however seldom straight, which creates a gap between theory and applications. After early steps in [4], several research efforts have been devoted to close this gap: for a general curved interface, transmission conditions involving the local interface curvature using micro-local analysis were derived in [1], but they are not optimal. When the curved interface is simple, for example a circle, it was shown in [5] and [7] that the curvature enters the transmission parameters and the corresponding estimates of the convergence factors, and that optimized transmission parameters can be well approximated using parameters from straight interface analysis, provided the curvature is included through a proper scaling. For cylindrical interfaces, see [8]. This analysis can however not show if any other geometric characteristics enter the optimized transmission parameters for a general curved interface, apart from the curvature. We examine here the situation of a parabolically shaped interface, and show that in addition to the interface curvature, other information of the interface will also enter the optimized transmission parameters. In applications with curved interfaces, optimized transmission parameters from the straight interface analysis are often used locally without any theoretical explanation and lead to fairly good performance, see for example [2]. We will also compare our new results with this approach.

1. Section de Mathématiques, Université de Genève, 2-4 rue du Lièvre, CP 64, CH-1211, Genève, Suisse. Martin.Gander@unige.ch 2. Corresponding author. School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China. yxxu@nenu.edu.cn, partly supported by NSFC-11671074, 11471047, CPSF-2012M520657 and the Science and Technology Development Planning of Jilin Province 20140520058JH.

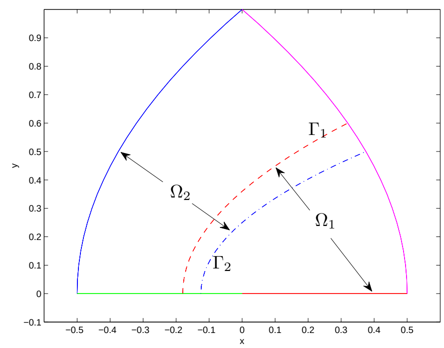

Fig. 1 Domain decomposition with parabolic interfaces.

2 Schwarz methods with parabolic interfaces

We consider the model problem

$$\begin{aligned} (\Delta - \eta)u &= f, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\eta > 0$ is a model parameter, $\Omega = \{(x,y) | x = \frac{1}{2}(\tau^2 - \sigma^2), y = \sigma\tau, \sigma \in (0,1), \tau \in (0,1)\}$. Using the so-called parabolic coordinates

$$y = \sigma\tau, \qquad x = \frac{1}{2}(\tau^2 - \sigma^2), \tag{2}$$

we have $\Omega = \{(x(\sigma,\tau), y(\sigma,\tau)) | 0 < \sigma < 1, 0 < \tau < 1\}$. We introduce the decomposition $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 = \{(x(\sigma,\tau), y(\sigma,\tau)) | 0 < \sigma < \sigma_0 + L, 0 < \tau < 1\}$ and $\Omega_2 = \{(x(\sigma,\tau), y(\sigma,\tau)) | \sigma_0 < \sigma < 1, 0 < \tau < 1\}$ where σ_0 is a constant satisfying $0 < \sigma_0 < 1$ and $L \geq 0$ is a constant that describes the overlap. If $L = 0$, there is no overlap. The curves $\Gamma_1 = \{(x(\sigma,\tau), y(\sigma,\tau)) | \sigma = \sigma_0 + L, 0 < \tau < 1\}$ and $\Gamma_2 = \{(x(\sigma,\tau), y(\sigma,\tau)) | \sigma = \sigma_0, 0 < \tau < 1\}$ are the artificial interfaces, see Fig. 1.

A general parallel Schwarz algorithm is then given by

$$\begin{aligned} (\Delta - \eta)u_i^n &= f && \text{in } \Omega_i, \\ u_i^n &= 0 && \text{on } \partial\Omega_i \backslash \Gamma_i, \\ \mathcal{B}_i(u_i^n) &= \mathcal{B}_i(u_j^{n-1}) && \text{on } \Gamma_i,\ 1 \leq i \neq j \leq 2, \end{aligned} \tag{3}$$

where $\mathcal{B}_i, i = 1,2$, are transmission conditions to be chosen. It is well known that for fast convergence, the transmission operators $\mathcal{B}_i$, $i = 1,2$ should be chosen as $\partial_{n_i} + \mathcal{S}_i$, with $\mathcal{S}_i$ local differential operators along the interfaces approximating the Dirichlet to Neumann operators [2, 3].

The Schwarz method (3) is usually analyzed with Fourier techniques, but in the case of parabolic interfaces this is not possible. Noting that the transform (2) is a conformal map with scale factor $H = \sqrt{\sigma^2+\tau^2}$, the model problem (1) becomes

$$\begin{aligned} (\tfrac{1}{\sigma^2+\tau^2}\Delta_{\sigma\tau} - \eta)u(\sigma,\tau) &= f(\sigma,\tau), && \text{in } \Omega,\\ u(\sigma,\tau) &= 0, && \text{on } \partial\Omega. \end{aligned} \tag{4}$$

Choosing the transmission operators $\mathcal{B}_i, i=1,2$ as $\mathcal{B}_i = \partial_\sigma + \mathcal{S}_i$, we then obtain the Schwarz method (3) as

$$\begin{aligned} (\tfrac{1}{\sigma^2+\tau^2}\Delta_{\sigma\tau} - \eta)u_i^n(\sigma,\tau) &= f(\sigma,\tau) && \text{in } \Omega_i,\\ u_i^n(\sigma,\tau) &= 0 && \text{on } \partial\Omega_i\backslash\Gamma_i,\\ (\partial_\sigma + \mathcal{S}_i)(u_i^n) &= (\partial_\sigma + \mathcal{S}_i)(u_j^{n-1}) && \text{on } \Gamma_i,\ 1 \le i \ne j \le 2. \end{aligned} \tag{5}$$

3 Optimized local transmission conditions

We now determine the optimized local operators $\mathcal{S}_i$, $i=1,2$. Since the Fourier transform can not be used, we apply the technique of separation of variables, which has been employed successfully in analyzing optimized Schwarz methods for model problems with variable reaction term in [6]. To this end, we assume that the function $u(\sigma,\tau)$ is separable, $u(\sigma,\tau)=\phi(\sigma)\psi(\tau)$, or equivalently, $u_i^n(\sigma,\tau) = \phi_i^n(\sigma)\psi(\tau)$, $i=1,2$. Inserting this ansatz into the first equation of (5) with homogeneous right hand side $f=0$ gives

$$-(\phi_i^n(\sigma))^{''}\psi(\tau) - \phi_i^n(\sigma)\psi^{''}(\tau) + (\sigma^2+\tau^2)\eta\psi_i^n(\sigma)\psi(\tau) = 0, \quad i=1,2.$$

Separating terms, we see that there must exist a positive constant α such that

$$-\frac{(\phi_i^n(\sigma))^{''}}{\phi_i^n(\sigma)} + \sigma^2\eta = \frac{\psi^{''}(\tau)}{\psi(\tau)} - \tau^2\eta = -\alpha, \quad i=1,2.$$

Together with the homogeneous boundary conditions, we obtain that α must be an eigenvalue of the Sturm-Liouville eigenvalue problem

$$\psi^{''}(\tau) + (\alpha - \tau^2\eta)\psi(\tau) = 0, \quad \psi(0)=\psi(1)=0. \tag{6}$$

Assuming that we use a uniform grid with mesh size $h=1/N$ in the τ-direction, we then have $\psi(\tau) = \sum_{j=1}^N \psi_j \sin j\pi\tau$. Using this ansatz and testing (6) with $\sin k\pi\tau$ for $k=1,\cdots,N$, we obtain for each k

$$(\alpha - k^2\pi^2)\psi_k - 2\eta\sum_{j=1}^N \psi_j \int_0^1 \tau^2 \sin j\pi\tau \sin k\pi\tau d\tau = 0.$$

Hence α represents eigenvalues of the matrix $\pi^2\mathtt{diag}(1^2,2^2,\cdots,N^2) + 2\eta M$, where M is a matrix with entries $M_{jk} = \int_0^1 \tau^2 \sin j\pi\tau \sin k\pi\tau d\tau$. We then denote the k-th eigenvalue by α_k, the smallest one by $\alpha_{\min}$ and the largest one by $\alpha_{\max}$.

For each eigenvalue α_k, $k = 1, \cdots, N$, we then need to consider

$$\begin{aligned} -(\phi_1^n(\sigma))'' + (\alpha_k + \sigma^2\eta)\phi_1^n(\sigma) = 0, \quad \phi_1^n(0) = 0, \\ -(\phi_2^n(\sigma))'' + (\alpha_k + \sigma^2\eta)\phi_2^n(\sigma) = 0, \quad \phi_2^n(1) = 0, \end{aligned}$$

whose basic solutions are known in closed form,

$$\begin{aligned} \phi_{in}(\sigma;\alpha,\eta) &= \frac{M(-\frac{1}{4}\frac{\alpha}{\sqrt{\eta}},\frac{1}{4},\sqrt{\eta}\sigma^2)}{\sqrt{\sigma}}, \\ \phi_{de}(\sigma;\alpha,\eta) &= \frac{W(-\frac{1}{4}\frac{\alpha}{\sqrt{\eta}},\frac{1}{4},\sqrt{\eta})}{M(-\frac{1}{4}\frac{\alpha}{\sqrt{\eta}},\frac{1}{4},\sqrt{\eta})}\frac{M(-\frac{1}{4}\frac{\alpha}{\sqrt{\eta}},\frac{1}{4},\sqrt{\eta}\sigma^2)}{\sqrt{\sigma}} + \frac{W(-\frac{1}{4}\frac{\alpha}{\sqrt{\eta}},\frac{1}{4},\sqrt{\eta}\sigma^2)}{\sqrt{\sigma}}, \end{aligned}$$

where W and M are Whittaker functions. Note that $\phi_{in}(\sigma;\alpha,\eta)$ increases monotonically in σ with $\phi_{in}(0;\alpha,\eta) = 0$ and $\phi_{de}(\sigma;\alpha,\eta)$ decreases monotonically in σ with $\phi_{de}(1;\alpha,\eta) = 0$.

Using the separation assumption $u_i(\sigma,\tau) = \phi_i(\sigma)\psi(\tau)$ also in the transmission conditions in (5) gives

$$\begin{aligned} (\partial_\sigma + \mathcal{S}_1)\phi_1^n(\sigma_0 + L)\psi(\tau) &= (\partial_\sigma + \mathcal{S}_1)\phi_2^{n-1}(\sigma_0 + L)\psi(\tau), \\ (\partial_\sigma + \mathcal{S}_2)\phi_2^n(\sigma_0)\psi(\tau) &= (\partial_\sigma + \mathcal{S}_2)\phi_1^{n-1}(\sigma_0)\psi(\tau). \end{aligned}$$

Inserting $\psi(\tau) = \sum_{j=1}^N \psi_j \sin j\pi\tau$ and testing these equations by $\sin k\pi\tau$ we obtain for each $k = 1, 2, \cdots, N$

$$\begin{aligned} (\partial_\sigma + \mu_1(k))\phi_1^n(\sigma_0 + L) &= (\partial_\sigma + \mu_1(k))\phi_2^{n-1}(\sigma_0 + L), \\ (\partial_\sigma + \mu_2(k))\phi_2^n(\sigma_0) &= (\partial_\sigma + \mu_2(k))\phi_1^{n-1}(\sigma_0), \end{aligned}$$

where $\mu_i(k)$, $i = 1, 2$ are the Fourier symbols of the operators $\mathcal{S}_i$.

Similar to the technique used in [6] (see also [2]), we then obtain the convergence factor of algorithm (5),

$$\rho(L,\mu_1(k),\mu_2(k)) := \frac{(\partial_\sigma + \mu_1(k))\phi_{de}(\sigma_0 + L)}{(\partial_\sigma + \mu_1(k))\phi_{in}(\sigma_0 + L)}\frac{(\partial_\sigma + \mu_2(k))\phi_{in}(\sigma_0)}{(\partial_\sigma + \mu_2(k))\phi_{de}(\sigma_0)}. \quad (7)$$

As local approximations of the Dirichlet to Neumann operators, we consider

$$\mu_1^{app}(k) = p_1 + q_1\alpha_k, \quad \mu_2^{app}(k) = -p_2 - q_2\alpha_k,$$

which correspond to the local operators along the interfaces Γ_1 and Γ_2,

$$\mathcal{S}_1 = p_1 - q_1\partial_{\tau\tau} + q_1\tau^2\eta, \quad \mathcal{S}_2 = -p_2 + q_2\partial_{\tau\tau} - q_2\tau^2\eta.$$

Inserting $\mu_i^{app}(k)$, $i = 1, 2$ into (7) leads to the convergence factor

$$\rho_{opt}(\alpha_k, L, p_1, p_2, q_1, q_2) := \frac{(\partial_\sigma + p_1 + q_1\alpha_k)\phi_{de}(\sigma_0 + L)}{(\partial_\sigma + p_1 + q_1\alpha_k))\phi_{in}(\sigma_0 + L)}\frac{(\partial_\sigma - p_2 - q_2\alpha_k)\phi_{in}(\sigma_0)}{(\partial_\sigma - p_2 - q_2\alpha_k)\phi_{de}(\sigma_0)}. \quad (8)$$

The best choice for the free parameters p_i, q_i, $i = 1, 2$, minimizes the convergence factor, i.e. it is solution of the min-max problem

$$\min_{p_i>0, q_i\geq 0, i=1,2} \max_{\alpha\in[\alpha_{\min},\alpha_{\max}]} |\rho_{opt}(\alpha, L, p_1, p_2, q_1, q_2)|. \tag{9}$$

Using the theory of ordinary differential equations, one can prove

Lemma 1. *a) For any fixed $\alpha, \eta > 0$, $\phi_{in}(\sigma; \alpha, \eta)$ is monotonically increasing in σ for $\sigma > 0$. For any fixed $\sigma, \eta > 0$, $\frac{\partial_\sigma \phi_{in}(\sigma;\alpha,\eta)}{\phi_{in}(\sigma;\alpha,\eta)}$ is monotonically increasing in α for $\alpha > 0$.*
b) For any fixed $\alpha, \eta > 0$, $\phi_{de}(\sigma; \alpha, \eta)$ is monotonically decreasing in σ for $\sigma \in (0, 1)$. For any fixed $\sigma, \eta > 0$, $-\frac{\partial_\sigma \phi_{de}(\sigma;\alpha,\eta)}{\phi_{de}(\sigma;\alpha,\eta)}$ is monotonically increasing in α for $\alpha > 0$.

Let $G(\sigma, \alpha, \eta) := \frac{\partial_\sigma \phi_{in}(\sigma;\alpha,\eta)}{\phi_{in}(\sigma;\alpha,\eta)} - \frac{\partial_\sigma \phi_{de}(\sigma;\alpha,\eta)}{\phi_{de}(\sigma;\alpha,\eta)}$ and $G_{\min} := G(\sigma_0; \alpha_{\min}, \eta)$.

Theorem 1. *For the OO0 (optimized of order 0) method, let $p_1 = p_2 = p > 0$ and $q_1 = q_2 = 0$. Then for small overlap, $L > 0$, the parameter $p^* = 2^{-1} G_{\min}^{\frac{2}{3}} L^{-\frac{1}{3}}$ solves asymptotically the min-max problem (9) and*

$$\max_{\alpha\in[\alpha_{\min},\alpha_{\max}]} |\rho_{opt}(\alpha, L, p^*, p^*, 0, 0)| = 1 - 4G_{\min}^{\frac{1}{3}} L^{\frac{1}{3}} + O(L^{\frac{2}{3}}). \tag{10}$$

Proof. Using Lemma 1, the results can be proved by the techniques used to prove Theorem 3.8 and Theorem 3.9 in [5].

Similar results can also be proved for the OO2 (optimized of order 2) method and the O2s (optimized two-sided Robin) method for overlapping, and non-overlapping domain decompositions. The corresponding results are summarized in Table 1.

	Type	Constraint	Optimized parameters	$\max\lvert\rho_{opt}\rvert$
$L > 0$	OO2	$p_1 = p_2 > 0$ $q_1 = q_2 > 0$	$p_1^* = p_2^* = 2^{-\frac{7}{5}} G_{\min}^{\frac{4}{5}} L^{-\frac{1}{5}}$ $q_1^* = q_2^* = 2^{\frac{1}{5}} G_{\min}^{-\frac{2}{5}} L^{\frac{3}{5}}$	$1 - 2^{\frac{12}{5}} G_{\min}^{\frac{1}{5}} L^{\frac{1}{5}} + O(L^{\frac{2}{5}})$
	O2s	$p_1 > 0, p_2 > 0$ $q_1 = q_2 = 0$	$p_1^* = 2^{-\frac{8}{5}} G_{\min}^{\frac{4}{5}} L^{-\frac{1}{5}}$ $p_2^* = 2^{-\frac{4}{5}} G_{\min}^{\frac{2}{5}} L^{-\frac{3}{5}}$	$1 - 2^{\frac{8}{5}} G_{\min}^{\frac{1}{5}} L^{\frac{1}{5}} + O(L^{\frac{2}{5}})$
$L = 0$	OO0	$p_1 = p_2 > 0$ $q_1 = q_2 = 0$	$p_1^* = p_2^* = 2^{-\frac{1}{2}} G_{\min}^{\frac{1}{2}} \alpha_{\max}^{\frac{1}{4}}$	$1 - 2^{\frac{3}{2}} G_{\min}^{\frac{1}{2}} \alpha_{\max}^{-\frac{1}{4}} + O(\alpha_{\max}^{-\frac{1}{2}})$
	OO2	$p_1 = p_2 > 0$ $q_1 = q_2 > 0$	$p_1^* = p_2^* = 2^{-\frac{5}{4}} G_{\min}^{\frac{3}{4}} \alpha_{\max}^{\frac{1}{8}}$ $q_1^* = q_2^* = 2^{-\frac{1}{4}} G_{\min}^{-\frac{1}{4}} \alpha_{\max}^{-\frac{3}{8}}$	$1 - 2^{\frac{9}{4}} G_{\min}^{\frac{1}{4}} \alpha_{\max}^{-\frac{1}{8}} + O(\alpha_{\max}^{-\frac{1}{4}})$
	O2s	$p_1 > 0, p_2 > 0$ $q_1 = q_2 = 0$	$p_1^* = 2^{-\frac{5}{4}} G_{\min}^{\frac{3}{4}} \alpha_{\max}^{\frac{1}{8}}$ $p_2^* = 2^{\frac{1}{4}} G_{\min}^{\frac{1}{4}} \alpha_{\max}^{\frac{3}{8}}$	$1 - 2^{\frac{5}{4}} G_{\min}^{\frac{1}{4}} \alpha_{\max}^{-\frac{1}{8}} + O(\alpha_{\max}^{-\frac{1}{4}})$

Table 1 Optimized transmission parameters and the corresponding convergence factor estimate.

4 Geometric characteristics entering the optimization

In Section 3 we obtained the optimized transmission conditions in the parabolic coordinates (σ,τ), where the interface is a line. In a real application, one would however compute in the standard Cartesian coordinates where the interface is a parabola in our model problem, and we study now how the optimized parameter of OO0 looks in the standard Cartesian coordinates to see how geometric characteristics enter the optimization of the transmission parameters. Without loss of generality, we consider only the interface Γ_1, where the optimized transmission condition is

$$(\partial_\sigma + p^*)u_1^n(\sigma_0+L,\tau) = (\partial_\sigma + p^*)u_2^{n-1}(\sigma_0+L,\tau). \tag{11}$$

A direct calculation gives $\partial_{n_1} = \frac{1}{\sqrt{\sigma^2+\tau^2}}\partial_\sigma$, and dividing both sides of (11) by $\sqrt{\sigma^2+\tau^2}$ we get

$$(\partial_{n_1} + \frac{1}{\sqrt{\sigma^2+\tau^2}}p^*)u_1^n(x,y) = (\partial_{n_1} + \frac{1}{\sqrt{\sigma^2+\tau^2}}p^*)u_2^{n-1}(x,y), \text{ on } \Gamma_1. \tag{12}$$

A further direct calculation shows that $\sigma^2+\tau^2 = \sqrt{x^2+y^2} - x + \frac{y^2}{\sqrt{x^2+y^2-x}}$, and hence in Cartesian coordinates the optimized transmission parameter is given by $(\sqrt{x^2+y^2} - x + \frac{y^2}{\sqrt{x^2+y^2-x}})p^*$, i.e. it varies along the interface, instead of being a constant. To see how the interface curvature enters this optimized transmission condition, we compute the curvature of the interface Γ_1 and obtain $\kappa = \frac{\sigma}{(\sigma^2+\tau^2)^{\frac{3}{2}}} = \frac{\sigma}{H^3}$ with $\sigma = \sigma_0 + L$. Hence the optimized parameter in Cartesian coordinates is given by $(\frac{\sigma_0+L}{\kappa})^{-\frac{1}{3}}p^*$. Note that the constant $\sigma_0 + L$ describes the position of the parabolically shaped interface. Therefore, in addition to the interface curvature, other geometric characteristics (here the constant σ_0+L) can enter as well the optimized transmission parameters.

5 Numerical experiments

To show that our predicted transmission parameter from Theorem 1 is indeed asymptotically optimal, we first consider the model problem (1) in the parabolic coordinates (σ,τ), i.e. the OO0 variant of the Schwarz algorithm (5), with $\sigma_0 = 0.5$ and $\mathcal{S}_i = p^*$, $i = 1,2$. We discretize (5) using FreeFem++, and start with a random initial guess on the interfaces, simulating directly the error equations, i.e. $f = 0$. The number of iterations required to reach an error reduction of $1e-6$ is shown in the first row of Table 2. A log-log plot of these results on the left in Fig. 2 shows good agreement with the estimate

Coordinates	N	20	40	80	160	320
Parabolic	#iter(OO0)	8	11	13	17	23
Cartesian	#iter(OO0)	8	12	14	19	24
	#iter(OO0-Scaled)	10	12	16	22	28
	#iter(OO0-Straight)	10	13	16	22	28

Table 2 Iteration numbers of the OO0 Schwarz method with overlap $1/N$ discretized in parabolic coordinates (first row), compared to discretization in Cartesian coordinates taking all geometric information into account (second row), and using the optimized parameter from the straight interface analysis [2] either locally scaled by the interface curvature (third row) or with $k_{\min} = \pi/c$, where c is the interface length (last row).

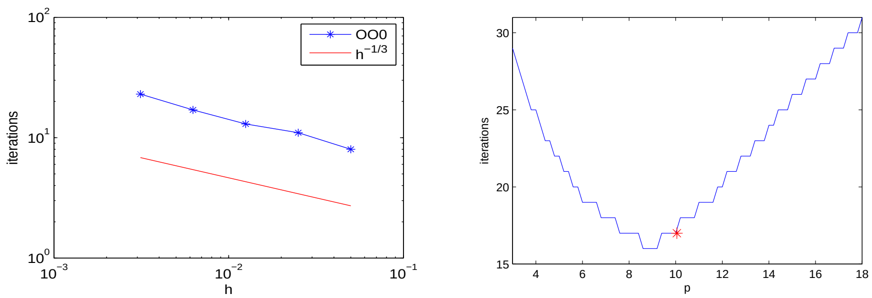

Fig. 2 Left: Log-log plot of the number of iterations from the first row in Table 2. Right: Number of iterations required by the OO0 Schwarz method in parabolic coordinates compared to other values of the Robin parameter p; the red star indicates our prediction p^*.

in Theorem 1. To show how our prediction p^* approximates the numerically optimal Robin parameter, we vary the Robin parameter p from 3 to 18 with 76 equidistant samples and record the corresponding number of iterations required by the Schwarz method with $N = 160$. The results are shown on the right in Fig. 2, and we see that our prediction p^* is very close to the numerically optimal Robin parameter.

We next solve the model problem (1) in Cartesian coordinates using Freefem++ like one would in a real application. We choose again the interface parameter $\sigma_0 = 0.5$, and use the transmission condition (12) on Γ_1 and a corresponding one on Γ_2. In this situation the overlap is the local distance between the interfaces Γ_1 and Γ_2. In Table 2 in the second row we show the number of iterations required by the optimized Schwarz method to reach an error reduction of $1e-6$. Comparing with the first row, we see that our prediction of the optimized Robin parameter taking into account all geometric characteristics performs basically as when computing in the parabolic coordinates. In the third and last row of Table 2, we show the results obtained with the strategy suggested in [7], i.e. to use the optimized transmission parameter from the straight interface analysis [2], either scaled locally by the interface curvature, or choosing $k_{\min} = \pi/c$ with c the length of the interface[1]. These last two approaches also reach the same asymptotic convergence order and are comparable, but more iterations are needed than for our new approach which takes more geometric features into account.

6 Conclusion

To get a better understanding on the influence of geometry on optimized transmission conditions, we studied a model problem using a domain decomposition with parabolically shaped interfaces. Using separation of variables, we showed that the optimized parameter in Cartesian coordinates varies along the interface, and not only the interface curvature comes in, but also further geometric characteristics of the interface appear. We then showed numerically that indeed taking all these geometric characteristics into account the new optimized parameter outperforms the strategy of using only the local curvature or interface length to scale appropriately an optimized parameter from a straight interface analysis.

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[1] The length of the interface $\sigma = \sigma_0$ is easy to calculate to be $\frac{\sigma_0^2}{2}\mathtt{arcsinh}(\frac{1}{\sigma_0}) + \frac{1}{2}\sqrt{\sigma_0^2+1}$.