1 Non Euclidean Geometry

1.1 Statement of the Problem

Ron is working on a problem in $\mathbb{R}^4$ with non euklidean distance. The distance between two points $p, q \in \mathbb{R}^4$ is defined as

$$d(p,q) := \inf_{\substack{\text{curves } \boldsymbol{x}(\text{t}) \\ \boldsymbol{x}(0)=\text{p},\ \boldsymbol{x}(1)=\text{q}}} L(\boldsymbol{x})$$

where the length of the parameterized curve $\boldsymbol{x}(t)$ from p to q is defined by

$$\begin{aligned} L(\boldsymbol{x}) &:= \int_0^1 ||x'(t)||dt \\ &= \int_0^1 \sqrt{x_1'(t)^2 + x_2'(t)^2 + x_3'(t)^2 + x_4'(t)^2}dt. \end{aligned}$$

The curves of this geometry are defined by the system of ODE's

$$\begin{aligned} x_1' &= h_1 \\ x_2' &= h_2 \\ x_3' &= h_3 + x_1 h_2 \\ x_4' &= h_4 + x_1 h_3 + \frac{x_1^2}{2} h_2 \\ h_1' &= -h_2 h_3 - h_3 h_4 \\ h_2' &= h_1 h_3 \\ h_3' &= h_1 h_4 \\ h_4' &= 0 \end{aligned} \tag{1}$$

Ron is interested in several problems. First he would like to know how the set with distance δ from zero looks like,

$$B(\delta) = \{\boldsymbol{x} \in \mathbb{R}^4 | d(0, \boldsymbol{x}) \leq \delta\}.$$

To do this, he takes N arbitrary points $h_0^i \in S^3$ as initial condition for $\boldsymbol{h}$ in (1) and then solves with $\boldsymbol{x}_0 = 0$ up to time $t = \delta$. Keeping all the point of the integration gives an impression of how $B(\delta)$ looks like.

The second problem is that for a given $\boldsymbol{x}_1$ he would like to find the path $\boldsymbol{x}(t)$ that leads from 0 to $\boldsymbol{x}_1$. In other words he wants to solve the boundary value problem (1) with given initial value $\boldsymbol{x}(0) = 0$ and end value $\boldsymbol{x}(1) = x_1$.

2 Advice given

For the first problem Ron already proposed a reasonable solution. For the boundary value problem there are three ways to find a solution, if a solution exists. Existence of a solution was analyzed by H. Keller and sufficient conditions are given in his book 'Numerical Methods for Two Point Boundary Value Problems'.

2.1 Shooting

For a given $\boldsymbol{x}_0$ and $\boldsymbol{h_0}$ equation (1) generates a solution path $\boldsymbol{x}(t, \boldsymbol{h}_0)$ where we emphasize the dependence on the parameter $\boldsymbol{h}_0$. To find a solution to the boundary value problem, we have to determine an $\boldsymbol{h}_0$ such that

$$\boldsymbol{\phi}(\boldsymbol{h}_0) := \boldsymbol{x}(1, \boldsymbol{h}_0) - \boldsymbol{x}_1 = 0. \tag{2}$$

Hence we have to find a zero of the nonlinear function $g : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$. This can be done using Newton's method, namely

$$d\boldsymbol{\phi}(\boldsymbol{h}_0^k)(\boldsymbol{h}_0^{k+1} - \boldsymbol{h}_0^k) = -\boldsymbol{\phi}(\boldsymbol{h}_0^k).$$

The function values of $\boldsymbol{\phi}$ are readily available solving (1), but the values of $d\boldsymbol{\phi}$ are not directly at hand. From (2) we have

$$d\boldsymbol{\phi}(\boldsymbol{h}_0^k) = d\boldsymbol{x}(1, \boldsymbol{h}_0^k).$$

We take the derivative of the ODE (1) written in compact notation as

$$\begin{aligned} \boldsymbol{x}' &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{h}) \\ \boldsymbol{h}' &= \boldsymbol{g}(\boldsymbol{h}) \end{aligned} \tag{3}$$

with respect to the initial data $\boldsymbol{h}_0$ and obtain

$$\begin{aligned} d\boldsymbol{x}' &= d^x \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{h}) d\boldsymbol{x} + d^h \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{h}) d\boldsymbol{h} \\ d\boldsymbol{h}' &= d\boldsymbol{g}(\boldsymbol{h}) d\boldsymbol{h}, \end{aligned} \tag{4}$$

together with the derivative of the initial condition $d\boldsymbol{x}(0) = 0$ and $d\boldsymbol{h}(0) = I$, the identity matrix.

Now one can obtain $\boldsymbol{\phi}$ and $d\boldsymbol{\phi}$ by solving (1) and (4) numerically for each Newton step.

2.2 Finite Difference

The problem can also be solved using finite differences, for example Backward Euler. Applied to the ODE (3) we get

$$\begin{aligned} \boldsymbol{x}_{j+1} - \boldsymbol{x}_j &= \Delta t \boldsymbol{f}(\boldsymbol{x}_{j+1}, \boldsymbol{h}_{j+1}) \\ \boldsymbol{h}_{j+1} - \boldsymbol{h}_j &= \Delta t \boldsymbol{g}(\boldsymbol{h}_{j+1}). \end{aligned} \tag{5}$$

Using N gridpoints (5) gives $4(N+1)+4(N+1)$ unknowns and $4N+4N$ equations. The missing 8 equations come from the given boundary conditions $\boldsymbol{x}_0$ and $\boldsymbol{x}_N$. Hence the nonlinear system is determined and can be solved, for example again by Newtons method.

2.3 Finite Elements

The finite element method leads to a similar system of equations like the finite difference method.