1 Flow in Porous Media

Oil companies often pump water into the cavities of the earth where the oil is situated to drive out the oil. In a simplified situation, as given in figure 1 we have a rectangular block of porous material filled with oil. Water is pumped in from the left, creating a presure difference between

Figure 1: Driving oil out of a porous material using water

the left and the right side of the block. This difference causes water to enter the block of porous material from the left and drive out the oil on the right, in the ideal case for the oil company without mixing oil and water together. In reality the water and the oil get mixed together, which is modeled by a saturation function S which varies between 0 for pure oil and 1 for pure water. Now suppose F is the flux of water. Conservation of mass implies that the difference of water flowing out of an infinitesimal volume element minus the water flowing into it equals the rate of change of the saturation, which leads to

$$\frac{\partial S}{\partial t} + \nabla \cdot \boldsymbol{f} = 0. \tag{1}$$

Assuming a functional relationship between F and S, $F = F(S)$, which depends on the permeability of the porous material as well as the viscosity of oil and water, equation (1) becomes a nonlinear, first order wave equation, namely

$$\frac{\partial S}{\partial t} + \boldsymbol{C}(S) \cdot \nabla S = 0. \tag{2}$$

where we've defined

$$C(S) := \frac{d\boldsymbol{f}}{dS}.$$

This equation describes the so called one phase flow in porous media, since we have assumed one porous material.

Leonid is interested in two phase flow in one dimension. We have two different materials and hence two equations, one for each material. A simple situation is given in figure 2. If the

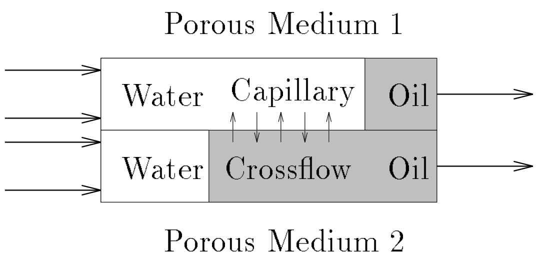

Figure 2: Two phase flow

permeability of one of the materials is much bigger than the one of the other, water can enter this material much easier and travels faster in this material. Hence at the interface between the two materials, the saturation in one of the materials may be much bigger than in the other one. This difference causes a capillary flow of water from the higher saturated material towards the lower saturated one. In several space dimensions this phenomenon would be modeled as a boundary condition, where as in the one dimensional case, we can model it as a source term within the partial differential equation. This leads to the following system of partial differential equations

$$\begin{aligned}
\frac{\partial S_1}{\partial t} + C_1(S)\frac{\partial S_1}{\partial x} &= F(S_1, S_2) \\
\frac{\partial S_2}{\partial t} + C_2(S)\frac{\partial S_2}{\partial x} &= -F(S_1, S_2).
\end{aligned}$$

The domain on which these equations hold is the first quarter plane in the xt plane, $[0, \infty] \times [0, \infty]$. Initial and boundary conditions are given by

$$\begin{aligned}
S_1(x, 0) &= 0 \\
S_1(0, t) &= 1 \\
S_2(x, 0) &= 0 \\
S_2(0, t) &= 1.
\end{aligned}$$

Leonid is interested in analytical solutions of this system.

2 Advice given

Since this is a first order PDE, the method of characteristics can be applied. But the chance to find an analytical solution is small, since for a system we get in general for each equation a different characteristic, along which the solution propagates. Using the method of characteristics leads to two systems of ODE's, the first one for the characteristics emerging from the initial line, namely

$$\begin{aligned}
\frac{\partial x_1}{\partial t} &= C_1(S_1); & x_1(0) = \phi \\
\frac{\partial S_1}{\partial t} &= F(S_1, S_2); & S_1(0) = 0 \\
\frac{\partial x_2}{\partial t} &= C_2(S_2); & x_2(0) = \phi \\
\frac{\partial S_2}{\partial t} &= -F(S_1, S_2); & S_2(0) = 0,
\end{aligned}$$

where ϕ is the parameter along the initial condition, and the second one for the characteristics emerging from the boundary at $x = 0$,

$$\begin{aligned}
\frac{\partial x_1}{\partial t} &= C_1(S_1); & x_1(0) = 0 \\
\frac{\partial S_1}{\partial t} &= F(S_1, S_2); & S_1(0) = 1 \\
\frac{\partial x_2}{\partial t} &= C_2(S_2); & x_2(0) = 0 \\
\frac{\partial S_2}{\partial t} &= -F(S_1, S_2); & S_2(0) = 1.
\end{aligned}$$

From the equations we see that S_1 and S_2 grow in opposite directions at the same rate,

$$\frac{\partial S_1}{\partial t} = -\frac{\partial S_2}{\partial t},$$

therefore we have $S_1 = -S_2$ along the corresponding characteristic emerging from the initial line at ϕ, because we have homogeneous data there. Furthermore if we assume that

$$C_1(S) \equiv C_2(S) \equiv C(S) \tag{3}$$

where $C(S)$ is an even function, the characteristics become the same for the two equations, permitting an analytical solution emerging from the initial condition. Unfortunately this method does not work from the boundary condition since it is inhomogeneous and therefore $S_1 \neq -S_2$ along the corresponding characteristic. Moreover assumption 3 has no physical justification according to Leonid.

3 Conclusion

After discussing the problem in our group meeting, we came to the conclusion that an analytical solution can not be found for this problem in general. Even with the simplification that the functions C_1, C_2 and F are linear in their arguments, an analytical solution does not exist. The main problem is that the method of characteristics does not work for systems of equations, because for every equation in the system one gets a corresponding characteristic, and they differ from the characteristic of the other equations. For example in our case we have two coupled equations, and we need the values of S_1 on the characteristic of S_2, which we don't have. Since the problem is already solved using various numerical methods, there is no more advice that can be given.