Noname manuscript No.
(will be inserted by the editor)

Power boundedness in the maximum norm of stability matrices for ADI methods

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Abstract The study of convergence of time integrators, applied to linear discretized PDEs, relies on the power boundedness of the stability matrix R. The present work investigates power boundedness in the maximum norm for ADI-type integrators in arbitrary space dimension m. Examples are the Douglas scheme, the Craig-Sneyd scheme, and W-methods with a low stage number. It is shown that for some important integrators $\|R^n\|_\infty$ is bounded in the maximum norm by a constant times $\min\big((\ln(1+n))^m, (\ln N)^m\big)$, where m is the space dimension of the PDE, and $N \geq 2$ is the space discretization parameter. For $m \leq 2$ sharper bounds are obtained that are independent of n and N.

Keywords Parabolic PDEs, time integration, Alternating Direction Implicit schemes, stability, power boundedness, maximum norm.

Mathematics Subject Classification (2000) 65M12, 65M20.

1 Introduction

We consider linear diffusion problems

$$\partial_t u(t,\mathbf{x}) = \sum_{j=1}^{m} \alpha_j \, \partial_{x_j x_j} u(t,\mathbf{x}) + c(t,\mathbf{x}), \qquad t \geq 0, \tag{1.1}$$

for $\mathbf{x} = (x_1, \ldots, x_m)^\top \in (0,1)^m$, with constants $\alpha_j > 0$, $1 \leq j \leq m$, and Dirichlet boundary conditions, where the space dimension m can be arbitrarily large. A second order central finite difference discretization on a uniform

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grid with spacing $\Delta x_j = 1/(N_j + 1)$ yields an ordinary differential system of equations

$$\dot{U} = D\,U + g(t), \qquad D = D_1 + \ldots + D_m, \tag{1.2}$$

where $D_j = \alpha_j\,(I_{N_m} \otimes \ldots \otimes D_{x_j x_j} \otimes \ldots \otimes I_{N_1})$. Here, the differentiation matrices $D_{x_j x_j}$ are tridiagonal with entries $(1, -2, 1)/\Delta x_j^2$, and $\otimes$ stands for the Kronecker product of matrices. The dimension of the system (1.2) is $N_x := N_1 \cdot \ldots \cdot N_m$ and we denote

$$\alpha := \max_{1 \le j \le m} \alpha_j.$$

Many numerical methods applied to (1.2) typically produce a recursion for the global errors $E_n := U(t_n) - U_n$, $n \ge 0$, of the form $E_{n+1} = RE_n + \delta_n$, where U_n stands for the numerical solution at t_n, δ_n, $n \ge 0$, are local errors and R is a certain stability matrix associated to the numerical integrator (see, e.g., [12, Sec. II.2.3]. For ADI-type integrators the stability matrix depends on $D_1, \ldots, D_m$ and not only on their sum D. A typical example is

$$R(\tau D_1, \ldots, \tau D_m) = I + \Pi(\theta)^{-1}\tau D, \tag{1.3}$$

where $\Pi(\theta) = (I - \theta\tau D_1) \ldots (I - \theta\tau D_m)$, which has the associated stability function of m complex variables

$$R(z_1, z_2, \ldots, z_m) = 1 + \frac{z}{P}, \quad z := \sum_{j=1}^{m} z_j, \quad P := \prod_{j=1}^{m}(1 - \theta z_j). \tag{1.4}$$

For ease of notation we will henceforth denote by R both the stability function (1.4) and the stability matrix (1.3) of the method. The difference between them can be observed from their arguments. For the choice $\theta = 1/2$ it is the stability matrix of the Peaceman–Rachford integrator [15], the Douglas scheme [5], the Crank-Nicolson scheme with locally one-dimensional splitting [14], and of the modified Craig–Sneyd scheme [3,13]. It is also the stability matrix of the one-stage W-method [7].

There is also much effort in the construction of ADI-type time integrators of order higher than 2, see e.g., [16,8]. The stability matrix of the so-called Hundsdorfer–Verwer scheme [12, Section IV.5.2], which is a 2-stage W-method of order 2 in general, and of order 3 for $\theta = (3 + \sqrt{3})/6$, is given by

$$R(\tau D_1, \ldots, \tau D_m) = I + 2\Pi(\theta)^{-1}\tau D - \Pi(\theta)^{-2}\tau D + \frac{1}{2}\big(\Pi(\theta)^{-1}\tau D\big)^2. \tag{1.5}$$

In this case the stability function is given by (with z and P defined in (1.4))

$$R(z_1, z_2, \ldots, z_m) = 1 + \frac{2z}{P} + \frac{z^2 - 2z}{2P^2}. \tag{1.6}$$

Outline of the paper. This article is devoted to bounds in the infinity (operator) norm of the nth power of stability matrices arising in ADI-type time integrators. The main results are presented in Section 2, where we prove power boundedness in space dimension $m \le 2$. For $m \ge 3$ we give two type of bounds. One of them shows a $\mathcal{O}\big((\ln(1+n))^m\big)$ behaviour that is independent of the space discretization parameter

$$N := \max(N_1, \dots, N_m) \ge 2,$$

whereas the other shows a $\mathcal{O}\big((\ln N)^m\big)$ bound which is independent on n. Numerical experiments (Section 3) for $m = 3$ indicate that an N-dependence is present in the norm of the nth power of the stability matrix. Section 5 provides the tools to prove the estimates, while Section 4 provides the preliminary work. The proof of Theorem 2.1 is then given in Section 6, and that of Theorem 2.2 in Sections 7 and 8. Finally, Section 9 explains the extension to more general stability matrices like that of (1.5).

2 Main results

We collect the bounds for powers of the stability matrix (1.3) which have important consequences in the convergence studies of PDEs. Our first result presents bounds that are independent of the space discretization parameter $N \ge 2$.

Theorem 2.1 *For the stability matrix* (1.3) *with* $\theta \ge 1/2$ *we have*

$$\big\|R(\tau D_1, \dots, \tau D_m)^n\big\|_\infty \le \begin{cases} c_m & \textit{for} \quad m = 1,2 \quad \textit{and} \quad n \ge 1, \\ c_m(1+\ln n)^m & \textit{for} \quad m \ge 3 \quad \textit{and} \quad n \ge 1, \end{cases}$$

where the constants c_m *only depend on* θ *and on* m.

In dimension $m = 1$ and for $\theta = 1/2$ the stability matrix (1.3) equals $R(\tau D_1) = (I - \frac{\tau}{2} D_1)^{-1}(I + \frac{\tau}{2} D_1)$ with $D_1 = \alpha_1 D_{x_1 x_1}$, which we denote by $Tr(\tau D_1)$, because it is the stability matrix for the trapezoidal rule (or Crank–Nicolson method). Power boundedness has been considered by several authors. An early contribution is [17], where the bound $c_1 = 23$ is proved. The improved bound $c_1 = 4.325$ has been obtained in [6]. For general $\theta \ge 1/2$ the stability function (where τD_1 is replaced by the complex variable z_1) can be written as

$$R(z_1) = 1 + \frac{z_1}{1 - \theta z_1} = 1 - \frac{1}{2\theta} + \frac{1}{2\theta} \frac{(1 + \theta z_1)}{(1 - \theta z_1)}$$

The nth power of the stability matrix thus satisfies (with $\tau_1 = 2\theta\tau\alpha_1$)

$$R(\tau D_1)^n = \sum_{i=0}^{n} \binom{n}{i} \Big(1 - \frac{1}{2\theta}\Big)^{n-i} \frac{1}{(2\theta)^i}\ Tr(\tau_1 D_{x_1 x_1})^i.$$

Applying the triangle inequality and using $\|Tr(\tau_j D_{x_j x_j})^i\|_\infty \le c_1$ we obtain for all $\theta \ge 1/2$ the same bound c_1 as for the case $\theta = 1/2$. Note that for the special case $\theta = 1$ (backward Euler method) we have contractivity in the maximum norm (see [18]), so that $c_1 = 1$ in that case.

In dimension $m = 2$ similar arguments are possible. The stability function (where τD_j is replaced by the complex variable z_j in (1.3)) satisfies the identity

$$R(z_1, z_2) = 1 + \frac{z_1 + z_2}{(1-\theta z_1)(1-\theta z_2)} = 1 - \frac{1}{2\theta} + \frac{1}{2\theta}\frac{(1+\theta z_1)(1+\theta z_2)}{(1-\theta z_1)(1-\theta z_2)}.$$

As before we have (with $\tau_j = 2\theta\tau\alpha_j$)

$$R(\tau D_1, \tau D_2)^n = \sum_{i=0}^{n} \binom{n}{i} \Big(1 - \frac{1}{2\theta}\Big)^{n-i} \frac{1}{(2\theta)^i}\; Tr(\tau_2 D_{x_2x_2})^i \otimes Tr(\tau_1 D_{x_1x_1})^i.$$

Applying the triangle inequality, using $\|A \otimes B\|_\infty = \|A\|_\infty \|B\|_\infty$ and the bound $\|Tr(\tau_j D_{x_jx_j})^i\|_\infty \le c_1$, we obtain the statement of the theorem with $c_2 = c_1^2$.[1] Unfortunately, this elegant proof does not extend to $m \ge 3$.

Sharper bounds than those of Theorem 2.1 are obtained in Theorem 2.2 for the case in which $m \ge 3$, $n > N$ and under the weak step size restriction

$$\tau \le c^* \mathit{\Delta} x_j \qquad \text{for} \quad j = 1, \dots, m. \tag{2.1}$$

The last inequality is typically assumed in the study of convergence for time integrators.

Theorem 2.2 *For the stability matrix* (1.3) *with* $\theta \ge 1/2$ *and for arbitrary* n, $\tau > 0$ *and* $N \ge 2$ *such that (2.1) is fulfilled, we have that*

$$\big\|R(\tau D_1, \dots, \tau D_m)^n\big\|_\infty \le \begin{cases} c'_m (\ln N)^m & \text{for} \quad m \ge 3 \quad \text{and} \quad n \ge 1, \\ c' \ln N & \text{for} \quad m = 3 \quad \text{and} \quad n \ge N^2, \end{cases}$$

where the constants c'_m, c' *only depend on* m, θ, α *and* c^*.

The proofs of these bounds for $m \ge 3$ are postponed to Section 6 (Theorem 2.1) and to Sections 7 and 8 (Theorem 2.2).

Remark 2.1 The bounds proposed above are relevant for the convergence analysis in arbitrary spatial dimensions and in the infinity norm to Alternating Direction Implicit methods, such as Peaceman–Rachford [15], Douglas [5], Crank-Nicolson with locally one-dimensional splitting [14], the modified Craig–Sneyd scheme [3,13] and the one-stage W-method [7]. In those cases, the logarithmic bounds for the growth of the powers of the stability matrix do not significantly deteriorate the second order of convergence in the infinity norm for PDE problems with time independent boundary conditions. These bounds can also be helpful for the convergence analysis for other stability matrices such as that corresponding to the Hundsdorfer-Verwer method [12, Section IV.5.2].

[1] For $m = 2$ and $\theta = 1/2$ power boundedness with $c_2 = c_1^2$ is proved in [2,7]. The proof for general $\theta \ge 1/2$ has been communicated to us by an anonymous referee.

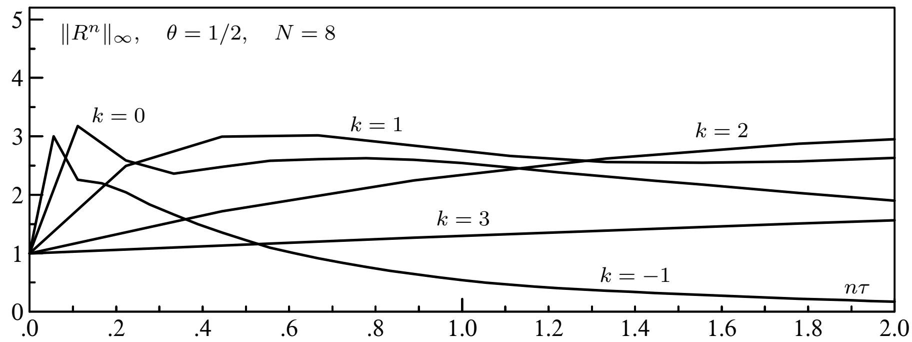

Fig. 3.1 Norm $\|R^n\|_\infty$ for the stability matrix (1.3) as a function of $n\tau$ with $m = 3$, $N = 8$, and $\tau = 2^k/(N+1)$ for $k = -1, 0, 1, 2, 3$.

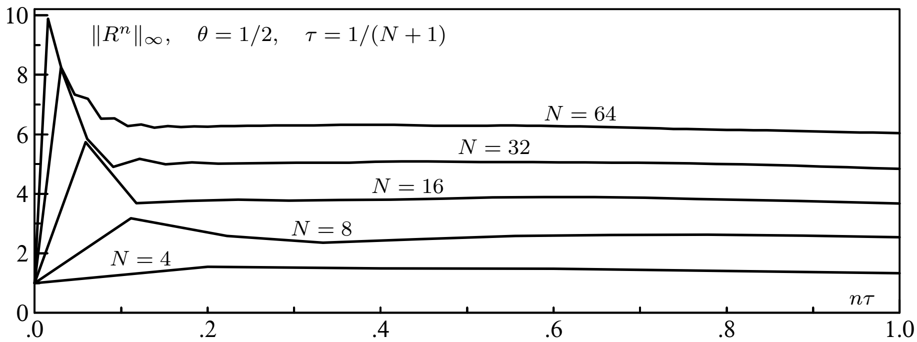

Fig. 3.2 Norm $\|R^n\|_\infty$ for the stability matrix (1.3) as a function of $n\tau$ with $m = 3$, and $\tau = 1/(N+1)$ for $N = 4, 8, 16, 32, 64$.

3 Numerical experiments

Since the modulus of the eigenvalues of $R = R(\tau D_1, \dots, \tau D_m)$ is, for fixed N and fixed τ, strictly smaller than 1, it holds $\|R^n\|_\infty \to 0$ for $n \to \infty$. Although the bounds of Section 2 are valid over a wide range of N and n, our main interest is in power boundedness for values of n such that τ is proportional to $1/(N+1)$. We restrict our experiments to dimension $m = 3$, because it is the first one, where a dependence on n or N should be observed. We also consider only the important case with $\theta = 1/2$.

In our first experiment (Figure 3.1) we consider a fixed space discretization $N = 8$ and we study $\|R^n\|_\infty$ as a function of $n\tau$ for various values of τ. We observe that the smaller the step size, the faster convergence towards zero occurs. For step sizes $\tau \le 1/(N+1)$ the maximum value is for $n = 1$, and for increasing step sizes $\tau > 1/(N+1)$ the maximum is for increasing n.

The dependence on the space discretization parameter N is studied in Figure 3.2. We consider $N = 4, 8, 16, 32, 64$ and we fix the time step size to $\tau = 1/(N+1)$. A clear dependence on N can be observed although it seems to be closer to $\ln N$ rather than to $(\ln N)^3$. From the figure we see that the maximum value is for $n = 1$, but since the dependence on n is not monotonic, it could take larger values for an increasing n. For convenience we have included the numbers for $\|R^n\|_\infty, (n = 1, 2, 3)$ up to $N = 128$ in Table 3.1.

dimension	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$	$N = 128$
$\|R\|_\infty$	1.547	3.175	5.741	8.224	9.877	10.839
$\|R^2\|_\infty$	1.488	2.586	3.806	5.846	8.246	9.645
$\|R^3\|_\infty$	1.481	2.361	3.765	4.902	7.330	9.959

Table 3.1 Values of $\|R^k\|_\infty$ $(m = 3, \theta = 1/2, \tau = 1/(N+1))$ for various N.

4 Resolvent bounds in the maximum norm

The proof for the resolvent bound in the maximum norm is based on the approach by Grigorieff [9], which relies on a result by Thomée [19]. It is closely related to the computations of Crouzeix [4]. The bound of Lemma 4.1 below is presented and used in [6] referring for its proof to an unpublished manuscript.

We denote the tridiagonal matrix appearing in (1.2) by $T = \text{tridiag}\,(1, -2, 1)$ (dimension N). Its resolvent is

$$(zI - T)^{-1} = G(z) = \big(g_{j,k}(z)\big)_{j,k=1}^{N}. \tag{4.1}$$

We are interested in sharp bounds for the resolvent in the maximum norm,

$$\big\|(zI - T)^{-1}\big\|_\infty = \max_{j=1,\ldots,N} \sum_{k=1}^{N} |g_{j,k}(z)|. \tag{4.2}$$

4.1 Explicit formulas for the coefficients of the resolvent

The relation $(zI - T)G(z) = I$ reads

$$-g_{j,k-1}(z) + (2+z)g_{j,k}(z) - g_{j,k+1} = \delta_{j,k}, \tag{4.3}$$

where $\delta_{j,k}$ is Kronecker's symbol. We use the convention that $g_{j,0} = g_{j,N+1} = 0$ for all j. For fixed j, (4.3) represents a 3-term recurrence relation with characteristic equation

$$\lambda^2 - (2+z)\,\lambda + 1 = 0. \tag{4.4}$$

Denoting the zeros of this polynomial by $\lambda_1(z)$, $\lambda_2(z)$, the solution of the homogeneous 3-term recursion (corresponding to (4.3)) is a linear combination of $\lambda_1(z)^k$ and $\lambda_2(z)^k$. Since the right-hand side of (4.3) is zero for all $k < j$, the solution of (4.3) satisfying $g_{j,0} = 0$ is given by the upper relation of

$$g_{j,k}(z) = \begin{cases} c_j(z)\,\big(\lambda_1^k(z) - \lambda_2^k(z)\big), & \text{for } k \le j \\ d_j(z)\,\big(\lambda_1^{N+1-k}(z) - \lambda_2^{N+1-k}(z)\big), & \text{for } k \ge j. \end{cases}$$

The lower relation follows from the fact that the right-hand side of (4.3) vanishes for $k > j$, that $g_{j,N+1} = 0$ and that $\lambda_1(z)\lambda_2(z) = 1$. In the following we suppress the dependence on z in the notation of c_j, d_j, and λ_1, λ_2. Equating the two formulas for $g_{j,j}(z)$ and using (4.3) with $k = j$ gives the equations

$$\begin{aligned} c_j\,(\lambda_1^j - \lambda_2^j) &= d_j\,(\lambda_1^{N+1-j} - \lambda_2^{N+1-j}) \\ -c_j\,(\lambda_1^{j-1} - \lambda_2^{j-1}) + (2+z)\,c_j\,(\lambda_1^j - \lambda_2^j) - d_j\,(\lambda_1^{N-j} - \lambda_2^{N-j}) &= 1 \end{aligned}$$

for the coefficients c_j, d_j. With $(2+z)\lambda_i^j = \lambda_i^{j+1} + \lambda_i^{j-1}$ for $i \in \{1,2\}$ as well as $\lambda_1\lambda_2 = 1$, a short calculation yields the coefficients c_j, d_j and we obtain [9]

$$g_{j,k}(z) = \frac{1}{(\lambda_1 - \lambda_2)(\lambda_1^{N+1} - \lambda_2^{N+1})} \begin{cases} (\lambda_1^{N+1-j} - \lambda_2^{N+1-j})(\lambda_1^k - \lambda_2^k) & \text{for } k \le j \\ (\lambda_1^j - \lambda_2^j)(\lambda_1^{N+1-k} - \lambda_2^{N+1-k}) & \text{for } k \ge j. \end{cases}$$

This explicit formula for the elements $g_{j,k}(z)$ permits us to bound the maximum norm (4.2). Without loss of generality we assume $|\lambda_1| \ge 1 \ge |\lambda_2|$. Using a geometric series for the case $|\lambda_1| > 1$, and the fact that $|\lambda_2|/\big(|\lambda_2| - 1\big) = 1/\big(1 - |\lambda_1|\big)$, we obtain

$$\begin{aligned} \sum_{k=1}^{j}\Big(|\lambda_1|^k + |\lambda_2|^k\Big) &= \frac{|\lambda_1|\big(|\lambda_1|^j - 1\big)}{|\lambda_1| - 1} + \frac{|\lambda_2|\big(|\lambda_2|^j - 1\big)}{|\lambda_2| - 1} \le \frac{|\lambda_1|^{j+1} - |\lambda_2|^j}{|\lambda_1| - 1}, \\ \sum_{k=j+1}^{N}\Big(|\lambda_1|^{N+1-k} + |\lambda_2|^{N+1-k}\Big) &= \sum_{r=1}^{N-j}\Big(|\lambda_1|^r + |\lambda_2|^r\Big) \le \frac{|\lambda_1|^{N-j+1} - |\lambda_2|^{N-j}}{|\lambda_1| - 1}. \end{aligned}$$

and consequently also

$$\begin{aligned} &\Big(|\lambda_1|^{N+1-j} + |\lambda_2|^{N+1-j}\Big)\sum_{k=1}^{j}\Big(|\lambda_1|^k + |\lambda_2|^k\Big) + \Big(|\lambda_1|^j + |\lambda_2|^j\Big)\cdot \\ &\qquad \sum_{k=j+1}^{N}\Big(|\lambda_1|^{N+1-k} + |\lambda_2|^{N+1-k}\Big) \le \frac{|\lambda_1| + 1}{|\lambda_1| - 1}\Big(|\lambda_1|^{N+1} - |\lambda_2|^{N+1}\Big). \end{aligned}$$

Using the triangle inequality in the explicit formula for $g_{j,k}(z)$ as well as $|\lambda_1^{N+1} - \lambda_2^{N+1}| \ge |\lambda_1|^{N+1} - |\lambda_2|^{N+1}$, the formula (4.2) leads to the bound

$$\big\|(zI - T)^{-1}\big\|_\infty \le \frac{|\lambda_1| + 1}{|\lambda_1 - \lambda_2|(|\lambda_1| - 1)}. \tag{4.5}$$

4.2 Resolvent bound depending on the angle of the argument

Here, we present a proof of the bound announced in [6]. It is common to use the notation, $\sec t \equiv 1/\cos t$.

Lemma 4.1 *We have the resolvent bound*

$$\left\|(zI-T)^{-1}\right\|_\infty \le \frac{\sec(\alpha/2)}{|z|} \qquad \textit{for} \quad z=\rho\,\mathrm{e}^{\mathrm{i}\alpha}\ne 0, \quad |\alpha|<\pi.$$

Proof Motivated by (4.5) we consider the function

$$\omega(\rho,\alpha)=\frac{|z|(|\lambda(z)|+1)}{(|\lambda(z)|-1)|\lambda(z)-1/\lambda(z)|}, \tag{4.6}$$

where $\lambda(z)$ is the root of maximum modulus in (4.4). To prove the statement of the lemma, we have to show that $\omega(\rho,\alpha)\le\sec(\alpha/2)$.

By taking the square root of $(\lambda-1)^2=z\lambda$ with $z=\rho\,\mathrm{e}^{\mathrm{i}\alpha}$, it follows that $\lambda=\lambda(z)=1+\sqrt{\rho}\,e^{i\alpha/2}+\mathcal{O}(\rho)$ for $\rho\to 0^+$. Inserted into (4.6), this implies

$$\lim_{\rho\to 0^+}\omega(\rho,\alpha)=\sec(\alpha/2). \tag{4.7}$$

For a fixed α and for $z=\rho\,\mathrm{e}^{\mathrm{i}\alpha}$ we write $\lambda(z)$ in polar coordinates as

$$\lambda(z)=x(\rho)\,\mathrm{e}^{\mathrm{i}\phi(\rho)}. \tag{4.8}$$

In part (A) below we prove that $x(\rho)$ is a strictly monotonically increasing function satisfying $x(0)=1$ and $x(\infty)=\infty$. We denote its inverse by $\rho(x)$. We then prove in part (B) that $\omega(x)=\omega\big(\rho(x),\alpha\big)$ is a monotonically decreasing function of x. From $\omega(\rho,\alpha)=\omega\big(x(\rho)\big)$ and from the chain rule we have

$$\partial_\rho\omega(\rho,\alpha)=\omega'\big(x(\rho)\big)x'(\rho).$$

Since $x'(\rho)>0$ by (A) and $\omega'(x)\le 0$ by (B), we have $\partial_\rho\omega(\rho,\alpha)\le 0$, so that $\omega(\rho,\alpha)$ is a monotonically decreasing function of ρ. From (4.7) it therefore follows $\omega(\rho,\alpha)\le\sec(\alpha/2)$, which proves the statement of the lemma.

(A) It follows from $(\lambda-1)^2=z\lambda$ that $x(0)=1$ and $x(\infty)=\infty$. Taking real and imaginary parts of $\lambda+\lambda^{-1}=2+z$ yields

$$(x^2+1)\cos\phi=2x+\rho x\cos\alpha, \qquad (x^2-1)\sin\phi=\rho x\sin\alpha. \tag{4.9}$$

Implicit differentiation with respect to ρ (with constant α) gives

$$\begin{aligned} 2xx'\cos\phi-(x^2+1)\phi'\sin\phi &= 2x'+(x+\rho x')\cos\alpha \\ 2xx'\sin\phi+(x^2-1)\phi'\cos\phi &= (x+\rho x')\sin\alpha. \end{aligned} \tag{4.10}$$

Assume now, by contradiction, that $x'(\rho)=0$ for some $\rho>0$. The relations (4.10) then read

$$x\cos\alpha=-(x^2+1)\phi'\sin\phi, \qquad x\sin\alpha=(x^2-1)\phi'\cos\phi.$$

Together with (4.9) we obtain

$$\tan\alpha = \frac{(1-x^2)\cos\phi}{(x^2+1)\sin\phi} = \frac{(x^2-1)\sin\phi}{(x^2+1)\cos\phi - 2x}. \tag{4.11}$$

Taking the cross product results in

$$(x^2-1)(x^2+1-2x\cos\phi) = 0,$$

which is a contradiction, because $x > 1$ for $|\alpha| < \pi$. Consequently, $x'(\rho) > 0$ for all $\rho > 0$, which proves the monotonicity of $x(\rho)$.

(B) Squaring (4.6) and using (4.8) we obtain

$$\omega^2(x-1)^2\big((x^2+1)^2 - 4x^2\cos^2\phi\big) = (\rho x)^2(x+1)^2. \tag{4.12}$$

By eliminating $\cos\phi$ and ρ in (4.12) with the help of (4.9), a straightforward calculation (we have used Mathematica [20]) leads to

$$\big((1+x)^4 - \omega^2(1+6x+x^4)\big)^2 = 16\omega^4 x^2(1+x^2)^2\cos^2\alpha.$$

Taking the square root (we have to take the plus sign, because $\omega \to \sec(\alpha/2)$ for $x \to 1$) we obtain the explicit formula

$$\omega^2 = \frac{(1+x)^4}{(1+x)^4 + 4x(1+x^2)(\cos\alpha - 1)}.$$

Its derivative

$$2\omega\omega' = \frac{\mathrm{d}\omega^2}{\mathrm{d}x} = \frac{4(1+x)^3(x-1)^3(\cos\alpha - 1)}{\big((1+x)^4 + 4x(1+x^2)(\cos\alpha-1)\big)^2}$$

is strictly negative for $x > 1$ and $0 < |\alpha| \le \pi$, which implies that $\omega(x)$ is monotonically decreasing from $\omega(1) = \sec(\alpha/2)$ to $\omega(\infty) = 1$. □

4.3 Resolvent bound depending on the modulus of the argument

The estimate of Lemma 4.1 is optimal for small $\rho = |z|$, but it can be much improved for large ρ close to the negative real axis.

Lemma 4.2 *We have the resolvent bound*

$$\big\|(zI-T)^{-1}\big\|_\infty \le \frac{1}{|z|-4} = \frac{\rho}{(\rho-4)\,|z|} \qquad \textit{for} \quad \rho = |z| > 4.$$

Proof Since $\|T\|_\infty = 4$, we have for $|z| > 4$ that $\|z^{-1}T\|_\infty < 1$. Applying Neumann's series

$$\big\|(zI-T)^{-1}\big\|_\infty = \Big\|z^{-1}\sum_{j\ge 0} z^{-j}T^j\Big\|_\infty \le |z|^{-1}\sum_{j\ge 0}|z|^{-j}\|T\|_\infty^j = \frac{1}{|z|-4}.$$

yields the desired bound. □

Now, from the lemmas above we have the following bound.

Lemma 4.3 *We have the resolvent bound*

$$\big\|\big(zI-\mu T\big)^{-1}\big\|_\infty \le \frac{\omega_{\rho,\alpha}}{|z|} \qquad \textit{for} \quad z=\rho\,\mathrm{e}^{\mathrm{i}\alpha},\ \rho>0,\ -\pi<\alpha\le\pi, \quad \mu>0, \tag{4.13}$$

where $\omega_{\rho,\alpha}=\min(\omega_\alpha,\omega_\rho)$ *with*

$$\omega_\alpha=\sec(\alpha/2), \qquad \omega_\rho=\begin{cases}\rho/(\rho-4\mu) & \textit{for}\ \ \rho>4\mu\\ \infty & \textit{for}\ \ \rho\le 4\mu.\end{cases}$$

Proof We note that $\big(zI-\mu T\big)^{-1}=\mu^{-1}\big(\mu^{-1}z\,I-T\big)^{-1}$ for $z=\rho\,\mathrm{e}^{\mathrm{i}\alpha}$. Consequently, the bound (4.13) with ω_α in place of $\omega_{\rho,\alpha}$ follows from Lemma 4.1, and with ω_ρ in place of $\omega_{\rho,\alpha}$ from Lemma 4.2. □

5 Using Cauchy's integral formula

The stability matrix (1.3) is a function of τD_j, where τ is the time step size and $D_j=(\alpha_j/\Delta x_j^2)(I_{N_m}\otimes\ldots\otimes T_{N_j}\otimes\ldots\otimes I_{N_1})$. We let T (omitting the subscript N_j) be the tridiagonal matrix of dimension N_j with entries $(1,-2,1)$. With the notation $\mu_j=\alpha_j\tau/\Delta x_j^2>0$ the eigenvalues of τD_j are real and lie in the interval $(-4\mu_j,0)$. We also denote

$$\mu := \alpha\tau(N+1)^2 \ge \max_{1\le j\le m}\mu_j.$$

Assume that $f(z)$ is analytic in the bounded open connected set $\Omega\subset\mathbb{C}$, and that the negative real interval $(-4\mu,0)\subset\Omega$ is a subset of Ω. With $\Gamma=\partial\Omega$ (smooth and positively oriented boundary) we then have

$$f(\mu T)=\frac{1}{2\pi\mathrm{i}}\int_\Gamma f(z)\big(zI-\mu T\big)^{-1}\,\mathrm{d}z. \tag{5.1}$$

An extension to functions of more than one variable is possible by applying the formula successively to all variables. For the nth power of the stability matrix (1.3)

$$f(z_1,z_2,\ldots,z_m)=R(z_1,z_2,\ldots,z_m)^n, \tag{5.2}$$

and $m=3$, we obtain the following formula

$$R(\tau D_1,\tau D_2,\tau D_3)^n=\frac{1}{(2\pi\mathrm{i})^3}\int_{\Gamma_1}\int_{\Gamma_2}\int_{\Gamma_3}R(z_1,z_2,z_3)^n \cdot\big(z_3I-\mu_3T_{N_3}\big)^{-1}\otimes\big(z_2I-\mu_2T_{N_2}\big)^{-1}\otimes\big(z_1I-\mu_1T_{N_1}\big)^{-1}\,\mathrm{d}z_3\,\mathrm{d}z_2\,\mathrm{d}z_1,$$

where $\Gamma_j=\partial\Omega_j$ is a closed, positively oriented path in the z_j-plane that surrounds the real interval $(-4\mu,0)$ and such that $\theta^{-1}\notin\Omega_j\cup\Gamma_j$ for $j=1,\ldots,m$. From the estimates of Section 4 we get the following bound

$$\big\|R(\tau D_1,\tau D_2,\tau D_3)^n\big\|_\infty\le\frac{\omega^3}{(2\pi)^3}\int_{\Gamma_1}\int_{\Gamma_2}\int_{\Gamma_3}\big|R(z_1,z_2,z_3)\big|^n\,\frac{|dz_3|}{|z_3|}\,\frac{|dz_2|}{|z_2|}\,\frac{|dz_1|}{|z_1|}.$$

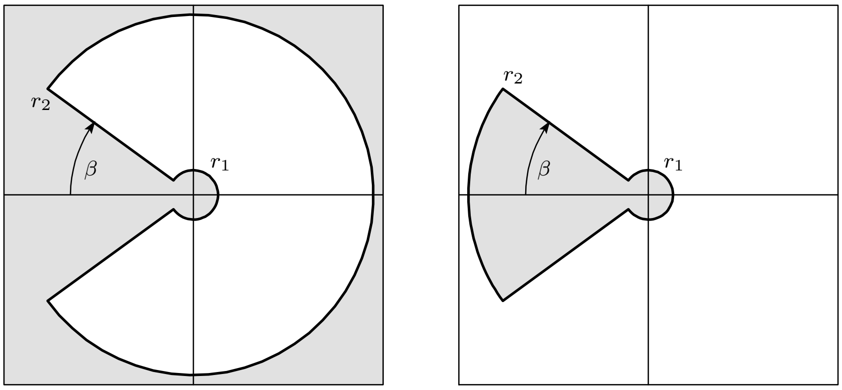

Fig. 5.1 Integration paths consist of arcs of circles of radius r_1 and r_2 and of straight lines; the open set Ω is the shaded region.

where

$$\omega = \max\left\{\omega_{\rho,\alpha} \,\middle|\, \rho\, \mathrm{e}^{\mathrm{i}\alpha} \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3\right\}.$$

For the paths of Figure 5.1, by assuming that $0 < r_1 < \theta^{-1} < r_2$, we have $\omega = \sec((\pi-\beta)/2)$ (left picture) and $\omega = \max(2, \sec((\pi-\beta)/2))$ (right picture) provided that $r_2 \geq 8\mu$.

If the set Ω is unbounded (left picture of Figure 5.1), the Cauchy's integral formula reads for $f(z)$ analytic in $\mathbb{C} \cup \{\infty\} \setminus \Omega$ (recall that $f(z)$ is analytic at ∞ if and only if $f(1/z)$ is analytic at $z = 0$)

$$f(\mu T) = f(\infty)\, I - \frac{1}{2\pi\mathrm{i}} \int_{\Gamma} f(z)\big(zI - \mu T\big)^{-1}\, \mathrm{d}z. \tag{5.3}$$

It can also be extended to functions of more than one variable. Thus, by considering the case (5.2) with $m = 2$, we have that (below for the cases $m = 2$ and $m = 3$, I represents identity matrices of appropriate dimensions)

$$\begin{aligned} R\big(\tau D_1, \tau D_2\big)^n &= R\big(\infty, \infty\big)^n I + \left(\frac{-1}{2\pi\mathrm{i}}\right) \oint_{\Gamma_1} R\big(z_1, \infty\big)^n \Big(I \otimes \big(z_1 I - \mu_1 T_{N_1}\big)^{-1}\Big)\, dz_1 \\ &+ \left(\frac{-1}{2\pi\mathrm{i}}\right) \oint_{\Gamma_2} R\big(\infty, z_2\big)^n \Big(\big(z_2 I -_2 T_{N_2}\big)^{-1} \otimes I\Big)\, dz_2 \\ + \left(\frac{-1}{2\pi\mathrm{i}}\right)^2 & \oint_{\Gamma_1} \oint_{\Gamma_2} R\big(z_1, z_2\big)^n \Big(\big(z_2 I - \mu_2 T_{N_2}\big)^{-1} \otimes \big(z_1 I - \mu_1 T_{N_1}\big)^{-1}\Big)\, dz_2 dz_1, \end{aligned} \tag{5.4}$$

We also include the case $m = 3$ which can directly be extended to a general formula for arbitrary m,

$$\begin{aligned}
R\big(\tau D_1, \tau D_2, \tau D_3\big)^n &= R\big(\infty,\infty,\infty\big)^n I + \left(\frac{-1}{2\pi \mathrm{i}}\right) \oint_{\Gamma_1} R\big(z_1,\infty,\infty\big)^n \left(I \otimes I \otimes \big(z_1 I - \mu_1 T_{N_1}\big)^{-1}\right) dz_1 \\
&+ \left(\frac{-1}{2\pi \mathrm{i}}\right) \oint_{\Gamma_2} R\big(\infty, z_2,\infty\big)^n \left(I \otimes \big(z_2 I - \mu_2 T_{N_2}\big)^{-1} \otimes I\right) dz_2 \\
&+ \left(\frac{-1}{2\pi \mathrm{i}}\right) \oint_{\Gamma_3} R\big(\infty,\infty, z_3\big)^n \left(\big(z_3 I - \mu_1 T_{N_3}\big)^{-1} \otimes I \otimes I\right) dz_3 \\
&+ \left(\frac{-1}{2\pi \mathrm{i}}\right)^2 \oint_{\Gamma_1}\oint_{\Gamma_2} R\big(z_1, z_2,\infty\big)^n \left(I \otimes \big(z_2 I - \mu_2 T_{N_2}\big)^{-1} \otimes \big(z_1 I - \mu_1 T_{N_1}\big)^{-1}\right) dz_2 dz_1 \\
&+ \left(\frac{-1}{2\pi \mathrm{i}}\right)^2 \oint_{\Gamma_1}\oint_{\Gamma_3} R\big(z_1,\infty, z_3\big)^n \left(\big(z_3 I - \mu_3 T_{N_3}\big)^{-1} \otimes I \otimes \big(z_1 I - \mu_1 T_{N_1}\big)^{-1}\right) dz_3 dz_1 \\
&+ \left(\frac{-1}{2\pi \mathrm{i}}\right)^2 \oint_{\Gamma_2}\oint_{\Gamma_3} R\big(\infty, z_2, z_3\big)^n \left(\big(z_3 I - \mu_3 T_{N_3}\big)^{-1} \otimes \big(z_2 I - \mu_2 T_{N_2}\big)^{-1} \otimes I\right) dz_3 dz_2 \\
&+ \left(\frac{-1}{2\pi \mathrm{i}}\right)^3 \oint_{\Gamma_1}\oint_{\Gamma_2}\oint_{\Gamma_3} R\big(z_1, z_2, z_3\big)^n \left(\big(z_3 I - \mu_3 T_{N_3}\big)^{-1} \otimes \big(z_2 I - \mu_2 T_{N_2}\big)^{-1} \otimes \big(z_1 I - \mu_1 T_{N_1}\big)^{-1}\right) dz_3 dz_2 dz_1 .
\end{aligned} \tag{5.5}$$

6 Proof of Theorem 2.1: power boundedness in dimension $m \geq 3$

The stability function corresponding to the stability matrix (1.3) is given by (1.4). Motivated by the integration paths of Figure 5.1 we also consider the sets

$$K_\theta = \Big\{ z \in \mathbb{C} \,\Big|\, |z| \le \frac{1}{2\theta} \ \text{ or } \ |z| \ge \frac{2}{\theta} \Big\}, \qquad L_\beta = \Big\{ z \in \mathbb{C} \,\Big|\, |\arg(-z)| \le \beta \Big\}.$$

We know from [10,11] that for $\theta \geq 1/2$

$$|R(z_1,\ldots,z_m)| \le 1 \quad \text{for} \quad z_j \in L_\beta \qquad \text{iff} \qquad \beta \le \frac{\pi}{2(m-1)}, \tag{6.1}$$

even if we consider 0 and ∞ as elements of L_β.

In the following we use the notation $\nu(z) = \min\{|z|, |z|^{-1}\}$.

Lemma 6.1 *For $\theta \geq 1/2$ and $0 < \beta \leq \frac{\pi}{2(m-1)}$ we have for $z_j \in K_\theta \cup L_\beta$ the following bound for the stability function* (1.4)

$$\big|R(z_1,\ldots,z_m)\big| \le 1 + C \sum_{z_j \in K_\theta \setminus L_\beta} \nu(z_j)$$

with a positive constant C depending on θ.

Proof Let m_0 be the number of z_j among $\{z_1,\ldots,z_m\} \subset K_\theta \cup L_\beta$ that satisfy $z_j \in K_\theta \setminus L_\beta$. Note that $0 \in L_\beta$ and $\infty \in L_\beta$.

We give a proof by induction on m_0. For $m_0 = 0$ we have $z_j \in L_\beta$ for all j, so that the statement is a consequence of (6.1). Assume next that $z_m \in K_\theta \backslash L_\beta$. We have to distinguish between $|z_m| \le 1/(2\theta)$ and $|z_m| \ge 2/\theta$.

If $z_m \in K_\theta \setminus L_\beta$ and $|z_m| \le 1/(2\theta)$ we write

$$R(z_1,\ldots,z_m) = R(z_1,\ldots,z_{m-1},0) + z_m S_0(z_1,\ldots,z_m)$$
$$\text{with} \quad S_0(z_1,\ldots,z_m) = \frac{1+\theta(z_1+\ldots+z_{m-1})}{(1-\theta z_1)\cdot\ldots\cdot(1-\theta z_m)}.$$

Since $\{z_1,\ldots,z_{m-1},0\}$ has one element less in $K_\theta \setminus L_\beta$, the statement follows from the induction hypothesis and from the boundedness of $S_0(z_1,\ldots,z_m)$.

If $z_m \in K_\theta \setminus L_\beta$ and $|z_m| \ge 2/\theta$ we write (with $w_m = z_m^{-1}$)

$$R(z_1,\ldots,z_m) = R(z_1,\ldots,z_{m-1},\infty) + w_m S_\infty(z_1,\ldots,z_m)$$
$$\text{with} \quad S_\infty(z_1,\ldots,z_m) = \frac{\theta^{-1}\big(1+\theta(z_1+\ldots+z_{m-1})\big)}{(1-\theta z_1)\cdot\ldots\cdot(1-\theta z_{m-1})(w_m-\theta)}.$$

The same argument as before concludes the proof. □

Proof of Theorem 2.1 for $m = 3$ (Similar ideas apply to arbitrary $m > 3$). Applying the Cauchy's integral formula (5.5) and using the estimate (4.13) for the resolvent yields the bound (here written for $m = 3$)

$$\begin{aligned}\big\|R(\tau D_1,\tau D_2,\tau D_3)^n\big\|_\infty &\le |R(\infty,\infty,\infty)|^n + \frac{\omega}{2\pi}\int_{\Gamma_1}|R(z_1,\infty,\infty)|^n\,\frac{|dz_1|}{|z_1|}\\ &+\ldots+\frac{\omega^2}{(2\pi)^2}\int_{\Gamma_1}\int_{\Gamma_2}|R(z_1,z_2,\infty)|^n\,\frac{|dz_2|}{|z_2|}\,\frac{|dz_1|}{|z_1|}+\ldots \qquad (6.2)\\ &+\frac{\omega^3}{(2\pi)^3}\int_{\Gamma_1}\int_{\Gamma_2}\int_{\Gamma_3}\big|R(z_1,z_2,z_3)\big|^n\,\frac{|dz_3|}{|z_3|}\,\frac{|dz_2|}{|z_2|}\,\frac{|dz_1|}{|z_1|},\end{aligned}$$

where $\omega = \sec((\pi-\beta)/2)$. To estimate this expression we use the bound of Lemma 6.1, which is of the form $1 + x$. Using $1 + x \le \mathrm{e}^x$ (for $x \in \mathbb{R}$), and consequently $1 + a_1 + \ldots + a_j \le \mathrm{e}^{a_1}\cdot\ldots\cdot\mathrm{e}^{a_j}$, the estimate of Lemma 6.1 yields a product of terms each of which depends only on one variable. Hence, the j-tuple integral over $|R(z_1,\ldots,z_m)|^n$ in (6.2) is reduced to a product of j simple path integrals.

For all variables we use the integration path Γ of the left picture of Figure 5.1 with

$$r_1 = c_1/n, \quad r_2 = c_2 n, \quad \text{where } 0 < c_1 < \theta^{-1} < c_2 \text{ are two constants.} \qquad (6.3)$$

Since the bounds of Lemma 6.1 only depend on the absolute value $|z_j|$, it is sufficient to consider the integrals over the upper half of the path Γ (for each variable). We split it into $\gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, where

$$\begin{aligned}\gamma_1 &= \{r_1\mathrm{e}^{\mathrm{i}\phi}\,;\,0\le\phi\le\pi-\beta\} & \gamma_2 &= \{\rho\mathrm{e}^{\mathrm{i}(\pi-\beta)}\,;\,r_1\le\rho\le 1\}\\ \gamma_3 &= \{\rho\mathrm{e}^{\mathrm{i}(\pi-\beta)}\,;\,1\le\rho\le r_2\} & \gamma_4 &= \{r_2\mathrm{e}^{\mathrm{i}\phi}\,;\,0\le\phi\le\pi-\beta\}.\end{aligned} \qquad (6.4)$$

Since the paths γ_2 and γ_3 are entirely in L_β, there is no contribution of the corresponding variable in the right-hand side of the estimate in Lemma 6.1. We are therefore concerned with the following path integrals:

$$\int_{\gamma_1} \mathrm{e}^{Cn|z|} \frac{|\mathrm{d}z|}{|z|} = \int_0^{\pi-\beta} \mathrm{e}^{Cnr_1}\,\mathrm{d}\phi = (\pi-\beta)\,\mathrm{e}^{Cc_1},$$
$$\int_{\gamma_2} \frac{|\mathrm{d}z|}{|z|} = \int_{r_1}^{1} \frac{\mathrm{d}\rho}{\rho} = -\ln r_1, \qquad \int_{\gamma_3} \frac{|\mathrm{d}z|}{|z|} = \int_{1}^{r_2} \frac{\mathrm{d}\rho}{\rho} = \ln r_2, \tag{6.5}$$
$$\int_{\gamma_4} \mathrm{e}^{Cn|z|^{-1}} \frac{|\mathrm{d}z|}{|z|} = \int_0^{\pi-\beta} \mathrm{e}^{Cn/r_2}\,\mathrm{d}\phi = (\pi-\beta)\,\mathrm{e}^{C/c_2}.$$

The first and fourth of these integrals are bounded independently of n. The second and third are bounded by $\mathcal{O}(1+\ln n)$. Since the bound (6.2) contains products of at most three integrals of the type (6.5) (and at most m integrals in dimension m), this completes the proof. □

7 Proof of Theorem 2.2: general bound

For the proof of Theorem 2.2 we need different bounds for the modulus $|R(z_1,\dots,z_m)|$ than those of Lemma 6.1. We denote

$$K_\theta^0 = \Big\{ z\in\mathbb{C} \,\Big|\, |z| \le \frac{1}{2\theta} \Big\}, \qquad L_\beta = \Big\{ z\in\mathbb{C} \,\Big|\, |\arg(-z)| \le \beta \Big\},$$

and we let $L_\beta^0 = \{z \in L_\beta \,|\, |z| \le 1\}$ and $L_\beta^\infty = \{z \in L_\beta \,|\, |z| \ge 1\}$ so that $L_\beta = L_\beta^0 \cup L_\theta^\infty$.

Lemma 7.1 *For $\theta \ge 1/2$ and $\beta = \frac{\pi}{4m}$ we consider $(z_1,\dots,z_m)$ with*

$$z_j \in K_\theta^0 \cup L_\beta, \quad |z_j| \le r_2 := c_2(N+1), \quad c_2 = 8\alpha c^*, \qquad (j=1,\dots,m) \tag{7.1}$$

and we denote by m_0 the number of elements in L_β^∞ among $\{z_1,\dots,z_m\}$. Then, we have the following bound for the stability function (1.4)

$$\big|R(z_1,\dots,z_m)\big| \le 1 + C_1 \sum_{z_j\in K_\theta^0\setminus L_\beta^0} |z_j| - \frac{C_2}{r_2^{m_0}} \sum_{z_j \in L_\beta^0} |z_j| \tag{7.2}$$

with positive constants C_1 and C_2 only depending on θ.

Remark 7.1 It should be observed that the same constants C_1 and C_2 in (7.2) are valid for any β fulfilling $0 < m\beta \le \pi/4$.

Proof An argument $z_j \in K_\theta^0 \setminus L_\beta^0$ can be treated as in the proof of the Lemma 6.1. This gives rise to the first sum in the estimate of the stability function. We can therefore assume without loss of generality that all z_j are in L_β and satisfy $|z_j| \le r_2$.

We assume that $z_1,\dots,z_k \in L_\beta^0$ and $z_{k+1},\dots,z_m \in L_\beta^\infty$, so that $m_0 = m-k$. The stability function can then be written as (recall that $w_j = z_j^{-1}$)

$$1 + \frac{z+w}{(1-\theta z_1)\cdot\ldots\cdot(1-\theta z_k)(\theta - w_{k+1})\cdot\ldots\cdot(\theta-w_m)}$$

$$z = (z_1+\ldots+z_k)(-w_{k+1})\cdot\ldots\cdot(-w_m)$$

$$w = -\sum_{j=k+1}^{m}(-w_{k+1})\cdot\ldots\cdot\widehat{(-w_j)}\cdot\ldots\cdot(-w_m),$$

where the hat on $(-w_j)$ indicates that this factor has to be omitted in building the product. For $z_1,\dots,z_k,w_{k+1},\dots,w_m$ close to zero, the stability function satisfies $R(z_1,\dots,z_m)\approx 1+\theta^{k-m}(z+w)$. By the assumption $z_j\in L_\beta$ with $m\beta\le\pi/4$, each summand u in z and in w is in $L_{\pi/4}$, so that $\Re u \le -2^{-1/2}|u|$. This implies that there exists a positive constant C_2', such that

$$|R(z_1,\dots,z_m)| \le 1 - C_2'\big(|z|+|w|\big) \tag{7.3}$$

for $z_1,\dots,z_k,w_{k+1},\dots,w_m$ close to zero. This together with (6.1) imply that the expression $\big(1-|R(z_1,\dots,z_m)|\big)/\big(|z|+|w|\big)$ is positive on the compact set $z_1,\dots,z_k,w_{k+1},\dots,w_m\in L_\beta^0$. Therefore, there exists (a possibly smaller) $C_2'>0$ such that (7.3) holds true on this compact set.

Since $|z_j|\le r_2$, we have $|w_j|\ge r_2^{-1}$ and $|z|\ge 2^{-1/2}r_2^{k-m}\big(|z_1|+\ldots+|z_m|\big)$. Consequently, it follows from (7.3) that

$$|R(z_1,\dots,z_m)| \le 1-C_2'\big(|z|+|w|\big)\le 1-C_2'|z| \le 1-C_2 r_2^{k-m}\big(|z_1|+\ldots+|z_m|\big)$$

with $C_2 = 2^{-1/2}C_2'$. This yields the statement of Lemma 7.1. □

Proof of Theorem 2.2 (first inequality).

Take r_2 as defined in (7.1), then it holds that

$$r_2 = c_2(N+1) = 8\alpha c^*(N+1)\ge 8\alpha\tau(N+1)^2 = 8\mu \ge 8\mu_j > 0,\ (j=1,\dots,m).$$

We use Cauchy's integral formula with the path of Figure 5.1 (right picture). We split the upper half of the path into four parts $\gamma_1,\gamma_2,\gamma_3,\gamma_4$, where the first three paths are as in (6.4), but with r_1 defined in (6.3) and

$$\gamma_4 = \{r_2 e^{i\phi}\,;\,\pi-\beta\le\phi\le\pi\},\quad r_2 = c_2(N+1). \tag{7.4}$$

With the help of Lemma 7.1 the estimate of the m-tuple integral reduces to a product of m factors among the four expressions

$$\int_{\gamma_1} e^{C_1 n|z|}\frac{|dz|}{|z|},\qquad \int_{\gamma_2} e^{-C_2 n|z|/r_2^{m_0}}\frac{|dz|}{|z|},\qquad \int_{\gamma_3}\frac{|dz|}{|z|},\qquad \int_{\gamma_4}\frac{|dz|}{|z|}. \tag{7.5}$$

As in Section 6 the first and the fourth of these integrals are bounded independently of n and N. For the second integral we have, with $r_1 = c_1/n$,

$$\int_{\gamma_2} e^{-C_2 n|z|/r_2^{m_0}}\frac{|dz|}{|z|} = \int_{r_1}^{1} e^{-C_2 n\rho/r_2^{m_0}}\frac{d\rho}{\rho} = E_1(c_1 C_2/r_2^{m_0})$$

which, by the estimates of Section 10, see (10.1)-(10.2), is bounded by $A + m_0 \ln(1+N)$, where A is some constant only depending on m, c_1 and C_2. Finally, the third integral is bounded by $\ln r_2 = \ln(c_2(1+N))$. Since the bound of $\|R(\tau D_1,\ldots,\tau D_m)^n\|_\infty$ is a sum of products that contain at most m factors from (7.5), it is bounded by $\mathcal{O}\big((\ln(1+N))^m\big)$. This completes the proof of the first statement of Theorem 2.2 for any dimension $m \ge 3$. □

8 Proof of Theorem 2.2: special case of dimension 3

This section is devoted to the improved bounds of Theorem 2.2 in dimension $m = 3$ for large values of n. The proof is based on Taylor series expansions around 0 and ∞. The stability function is (with $\theta \ge 1/2$)

$$R(z_1,z_2,z_3) = 1 + \frac{z_1+z_2+z_3}{(1-\theta z_1)(1-\theta z_2)(1-\theta z_3)}, \tag{8.1}$$

and, if some of the arguments are replaced by $w_j = 1/z_j$, it becomes

$$\begin{aligned} R(z_1,z_2,z_3) &= \Big(1-\frac{1}{\theta}\Big) - \frac{w_3 + \theta^2(z_1+z_2) - \theta^2(\theta-w_3)z_1z_2}{\theta(1-\theta z_1)(1-\theta z_2)(\theta-w_3)} \\ R(z_1,z_2,z_3) &= 1 + \frac{z_1w_2w_3 + w_2 + w_3}{(1-\theta z_1)(\theta-w_2)(\theta-w_3)} \\ R(z_1,z_2,z_3) &= 1 - \frac{w_1w_2 + w_1w_3 + w_2w_3}{(\theta-w_1)(\theta-w_2)(\theta-w_3)}. \end{aligned}$$

We still consider the sets K_θ^0, L_β^0, and L_β^∞ as introduced in Section 7, and we continue to use the notation $\nu(z) = \min\{|z|,|z|^{-1}\}$.

Lemma 8.1 *For $\theta \ge 1/2$ and $0 < 3\beta \le \pi/4$, we have for the stability function* (8.1) *the following bounds with positive constants C_1 and C_2 only depending on θ and β,*

$$\begin{aligned} &1 + C_1\big(|z_1|+|z_2|+|z_3|\big) && \text{for} \quad z_1,z_2,z_3 \in K_\theta^0 \\ &1 + C_1\big(|z_1|+|z_2|\big) - C_2\nu(z_3) && \text{for} \quad z_1,z_2 \in K_\theta^0, \quad z_3 \in L_\beta \\ &1 + C_1|z_1| - C_2\big(\nu(z_2)+\nu(z_3)\big) && \text{for} \quad z_1 \in K_\theta^0, \quad z_2,z_3 \in L_\beta \\ &1 - C_2\big(|z_1|+|z_2|+\nu(z_3)\big) && \text{for} \quad z_1,z_2 \in L_\beta^0, \quad z_3 \in L_\beta \\ &1 - C_2\big(|z_1w_2w_3|+|w_2|+|w_3|\big) && \text{for} \quad z_1 \in L_\beta^0, \quad z_2,z_3 \in L_\beta^\infty \\ &1 - C_2\big(|w_1w_2|+|w_1w_3|+|w_2w_3|\big) && \text{for} \quad z_1,z_2,z_3 \in L_\beta^\infty. \end{aligned}$$

Proof The first bound follows from the triangle inequality applied to (8.1).

We next consider the fourth bound. Assume first that $z_1,z_2,z_3 \in L_\beta^0$. We note that $R(z_1,z_2,z_3) = 1 + z_1 + z_2 + z_3 + \mathcal{O}\big(|z_1|^2+|z_2|^2+|z_3|^2\big)$ close to the origin. This implies $\big|R(z_1,z_2,z_3)\big| \le 1 - C_2\big(|z_1|+|z_2|+|z_3|\big)$ for

$z_1, z_2, z_3 \in L_\beta \cap \{z; |z| \leq c\}$ with $c > 0$ sufficiently small. It follows from [11] that $\big|R(z_1,z_2,z_3)\big| < 1$ for $z_1, z_2, z_3 \in L_\beta \setminus \{0\}$. Consequently, a compactness argument yields $\big(1 - |R(z_1,z_2,z_3)|\big)/(|z_1|+|z_2|+|z_3|) \geq C_2 > 0$ for $z_j \in L_\beta \cap \{z; c \leq |z| \leq 1]\}$ with a possibly different value for C_2.

For $z_1, z_2 \in L_\beta^0$, but $z_3 \in L_\beta^\infty$ we have that (recall $w_3 = z_3^{-1}$)

$$R(z_1,z_2,z_3) = (1-\theta^{-1}) - z_1 - z_2 - \theta^{-2} w_3 + \mathcal{O}\big(|z_1|^2 + |z_2|^2 + |w_3|^2\big),$$

so that $\big|R(z_1,z_2,z_3)\big| \leq 1 - C_2\big(|z_1| + |z_2| + |w_3|\big)$ close to $(0,0,\infty)$. This follows from the fact that $1-\theta^{-1} \in [-1,0)$ is negative for $1/2 \leq \theta < 1$, and $1-\theta^{-1} \in [0,1)$ is strictly smaller than one for $\theta \geq 1$. This completes the proof for the fourth bound. We remark that this bound also holds if either z_1 or z_2 or both are replaced by 0.

For the second bound we write $R(z_1,z_2,z_3) = R(0,0,z_3) + z_1 S_1(z_1,z_2,z_3) + z_2 S_2(z_1,z_2,z_3)$ with bounded functions $S_1(z_1,z_2,z_3)$ and $S_2(z_1,z_2,z_3)$, and we use the fact that $|R(0,0,z_3)| \leq 1 - C_2 \nu(z_3)$.

For the third bound we use $R(z_1,z_2,z_3) = R(0,z_2,z_3) + z_1 S(z_1,z_2,z_3)$ with bounded $S(z_1,z_2,z_3)$ together with $|R(0,z_2,z_3)| \leq 1 - C_2(\nu(z_2) + \nu(z_3))$. For $z_2 \in L_\beta^0, z_3 \in L_\beta$ this inequality is a consequence of the fourth bound by putting $z_1 = 0$. If both z_2, z_3 are in L_β^∞, the inequality follows from the above compactness argument, because $R(0,z_2,z_3) = 1 + \theta^{-2}(w_2 + w_3) + \mathcal{O}(|w_2|^2 + |w_3|^2)$.

For the fifth bound we put $w = z_1 w_2 w_3 + w_2 + w_3$, and we note that the product $z_1 w_2 w_3$ and also w are in the sector $L_{\pi/4}$. Since $R(z_1,z_2,z_3) = 1 + w + \mathcal{O}\big(|w| \cdot \max(|z_1|,|w_2|,|w_3|)\big)$, it holds $|R(z_1,z_2,z_3)| \leq 1 - C_2'|w|$ for (z_1,z_2,z_3) close to $(0,\infty,\infty)$. Moreover,

$$|w| \geq |\Re w| = |\Re(z_1 w_2 w_3)| + |\Re w_2| + |\Re w_3| \geq 2^{-1/2}\big(|z_1 w_2 w_3| + |w_2| + |w_3|\big),$$

so that $\big|R(z_1,z_2,z_3)\big| \leq 1 - C_2\big(|z_1 w_2 w_3| + |w_2| + |w_3|\big)$ close to $(0,\infty,\infty)$. The same compactness argument as before proves the bound for all $z_1 \in L_\beta^0$ and $z_2, z_3 \in L_\beta^\infty$.

For the last bound we put $w = w_1 w_2 + w_1 w_3 + w_2 w_3$. Here, the negative products $-w_i w_j$ and $-w$ are in the sector $L_{\pi/4}$. Since $R(z_1,z_2,z_3) = 1 - w + \mathcal{O}\big(|w| \cdot \max(|w_1|,|w_2|,|w_3|)\big)$, the modulus of the stability function is bounded by $1 - C_2\big(|w_1 w_2| + |w_1 w_3| + |w_2 w_3|\big)$ for (z_1,z_2,z_3) close to (∞,∞,∞). Again, the compactness argument concludes the proof. □

Proof of Theorem 2.2 (second inequality).
We assume $n^2 > N$. Now, for $\big\|R(\tau D_1, \tau D_2, \tau D_3)^n\big\|_\infty$ we use Cauchy's integral formula with the integration path of the right picture of Figure 5.1, and we split the upper half into $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ as in Section 7, see (7.4).

It is sufficient to consider the integrals

$$I_{k,l,m} = \int_{\gamma_k} \int_{\gamma_l} \int_{\gamma_m} \big|R(z_1,z_2,z_3)\big|^n \frac{|dz_3|}{|z_3|} \frac{|dz_2|}{|z_2|} \frac{|dz_1|}{|z_1|}, \tag{8.2}$$

for $1 \le k \le l \le m \le 4$. If one of the first four bounds of Lemma 8.1 is relevant the triple integral in (8.2) reduces to a product of three simple path integrals of the form

$$\int_{\gamma_1} \mathrm{e}^{C_1 n|z|} \frac{|\mathrm{d}z|}{|z|}, \quad \int_{\gamma_2} \mathrm{e}^{-C_2 n|z|} \frac{|\mathrm{d}z|}{|z|}, \quad \int_{\gamma_3} \mathrm{e}^{-C_2 n|z|^{-1}} \frac{|\mathrm{d}z|}{|z|}, \quad \int_{\gamma_4} \mathrm{e}^{-C_2 n|z|^{-1}} \frac{|\mathrm{d}z|}{|z|}.$$

The first and the fourth integrals are bounded independently of n and N as in (6.5). For the second integral we have

$$\int_{\gamma_2} \mathrm{e}^{-C_2 n|z|} \frac{|\mathrm{d}z|}{|z|} = \int_{r_1}^{1} \mathrm{e}^{-C_2 n\rho} \frac{\mathrm{d}\rho}{\rho} \le E_1(C_2 n r_1) = E_1(C_2 c_1), \tag{8.3}$$

which is also bounded. The third integral

$$\int_{\gamma_3} \mathrm{e}^{-C_2 n|z|^{-1}} \frac{|\mathrm{d}z|}{|z|} = \int_{1}^{r_2} \mathrm{e}^{-C_2 n\rho^{-1}} \frac{\mathrm{d}\rho}{\rho} = \int_{r_2^{-1}}^{1} \mathrm{e}^{-C_2 n\rho} \frac{\mathrm{d}\rho}{\rho} \le E_1(C_2 n r_2^{-1})$$

is bounded by $E_1(C_2/c_2)$ for $n > N$, see (7.4).

From these arguments it is not hard to show that all $I_{i,j,k}$ integrals are uniformly bounded except the following cases that have to be considered in more detail, $I_{2,3,3}$, $I_{2,3,4}$, $I_{2,4,4}$, $I_{3,3,3}$, $I_{3,3,4}$, $I_{3,3,4}$ and $I_{4,4,4}$.

By item (8.3) we have for parameters $w_2, w_3 \in \gamma_3 \cup \gamma_4$ that

$$\int_{\gamma_2} \mathrm{e}^{-C_2 n|z w_2 w_3|} \frac{|\mathrm{d}z|}{|z|} \le E_1(C_2 |w_2 w_3|) \le A + 2\ln(1+N),$$

because

$$1 \ge |w_2 w_3| \ge \left(\frac{1}{r_2}\right)^2 = \left(\frac{1}{c_2(N+1)}\right)^2.$$

This yields bounds of size $\mathcal{O}(\ln(1+N))$ for the cases $I_{2,3,3}, I_{2,3,4}, I_{2,4,4}$.

The triple integral $I_{3,3,3}$, if we let $\rho_j = |w_j| = |z_j|^{-1}$, becomes

$$I_{3,3,3} \le \int_{r_2^{-1}}^{1} \int_{r_2^{-1}}^{1} \int_{r_2^{-1}}^{1} \mathrm{e}^{-C_2 n(\rho_1\rho_2 + \rho_1\rho_3 + \rho_2\rho_3)} \frac{d\rho_1}{\rho_1} \frac{d\rho_2}{\rho_2} \frac{d\rho_3}{\rho_3}.$$

Introducing the variables $u_1 = \rho_2\rho_3$, $u_2 = \rho_1\rho_3$, $u_3 = \rho_1\rho_2$, whose Jacobian determinant is $|\partial u/\partial \rho| = 2\rho_1\rho_2\rho_3$, we obtain

$$I_{3,3,3} \le \frac{1}{2} \int_{r_2^{-2}}^{1} \int_{r_2^{-2}}^{1} \int_{r_2^{-2}}^{1} \mathrm{e}^{-C_2 n(u_1+u_2+u_3)} \frac{du_1}{u_1} \frac{du_2}{u_2} \frac{du_3}{u_3} \le \frac{1}{2} E_1(C_2 n r_2^{-2})^3,$$

because $(\rho_1\rho_2\rho_3)^2 = u_1 u_2 u_3$. This expression is bounded independently of N and n for $n \ge (N+1)^2$, due to the fact that

$$E_1(C_2 n r_2^{-2}) \le E_1\left(\frac{C_2}{c_2^2}\right).$$

For an estimate of $I_{3,3,4}$ we first integrate over the path γ_4. This leads to the double integral

$$\int_{r_2^{-1}}^{1}\int_{r_2^{-1}}^{1} \mathrm{e}^{-C_2 n(\rho_1\rho_2 + r_2^{-1}(\rho_1+\rho_2))}\,\frac{d\rho_1}{\rho_1}\,\frac{d\rho_2}{\rho_2} \le \int_{r_2^{-1}}^{1}\int_{r_2^{-1}}^{1} \mathrm{e}^{-C_2 n r_2^{-1}(\rho_1+\rho_2)}\,\frac{d\rho_1}{\rho_1}\,\frac{d\rho_2}{\rho_2},$$

which is bounded by $E_1(C_2 n r_2^{-2})$, similar as for $I_{3,3,3}$.

For an estimate of $I_{3,4,4}$ we integrate over both paths γ_4, and obtain the simple integral

$$\int_{r_2^{-1}}^{1} \mathrm{e}^{-C_2 n(2r_2^{-1}\rho_1 + r_2^{-2})}\,\frac{d\rho_1}{\rho_1} \le \int_{r_2^{-1}}^{1} \mathrm{e}^{-2C_2 n r_2^{-1}\rho_1}\,\frac{d\rho_1}{\rho_1} \le E_1(2C_2 n r_2^{-2}).$$

Finally, for the proof of the boundedness of $I_{4,4,4}$ it is sufficient to apply the bound $|R(z_1,z_2,z_3)| \le 1$ for $z_1,z_2,z_3 \in \gamma_4$. This concludes the proof. □

9 Power boundedness of further stability matrices

The technique of proof for the main results of the present article is not restricted to the simple stability matrix (1.3), but extends straightforwardly to a much larger class of stability matrices.

Let us illustrate this claim at the stability matrix (1.5) of the Hundsdorfer–Verwer scheme [12, Section IV.5.2]. For the value $\theta = (3+\sqrt{3})/6$ it defines a method of (classical) order 3, otherwise it is of order 2.

The most difficult step in proving power boundedness of the stability matrix is to prove for the stability function given in (1.6) that $|R(z_1,\ldots,z_m)| \le 1$ in a sector $z_1,\ldots,z_m \in L_\beta$ with positive β. According to the numerical computations of [12, Section IV.5.2], this is satisfied for $m = 1, \theta \ge 1/4$ with $\beta = \pi/2$, for $m = 2, \theta \ge (3+\sqrt{3})/6$ with $\beta = \pi/2$ and for $m = 3, \theta \ge 1/2$ with $\beta = \pi/4$.

The next step is to verify the bounds of Lemma 6.1 and Lemma 7.1. Minor changes in the proofs yield these bounds for (1.6). The rest is identical to the proof for (1.3), because the application of Cauchy's integral formula and the estimates of the resolvent do not depend on the particular stability function. This proves the bounds $\mathcal{O}\big(\ln(1+n))^m\big)$ and $\mathcal{O}\big((\ln(1+N))^m\big)$ of Section 2 also for (1.5).

9.1 Improved bounds in dimension $m = 2$

In dimension $m = 2$ we are not aware of a representation for (1.6) that permits a simple proof of power boundedness like that of Section 2 for the stability matrix (1.3). To prove power boundedness of the stability matrix (1.5) with bounds that are independent of n and N we need the following lemma, which improves the bound of Lemma 6.1. The sets K_θ and L_β as well as the function $\nu(z)$ are same as in Lemma 6.1.

Lemma 9.1 *For $\theta > 1/2$ and $0 < \beta < \pi/4$ we have for $z_j \in K_\theta \cup L_\beta$ the following bound for the stability function* (1.6) *in dimension $m = 2$,*

$$|R(z_1, z_2)| \le 1 + C_1 \sum_{z_j \in K_\theta \setminus L_\beta} \nu(z_j) - C_2 \sum_{z_j \in L_\beta} \nu(z_j) \tag{9.1}$$

with positive constants C_1, C_2 only depending on θ and β.

Proof There are several cases to be considered: zero, one, or two elements among $\{z_1, z_2\}$ can lie in L_β, and each of them can have a modulus larger or smaller than 1. If both, z_1 and z_2, are in K_θ but not in L_β, the proof is the same as that for Lemma 6.1. Also the proof of the first four bounds of Lemma 8.1 extends straight-forwardly to the stability function (1.6) by putting $z_1 = 0$, because only the dominant terms in Taylor series expansions play a role.

It remains to consider the situation, where $z_1 \in K_\theta^\infty$ and $z_2 \in L_\beta$. We write $R(z_1, z_2) = R(\infty, z_2) + z_1^{-1} S(z_1, z_2)$, where $S(z_1, z_2)$ is bounded and

$$R(\infty, z_2) = 1 - \frac{2}{\theta(1-\theta z_2)} + \frac{1}{2\theta^2 (1-\theta z_2)^2}.$$

The proof of $|R(\infty, z_2)| \le 1 - C_2 \nu(z_2)$ is by the same compactness argument as in the previous lemmata. For $z_2 \in L_\beta^\infty$ we have $R(\infty, z_2) = 1 + 2\theta^{-2} z_2^{-1} + \mathcal{O}(|z_2|^{-2})$, so that $(1 - |R(\infty, z_2)|)/|z_2^{-1}| \ge C_2' > 0$ for z_2 close to ∞. For $z_2 \in L_\beta^0$ we have $R(\infty, z_2) = p(\theta) + (\theta^{-1} - 2) z_2 + \mathcal{O}(|z_2|^2)$ with $p(\theta) = 1 - \frac{2}{\theta} + \frac{1}{2\theta^2}$. The inequality follows from the fact that $|p(\theta)| < 1$ for $\theta > 1/2$. □

The estimate of Lemma 9.1 permits us to prove power boundedness for the stability matrix (1.6). The proof shows that we have power boundedness for all stability matrices that correspond to stability functions satisfying the estimate of Lemma 9.1.

Theorem 9.1 *In dimension $m = 2$ the stability matrix* (1.5) *(with $\theta \ge 1/2$) satisfies*

$$\big\| R(\tau D_1, \tau D_2)^n \big\|_\infty \le c_2 \qquad \text{for} \quad n \ge 1,$$

where the constant c_2 only depends on θ.

Proof From the Cauchy's integral formula (5.4), by using the estimate for the resolvent to get the bound

$$\begin{aligned} \big\| R(\tau D_1, \tau D_2)^n \big\|_\infty \le{}& |R(\infty, \infty)|^n \\ &+ \frac{\omega}{2\pi} \Big(\int_{\Gamma_1} |R(z_1, \infty)|^n \frac{|dz_1|}{|z_1|} + \int_{\Gamma_2} |R(\infty, z_2)|^n \frac{|dz_2|}{|z_2|} \Big) \\ &+ \frac{\omega^2}{(2\pi)^2} \int_{\Gamma_1} \int_{\Gamma_2} \big| R(z_1, z_2) \big|^n \frac{|dz_2|}{|z_2|} \frac{|dz_1|}{|z_1|}, \end{aligned} \tag{9.2}$$

where $\omega = \sec((\pi - \beta)/2)$ with β considered in Lemma 9.1. We again use the integration path Γ of the left picture of Figure 5.1 with r_1 and r_2 defined in

(6.3), and we split it into $\gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ as in (6.4). Using the estimate of Lemma 9.1 it follows from (9.2) that we have power boundedness if the four integrals

$$\int_{\gamma_1} \mathrm{e}^{C_1 n|z|} \frac{|\mathrm{d}z|}{|z|}, \quad \int_{\gamma_2} \mathrm{e}^{-C_2 n|z|} \frac{|\mathrm{d}z|}{|z|}, \quad \int_{\gamma_3} \mathrm{e}^{-C_2 n|z|^{-1}} \frac{|\mathrm{d}z|}{|z|}, \quad \int_{\gamma_4} \mathrm{e}^{C_1 n|z|^{-1}} \frac{|\mathrm{d}z|}{|z|}$$

are bounded independently of n. For the first and fourth of these integrals this follows from (6.5). For the second and third integrals we have

$$\int_{\gamma_2} \mathrm{e}^{-C_2 n|z|} \frac{|\mathrm{d}z|}{|z|} = \int_{r_1}^{1} \mathrm{e}^{-C_2 n\rho} \frac{\mathrm{d}\rho}{\rho} \le E_1(C_2 n r_1) = E_1(C_2 c_1)$$

$$\int_{\gamma_3} \mathrm{e}^{-C_2 n|z|^{-1}} \frac{|\mathrm{d}z|}{|z|} = \int_{1}^{r_2} \mathrm{e}^{-C_2 n\rho^{-1}} \frac{\mathrm{d}\rho}{\rho} = \int_{r_2^{-1}}^{1} \mathrm{e}^{-C_2 n\rho} \frac{\mathrm{d}\rho}{\rho} \le E_1(C_2/c_2)$$

which are bounded by the exponential integral (see Section 10). Using these bounds in the estimate (9.2) for $\|R(\tau D_1, \tau D_2)^n\|_\infty$ concludes the proof of Theorem 9.1. □

We note that in dimension $m = 2$ the stability function (1.4) also satisfies the estimate of Lemma 9.1. Since the proof of Theorem 9.1 only depends on (9.1) and not on the special form of the stability function, we have an alternative proof of Theorem 2.1 for $m = 2$.

10 Appendix

The exponential integral is defined by the strictly decreasing function

$$E_1(x) = \int_x^\infty \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t \qquad \text{for} \qquad x > 0.$$

We have the estimate (see [1, p. 229, Formula 5.1.20])

$$\frac{1}{2}\mathrm{e}^{-x} \ln\Big(1 + \frac{2}{x}\Big) < E_1(x) < \mathrm{e}^{-x} \ln\Big(1 + \frac{1}{x}\Big). \tag{10.1}$$

A change of variables immediately leads to

$$\int_a^\infty \frac{\mathrm{e}^{-ct}}{t}\,\mathrm{d}t = E_1(ac), \quad \text{whenever } a > 0,\ c > 0. \tag{10.2}$$

Acknowledgement. This work was partially supported by the Ministerio de Industria, Economía y Competitividad (España) MTM2016-77735-C3-3-P, and by the Swiss National Science Foundation, grant No. 200020_178752.

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