Version of January 9, 2023, file: class-codim3.tex

Sliding modes of high codimension in piecewise-smooth dynamical systems

Nicola Guglielmi[1], **Ernst Hairer**[⋆2]

[1] Gran Sasso Science Institute, Viale Francesco Crispi 7, I-67100 L'Aquila, Italy. e-mail: nicola.guglielmi@gssi.it

[2] Dept. de Mathématiques, Univ. de Genève, CH-1211 Genève 24, Switzerland, e-mail: ernst.hairer@unige.ch

Received: 2022 / Revised version: date

Summary We consider piecewise-smooth dynamical systems, i.e., systems of ordinary differential equations switching between different sets of equations on distinct domains, separated by hyper-surfaces. As is well-known, when the solution approaches a discontinuity manifold, a classical solution may cease to exist. For this reason, starting with the pioneering work of Filippov, a concept of weak solution (also known as sliding mode) has been introduced and studied. Nowadays, the solution of piecewise-smooth dynamical systems in and close to discontinuity manifolds is well understood, if the manifold consists locally of a single discontinuity hyper-surface or of the intersection of two discontinuity hyper-surfaces. The present work presents partial results on the solution in and close to discontinuity manifolds of codimension 3 and higher.

Keywords Piecewise-smooth systems · Filippov solution · Sliding modes in high codimension · Regularization · Hidden dynamics

Mathematics Subject Classification (2010) 34A36 · 34A60 · 34C40

1 Introduction

In this article we consider piecewise-smooth dynamical systems, i.e., systems of ODEs which are characterized by a vector field that has jump discontinuities along certain hyper-surfaces of the Euclidean

⋆ Corresponding author

space. As general references we address the reader to [2,13,15] and - from a more computational perspective - to [1]. There are several examples of this kind of systems in the scientific literature. E.g., many biological systems are represented in terms of switching functions (see e.g. [20, Section 6]), which describe the effect of a certain variable on another, based on the fact that a certain threshold is reached and activates the coupling. Switching functions are frequently described by sigmoid (continuous) functions, but in many cases it is convenient to consider them as step functions (i.e., discontinuous ones). Gene regulatory networks (see e.g., [19,16]), which are sets of genes (or parts of genes) that interact with each other to control a specific cell function, are also typically modeled by step functions that describe the switching interaction of this class of complex biological networks. Moreover, a strong interest in sliding mode control (see e.g., [18]) has been shown in recent years, where the finite-time reaching of a certain discontinuity (sliding) manifold is the goal of a control system. Finally, when considering models based on state-dependent differential-algebraic delay equations, the manifolds are induced by the breaking points which characterize the solution [7]. The presence of discontinuities in all these models may give rise to termination or bifurcation phenomena. The lack of smoothness when the solution reaches one of the discontinuity hyper-surfaces leads to difficulties in dealing with its analysis, because the classical theory does not apply.

To circumvent this problem the concept of weak solution (sliding mode) has been introduced [6,23]. Two methods have been proposed to analyze this situation: the regularization framework based on singular perturbation techniques (see e.g. [22]) and the Filippov-like theory of differential inclusions. The theory developed by Filippov covers successfully the codimension-1 case, which is determined when a classical solution meets a single discontinuity hyper-surface. In such a case it provides a concept of weak solution (a codimension-1 motion called sliding mode) in those cases where a classical solution ceases to exist. However Filippov theory is not able to treat the higher codimension case, where the analysis is increasingly more difficult when the solution approaches the intersection of 2 or more discontinuity hyper-surfaces, which - even less common - can likely happen in the dynamical systems described above.

In some recent articles the case has been considered, where a solution approaches the intersection of a pair of such discontinuity hypersurfaces, with the possible occurrence of codimension-2 sliding modes. Due to the lack of uniqueness in classical Filippov theory the analysis of the dynamics is non trivial. It is based either on the analysis of regularizations of the piecewise-smooth dynamical system by a suitable singular perturbation or on the application of some specific selection principles. In [4], [3] several possibilities have been proposed, based on some suitable criteria decided a priori, on how to select a Filippov sliding vector field on a codimension-2 discontinuity surface. Differently, a complete taxonomy has been obtained in [8], based on the analysis of a natural regularization which replaces step functions by continuous functions. A common feature of regularization techniques (e.g., [12,8,14]) is that of obtaining two time-scales for the dynamics, the faster of which describes the so-called hidden dynamics of the system and in many cases allows - passing to the limit with respect to the regularization parameter - to select a proper sliding mode. Based on the analysis of [8], a complete algorithm for the integration of piecewise-smooth dynamical systems with sliding modes of codimension at most 2 has been proposed in [10]. This algorithm is, up to our knowledge, the only one available to compute codimension-1 and 2 sliding modes in the numerical integration of a piecewise-smooth dynamical system. This is preferrable to the direct integration of the regularized system, because the stiffness of the singularly perturbed problem and the possible presence of high oscillations make the numerical integration very challenging.

This article is devoted to the case of higher dimensional intersections and related higher codimension sliding modes. Unfortunately the analysis of the hidden dynamics is much more complicated and a complete classification is not possible. Nevertheless we are able to prove some interesting results and to analyze to some detail special cases which are important in applications. E.g., situations where the vector fields have specific geometric properties (like the so-called nodal attractivity [3]) and where the systems are characterized by certain properties (like cooperativity [11]).

Organization of the article. In Section 2 we introduce the main concepts (sliding modes, regularization, and hidden dynamics). The main part of the paper is devoted to the study of the hidden dynamics in codimension 3 and higher. The maximal number of stationary points (which corresponds to sliding modes of maximal codimension) is studied in Section 3 and an example with six stationary points in codimension 3 is presented. This is obtained by applying Bezout's theory. Section 4 discusses the case of centrally symmetric vector fields and Section 5 the case of nodally attractive vector fields. An interesting class of examples shows that a vector field that is centrally symmetric and nodally attractive can have more than one stationary point. The study of the limit behaviour for $\tau \to \infty$ of the solution of the hidden dynamics is very challenging for $d \geq 3$, because geometric arguments (e.g., Poincaré–Bendixson theorem) are not available or difficult to apply. A positive convergence result for nodally attractive vector fields with a hidden dynamics that is strictly cooperative, is given in the final Section 6.

2 Main concepts

We closely follow the notation of [10].

• *Piecewise-smooth dynamical systems.* We consider discontinuity hyper-surfaces

$$\Sigma_j = \{y \in \mathbb{R}^n \,|\, \alpha_j(y) = 0\}, \qquad j = 1, \dots, d, \tag{1}$$

where $\alpha : \mathbb{R}^n \to \mathbb{R}^d$ (with $d < n$) is sufficiently differentiable and the hyper-surfaces intersect transversally. They divide the phase space into 2^d open regions

$$\mathcal{R}^{\mathbf{k}} = \big\{y \in \mathbb{R}^n \,\big|\, k_j \alpha_j(y) > 0 \text{ for } j = 1, \dots, d\big\}, \tag{2}$$

where $\mathbf{k} = (k_1, \dots, k_d)$ is a multi-index with $k_j \in \{-1, 1\}$. The piecewise-smooth dynamical system is then given by

$$\dot y = F^{\mathbf{k}}(y) \qquad \text{for} \qquad y \in \mathcal{R}^{\mathbf{k}}. \tag{3}$$

The smooth functions $F^{\mathbf{k}}(y)$ are assumed to be defined in a neighbourhood of the closure of $\mathcal{R}^{\mathbf{k}}$. In the discontinuity set $\bigcup_{j=1}^d \Sigma_j$ the right-hand side of (3) is thus multi-valued with values from the neighbouring domains.

• *Sliding modes.* We extend the concept of solution to the discontinuity set in the spirit of Filippov [5]. For an index vector $\mathbf{k} = (k_1, \dots, k_d)$ with $k_j \in \{-1, 0, 1\}$ we consider

$$\mathcal{R}^{\mathbf{k}} = \Big\{y \in \mathbb{R}^n \,\big|\, \alpha_j(y) = 0 \text{ if } k_j = 0,\ k_j\alpha_j(y) > 0 \text{ if } k_j \neq 0\Big\} \tag{4}$$

which is a discontinuity set, if at least one component k_j vanishes. It reduces to (2) if all components are non-zero. For $\mathbf{k} = (k_1, \dots, k_d)$ we define $\mathcal{I}^{\mathbf{k}} = \{j \,|\, k_j = 0\}$, and we let

$$\mathcal{N}^{\mathbf{k}} = \Big\{\boldsymbol{\ell} \in \{-1, 1\}^d \,\Big|\, \ell_j \in \{-1, 1\} \text{ if } k_j = 0,\ \ell_j = k_j \text{ if } k_j \neq 0\Big\}$$

which collects the index vectors $\boldsymbol{\ell}$ such that $\mathcal{R}^{\boldsymbol{\ell}}$ touches $\mathcal{R}^{\mathbf{k}}$. With this notation we consider the differential-algebraic equation (DAE)

$$\begin{aligned} \dot{y} &= \sum_{\boldsymbol{\ell}\in\mathcal{N}^{\mathbf{k}}} \Bigl(\prod_{j\in\mathcal{I}^{\mathbf{k}}} \frac{(1+\ell_j\lambda_j)}{2} \Bigr) F^{\boldsymbol{\ell}}(y) \\ 0 &= \alpha_j(y), \qquad j\in\mathcal{I}^{\mathbf{k}} \end{aligned} \tag{5}$$

with algebraic variables $\lambda_j, j\in\mathcal{I}^{\mathbf{k}}$. For $\lambda_j\in[-1,1]$ the vector field in (5) is a convex combination of the vector fields $F^{\boldsymbol{\ell}}(y)$ (with $\boldsymbol{\ell}\in\mathcal{N}^k$). The solution of (5) is therefore a Filippov solution of (3).

- *Regularization.* On the ε-neighbourhood

$$\mathcal{R}^{\mathbf{k}}_{\varepsilon} = \Bigl\{ y\in\mathbb{R}^n \,\Big|\, |\alpha_j(y)|\le\varepsilon \text{ if } k_j=0,\ k_j\alpha_j(y)>0 \text{ if } k_j\ne 0 \Bigr\} \tag{6}$$

of the discontinuity set (4) we consider the differential equation

$$\dot{y} = \sum_{\boldsymbol{\ell}\in\mathcal{N}^{\mathbf{k}}} \Bigl(\prod_{j\in\mathcal{I}^{\mathbf{k}}} \frac{(1+\ell_j\pi(u_j))}{2} \Bigr) F^{\boldsymbol{\ell}}(y) \qquad \text{with} \quad u_j=\frac{\alpha_j(y)}{\varepsilon}, \tag{7}$$

where the transition function $\pi(u)$ is continuous, piecewise-smooth, and satisfies $\pi(u)=-1$ for $u\le -1$ and $\pi(u)=1$ for $u\ge 1$. For ease of presentation we assume in the following that $\pi(u)=u$ for $|u|\le 1$. This ordinary differential equation is on the set $\mathcal{R}^{\mathbf{k}}_{\varepsilon}$ a regularization of the discontinuous problem (3).

- *Hidden dynamics.* After entering an intersection of discontinuity hyper-surfaces (say at $y^*\in\mathcal{R}^{\mathbf{k}}$) there are typically more than one Filippov solutions, and it is of interest to study which solution can be interpreted as the limit $\varepsilon\to 0$ of the regularized differential equation. In the region $\mathcal{R}^{\mathbf{k}}_{\varepsilon}$ it follows from (7) that $u_i=\alpha_i(y)/\varepsilon$ satisfies

$$\varepsilon\,\dot{u}_i = \sum_{\boldsymbol{\ell}\in\mathcal{N}^{\mathbf{k}}} \Bigl(\prod_{j\in\mathcal{I}^{\mathbf{k}}} \frac{(1+\ell_j\pi(u_j))}{2} \Bigr) \alpha_i'(y)F^{\boldsymbol{\ell}}(y), \qquad i\in\mathcal{I}^{\mathbf{k}}, \tag{8}$$

which is a singularly perturbed problem. For its study we introduce the fast time $\tau=t/\varepsilon$, we denote the derivative with respect to τ by a prime, and we substitute the constant vector y^* for y. This yields

$$u_i' = \sum_{\boldsymbol{\ell}\in\mathcal{N}^{\mathbf{k}}} \Bigl(\prod_{j\in\mathcal{I}^{\mathbf{k}}} \frac{(1+\ell_j\pi(u_j))}{2} \Bigr) \alpha_i'(y^*)F^{\boldsymbol{\ell}}(y^*), \qquad i\in\mathcal{I}^{\mathbf{k}}, \tag{9}$$

which is a regular dynamical system for $u_i, i\in\mathcal{I}^{\mathbf{k}}$. It is called *hidden dynamics* (a term coined in [12]). We note that this definition and all results of the present work are of local nature, because they depend on the fixed vector y^*.

The initial value for (9) is determined by the incoming solution at $y^* \in \mathcal{R}^{\mathbf{k}}$. Let $\mathbf{k}^-$ be an index vector corresponding to a manifold of codimension one less, i.e., there is $i^* \in \mathcal{I}^{\mathbf{k}}$, such that $k_j^- = k_j$ for $j \in \{1,\ldots,d\} \setminus \{i^*\}$, and $k_{i^*}^- \in \{-1,1\}$. Assuming that the solution lies in $\mathcal{R}^{\mathbf{k}^-}$ before entering $\mathcal{R}^{\mathbf{k}}$ at y^*, then the initial value for (9) is given by $u_{i^*}(0) = k_{i^*}$, and the remaining $u_i(0)$ are such that the right-hand side of (9) vanishes for $i \neq i^*$.

The hidden dynamics gives us much insight into the kind of solution leaving y^* (cf. [8, Section 5.2]). We have:

- sliding in $\mathcal{R}^{\mathbf{k}}$, if the solution of (9) converges, for $\tau \to \infty$, to a stationary point with all components inside $(-1,1)$;
- sliding in $\mathcal{R}^{\mathbf{k}^+}$, where $k_i^+ = k_i$ for all $i \in \{1,\ldots,d\}$ with the exception of those $i \in \mathcal{I}^{\mathbf{k}}$ for which $u_i(\tau)$ either tends to $-\infty$ (then $k_i^+ = -1$) or to $+\infty$ (then $k_i^+ = 1$). For the remaining $i \in \mathcal{I}^{\mathbf{k}}$ the solution $u_i(\tau)$ has a limit in $(-1,1)$;
- classical solution in $\mathcal{R}^{\mathbf{k}^+}$, where $k_i^+ = k_i$ for $i \notin \mathcal{I}^{\mathbf{k}}$ and, for $i \in \mathcal{I}^{\mathbf{k}}$, we have $k_i^+ = -1$ or $+1$, if $u_i(\tau)$ tends to $-\infty$ or $+\infty$, respectively.

If all expressions $\alpha_i'(y)F^{\boldsymbol{\ell}}(y)$ are constant in a neighbourhood of y^*, then the differential equations (9) are exact, and the above statements are obvious. The general case is less trivial, but we expect that the system (9) credibly describes the transient behaviour of the solution of the regularized differential equation. A rigorous treatment of the limit behaviour $\varepsilon \to 0$ of the regularized solution is challenging, and even for the codimension 2 case it is still an open problem (see [9]).

• *Special case: codimension* 3. Most of the present article deals with sliding modes of codimension 3. With the transition function $\pi(u) = u$ for $u \in [-1,1]$ and for $d = 3$ the differential equations of the hidden dynamics (9) becomes (for $i = 1,2,3$ and $u_i \in [-1,1]$)

$$
\begin{aligned}
u_i' = \Big(\big(1+u_1\big)\big(1+u_2\big) \Big(\big(1+u_3\big) f_i^{+++} + \big(1-u_3\big) f_i^{++-} \Big) \\
+ \ldots + \big(1-u_1\big)\big(1-u_2\big)\big(1-u_3\big)\, f_i^{---} \Big) \Big/ 8.
\end{aligned}
\tag{10}
$$

We use the notation $f_i^{\mathbf{k}} = \alpha_i'(y^*)F^{\mathbf{k}}(y^*)$, and for the index vector $\mathbf{k} = (k_1,k_2,k_3)$ we only write the sign of k_j. The hidden dynamics is completely described by the eight 3-dimensional vectors

$$
f^{---},\ f^{--+},\ f^{-+-},\ f^{-++},\ f^{+--},\ f^{+-+},\ f^{++-},\ f^{+++}. \tag{11}
$$

Sometimes it is convenient to write the equation (10) as

$$u_i' = g_i(u_1, u_2, u_3) \tag{12}$$

with the multilinear polynomial

$$\begin{aligned} g_i(u_1, u_2, u_3) = a_i\, u_1 u_2 u_3 + b_{i1}\, u_2 u_3 + b_{i2}\, u_1 u_3 + b_{i3}\, u_1 u_2 \\ + c_{i1}\, u_1 + c_{i2}\, u_2 + c_{i3}\, u_3 + d_i. \end{aligned} \tag{13}$$

The coefficients of this polynomial representation are given by

$$\begin{aligned}
a &= \tfrac{1}{8}\big(f^{+++} - f^{++-} - f^{+-+} + f^{+--} - f^{-++} + f^{-+-} + f^{--+} - f^{---}\big) \\
b_1 &= \tfrac{1}{8}\big(f^{+++} - f^{++-} - f^{+-+} + f^{+--} + f^{-++} - f^{-+-} - f^{--+} + f^{---}\big) \\
b_2 &= \tfrac{1}{8}\big(f^{+++} - f^{++-} + f^{+-+} - f^{+--} - f^{-++} + f^{-+-} - f^{--+} + f^{---}\big) \\
b_3 &= \tfrac{1}{8}\big(f^{+++} + f^{++-} - f^{+-+} - f^{+--} - f^{-++} - f^{-+-} + f^{--+} + f^{---}\big) \\
c_1 &= \tfrac{1}{8}\big(f^{+++} + f^{++-} + f^{+-+} + f^{+--} - f^{-++} - f^{-+-} - f^{--+} - f^{---}\big) \\
c_2 &= \tfrac{1}{8}\big(f^{+++} + f^{++-} - f^{+-+} - f^{+--} + f^{-++} + f^{-+-} - f^{--+} - f^{---}\big) \\
c_3 &= \tfrac{1}{8}\big(f^{+++} - f^{++-} + f^{+-+} - f^{+--} + f^{-++} - f^{-+-} + f^{--+} - f^{---}\big) \\
d &= \tfrac{1}{8}\big(f^{+++} + f^{++-} + f^{+-+} + f^{+--} + f^{-++} + f^{-+-} + f^{--+} + f^{---}\big)
\end{aligned}$$

In the following we also consider the surfaces

$$\Gamma_i = \big\{(u_1, u_2, u_3) \,\big|\, g_i(u_1, u_2, u_3) = 0\big\}, \qquad i = 1, 2, 3, \tag{14}$$

on which the flow (12) is orthogonal to the u_i-axis. The set of stationary points is $\mathcal{S} = \Gamma_1 \cap \Gamma_2 \cap \Gamma_3$.

3 Maximal number of sliding modes

In codimension 2 (intersection of two discontinuity hyper-surfaces) there are at most two stationary points in the hidden dynamics – one unstable and the other stable. Here we are interested in the maximal number of stationary points (i.e., sliding modes) in codimension three and higher. Without loss of generality we consider the transition function $\pi(u) = u$ for $|u| \le 1$. We also assume that the index set in (9) is $\mathcal{I}^{\mathbf{k}} = \{1, \ldots, d\}$.

Theorem 1 *Without any assumptions on the vector fields, the hidden dynamics* (9) *in codimension d can have at most $d!$ isolated stationary points in $\mathbb{R}^d$.*

Proof This bound is a consequence of Bézout's Theorem (see [17], where a proof based on basic results in algebraic geometry [21] is indicated). Introducing additional variables $u_{10}, \dots, u_{d0}$, substituting in the polynomial system (9) the variables $\pi(u_j) = u_j$ by u_j/u_{j0}, and multiplying each equation by the product $u_{10} \cdot \ldots \cdot u_{d0}$ yields homogeneous polynomials of degree 1 in the variables (u_j, u_{j0}). Bézout's Theorem as formulated in [17] states that there are at most $d!$ isolated solutions of the polynomial system.

For convenience of the reader we give an elementary proof in codimension $d = 3$ which is also useful for the construction of the stationary points. We consider the product u_1u_2 as an independent variable, and we write the condition for a stationary point of (13) as

$$\begin{pmatrix} b_{13} + a_1u_3 & c_{11} + b_{12}u_3 & c_{12} + b_{11}u_3 \\ b_{23} + a_2u_3 & c_{21} + b_{22}u_3 & c_{22} + b_{21}u_3 \\ b_{33} + a_3u_3 & c_{31} + b_{32}u_3 & c_{32} + b_{31}u_3 \end{pmatrix} \begin{pmatrix} u_1u_2 \\ u_1 \\ u_2 \end{pmatrix} = - \begin{pmatrix} d_1 + c_{13}u_3 \\ d_2 + c_{23}u_3 \\ d_3 + c_{33}u_3 \end{pmatrix}.$$

Solving this linear system by Cramer's rule shows that

$$u_1 = \frac{p(u_3)}{d(u_3)}, \quad u_2 = \frac{q(u_3)}{d(u_3)}, \quad u_1u_2 = \frac{r(u_3)}{d(u_3)}, \tag{15}$$

where $p(u_3), q(u_3), r(u_3), d(u_3)$ are polynomials of degree 3. The necessary condition $u_1 \cdot u_2 = u_1u_2$ then yields the polynomial equation

$$p(u_3)q(u_3) - r(u_3)d(u_3) = 0 \tag{16}$$

of degree 6, which has at most 6 solutions. The stationary points of (13) are then (u_1, u_2, u_3), where u_3 is a root of (16) and u_1, u_2 are given by (15). □

The following example shows that there exist problems having 6 stationary points in the unit cube.

Example 1 We let $\Sigma_j = \{y \in \mathbb{R}^3 \,|\, y_j = 0\}$ (for $i = 1, 2, 3$), so that $f^{\mathbf{k}} = F^{\mathbf{k}}(y^*)$. With y^* at the origin, we assume that the eight vector fields (in the ordering of (11)) are given by

$$\begin{pmatrix} 3.8 \\ -4.9 \\ 3.5 \end{pmatrix} \begin{pmatrix} -1.0 \\ 0.8 \\ -0.3 \end{pmatrix} \begin{pmatrix} -1.5 \\ -0.5 \\ 0.9 \end{pmatrix} \begin{pmatrix} 2.2 \\ 1.0 \\ -3.7 \end{pmatrix} \begin{pmatrix} -1.3 \\ -4.0 \\ -4.5 \end{pmatrix} \begin{pmatrix} -2.6 \\ 3.1 \\ 4.9 \end{pmatrix} \begin{pmatrix} 2.3 \\ 3.0 \\ -0.4 \end{pmatrix} \begin{pmatrix} -1.3 \\ -1.6 \\ 0.8 \end{pmatrix}.$$

The corresponding hidden dynamics (12) has 6 stationary points, which are given by (rounded to 3 digits)

$$\begin{pmatrix} 0.076 \\ 0.263 \\ 0.203 \end{pmatrix} \begin{pmatrix} 0.3917 \\ 0.274 \\ -0.056 \end{pmatrix} \begin{pmatrix} -0.928 \\ -0.713 \\ 0.653 \end{pmatrix} \begin{pmatrix} 0.640 \\ 0.934 \\ 0.363 \end{pmatrix} \begin{pmatrix} -0.543 \\ 0.822 \\ -0.542 \end{pmatrix} \begin{pmatrix} -0.029 \\ 0.191 \\ 0.276 \end{pmatrix}. \tag{17}$$

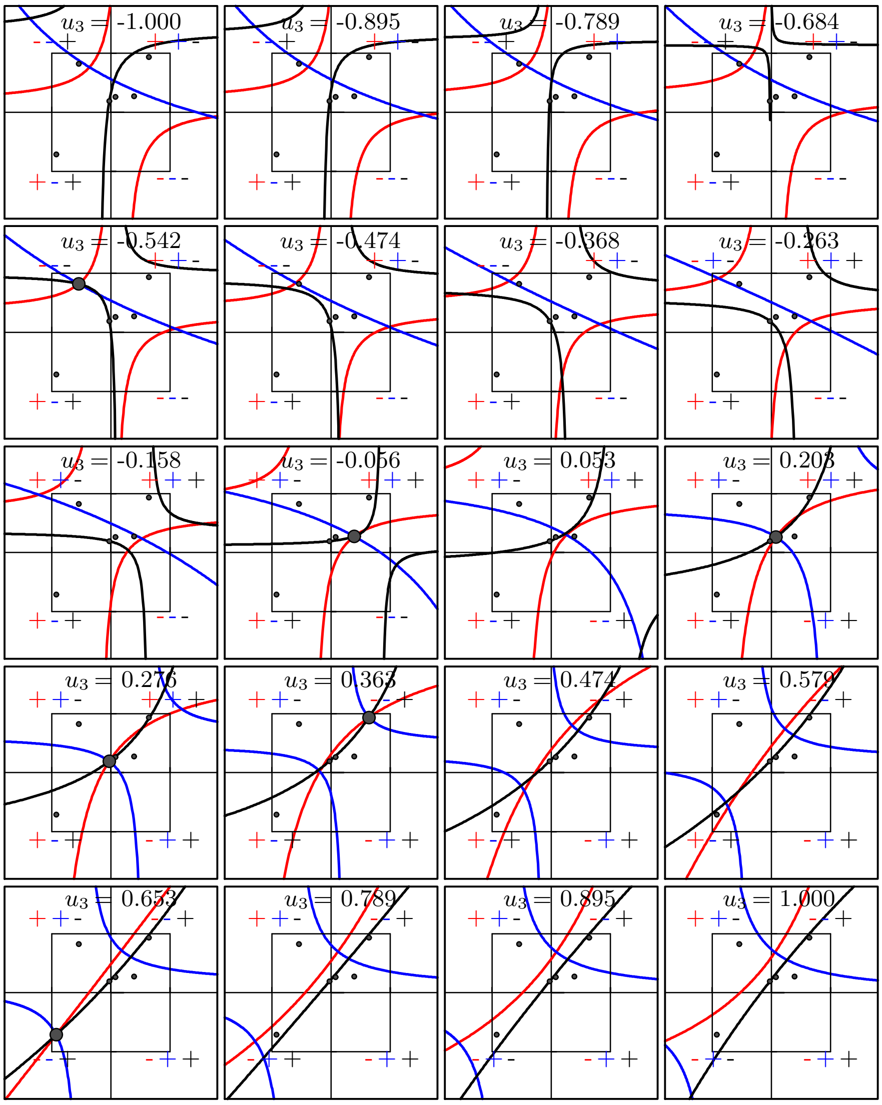

Fig. 1. Cross sections with the horizontal planes $u_3 = \text{const}$ for the problem with 6 stationary points, Γ_1 (red), Γ_2 (blue), Γ_3 (black).

By checking the eigenvalues of the Jacobian matrix of (12) we find that only the last stationary point in this list is asymptotically stable (one real and a complex pair of eigenvalues in the left complex half-plane), all others are unstable.

For an illustration of the dynamics of (12) we consider the surfaces Γ_i of (14), and we note that the set of stationary points is $\mathcal{S}$ =

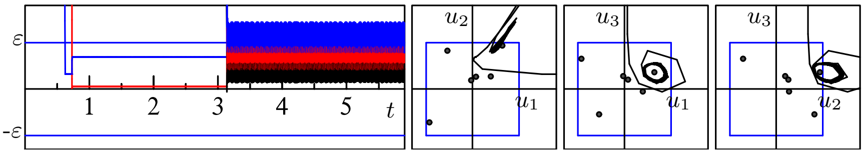

Fig. 2. The first (red), second (blue), and third (black) component of the solution of the problem of Example 1 as a function of time. The 3 pictures to the right show the solution in the phase space.

$\Gamma_1 \cap \Gamma_2 \cap \Gamma_3$. Figure 1 shows for 20 different values of u_3 cross sections of the surfaces Γ_1 (red), Γ_2 (blue), Γ_3 (black) with horizontal planes (constant u_3). They appear as hyperbolas with horizontal and vertical asymptotes. The projection of the stationary points to these planes is plotted as small circles. If a stationary point lies on such a horizontal plane, it is highlighted by a larger circle. Therefore we have included the values u_3 from (17) in the figure. The sign pattern of the vector field at the corners of the square in the (u_1, u_2)-space with fixed u_3 is included. Note that for some value of u_3 close to -0.7 (between the 3rd and 4th pictures if Figure 1) the hyperbola corresponding to Γ_3 (black curve) degenerates to the union of the two lines (its two asymptotes).

We are curious to see the behaviour of the solution of the regularized differential equation, when it enters the intersection of all three discontinuity surfaces. We use $\varepsilon = 10^{-3}$, but due to the relative scaling any other value would give the same pictures in Figure 2. With the initial value $y(0) = (1,1,1)^\top$, the solution approaches Σ_2 at $(0.1875, 0, 0.5)$ for $t = 0.625$. It remains $\mathcal{O}(\varepsilon)$-close to it (which, in the limit $\varepsilon \to 0$, is a codimension 1 sliding on Σ_2) until $t \approx 0.733$. Then, the solution approaches $\Sigma_1 \cap \Sigma_2$ and remains $\mathcal{O}(\varepsilon)$-close to it (i.e., a codimension 2 sliding in the limit) until it approaches the intersection $\Sigma_1 \cap \Sigma_2 \cap \Sigma_3$ slightly after $t = 3$, as can be seen in the left picture of Figure 2. From there on the solution stays $\mathcal{O}(\varepsilon)$-close to $\Sigma_1 \cap \Sigma_2 \cap \Sigma_3$ (i.e., a codimension 3 sliding in the limit $\varepsilon \to 0$).

The right pictures show the dynamics of the regularized problem close to the codimension 3 sliding. The scaled solution $u_i = y_i/\varepsilon$ does not approach the stable stationary point (as could be expected), but it remains close to a periodic solution around the 4th stationary point of (17), which is unstable (one negative real eigenvalue and a complex pair of eigenvalues with positive real part).

4 Centrally symmetric vector fields

Sharper bounds on the maximal number of sliding modes (stationary points of the hidden dynamics) can be obtained, when the class of vector fields is restricted. A vector field is called *centrally symmetric* at $y^* \in \mathcal{R}^{\mathbf{k}}$, if

$$f_i^{-\boldsymbol{\ell}} = -f_i^{\boldsymbol{\ell}} \qquad \text{for} \quad i \in \mathcal{I}^{\mathbf{k}}, \quad \boldsymbol{\ell} \in \mathcal{N}^{\mathbf{k}} \tag{18}$$

(as in (10) we use the notation $f_i^{\boldsymbol{\ell}} = \alpha_i'(y^*) F^{\boldsymbol{\ell}}(y^*)$). This means that close to $y^* \in \mathcal{R}^{\mathbf{k}}$ opposite vectors among $\{f^{\boldsymbol{\ell}} \,|\, \boldsymbol{\ell} \in \mathcal{N}^{\mathbf{k}}\}$ differ only in its sign. For the equations of the hidden dynamics this implies that a substitution $\boldsymbol{\ell} \leftrightarrow -\boldsymbol{\ell}$ is equivalent to substituting $u_j \leftrightarrow -u_j$ and multiplying the right-hand side by -1, so that only products with an odd number of factors can appear in a polynomial representation.

In codimension 3 the hidden dynamics thus becomes (12) with

$$g_i(u_1,u_2,u_3) = a_i\, u_1 u_2 u_3 + c_{i1}\, u_1 + c_{i2}\, u_2 + c_{i3}\, u_3 \tag{19}$$

with vectors $a = (a_1,a_2,a_3)^\top$ and $c_j = (c_{1j}, c_{2,j}, c_{3j})^\top$ given by the formulas after equation (13). For the statement of the following results we consider the expression

$$D = \frac{\det(a,c_1,c_2)\det(c_1,c_2,c_3)}{\det(a,c_2,c_3)\det(a,c_1,c_3)}. \tag{20}$$

Theorem 2 *Let D be the expression of* (20)*.*

- *If $D < 0$, the origin is the only (real) stationary point of* (19)*.*
- *If $D > 0$, we have three stationary points, one is at the origin, and the other two are*

$$u_3 = \pm\sqrt{D}, \quad u_1 = u_3 \frac{\det(a,c_2,c_3)}{\det(a,c_1,c_2)}, \quad u_2 = -u_3 \frac{\det(a,c_1,c_3)}{\det(a,c_1,c_2)}. \tag{21}$$

The nonzero stationary points can be in or outside the unit cube.

Proof For a stationary point (u_1,u_2,u_3) the hidden dynamics (12) with (19) yields (see the proof of Theorem 1)

$$\begin{pmatrix} a_1u_3 & c_{11} & c_{21} \\ a_2u_3 & c_{12} & c_{22} \\ a_3u_3 & c_{13} & c_{23} \end{pmatrix} \begin{pmatrix} u_1u_2 \\ u_1 \\ u_2 \end{pmatrix} = - \begin{pmatrix} c_{31}u_3 \\ c_{32}u_3 \\ c_{33}u_3 \end{pmatrix}. \tag{22}$$

An application of Cramer's rule gives the second and third relations of (21) and a formula for u_1u_2, which leads to $u_3^2 = D$. □

For the study of stability of the dynamical system (12) we consider its Jacobian matrix

$$J(u_1,u_2,u_3) = (c_1,c_2,c_3) + a z^\top, \qquad z = (u_2u_3, u_1u_3, u_1u_2)^\top. \quad (23)$$

Theorem 3 *Let* $(u_1^*,u_2^*,u_3^*) \in \mathbb{C}^3$ *be a stationary point that is different from the origin. We then have*

$$\det J(u_1^*,u_2^*,u_3^*) = -2 \det J(0,0,0).$$

Proof The determinant of the rank-one perturbation (23) is given by the matrix determinant lemma as

$$\det J(u_1,u_2,u_3) = \det J(0,0,0) + z^\top \mathrm{adj}(J^0)\, a,$$

where $\mathrm{adj}(J^0)$ is the adjugate matrix of $J^0 = J(0,0,0)$. At the nonzero stationary points we have from (22)

$$u_2^*u_3^* = -\frac{\det(c_1,c_2,c_3)}{\det(a,c_2,c_3)}, \quad u_1^*u_3^* = \frac{\det(c_1,c_2,c_3)}{\det(a,c_1,c_3)}, \quad u_1^*u_2^* = -\frac{\det(c_1,c_2,c_3)}{\det(a,c_1,c_2)}.$$

The formula for the adjugate matrix of J^0 yields

$$(u_2^*u_3^*, u_1^*u_3^*, u_1^*u_2^*)^\top \mathrm{adj}(J^0)\, a = -3\, \det(c_1,c_2,c_3) = -3\, \det J(0,0,0),$$

which proves the statement of the theorem. □

As a consequence there cannot exist three asymptotically stable stationary points in $\mathbb{R}^3$.

5 Nodally attractive vector fields

A vector field is called *nodally attractive* at $y^* \in \mathcal{R}^{\mathbf{k}}$, if

$$\mathrm{sgn}\, f_i^{\boldsymbol{\ell}} = -\,\mathrm{sgn}\, \ell_i \qquad \text{for} \quad i \in \mathcal{I}^{\mathbf{k}}, \;\; \boldsymbol{\ell} \in \mathcal{N}^{\mathbf{k}}. \qquad (24)$$

This means that all neighbouring vector fields point towards y^* (recall that $f_i^{\boldsymbol{\ell}} = \alpha_i'(y^*) F^{\boldsymbol{\ell}}(y^*)$). It is natural to consider such vector fields, because every solution of (3) entering such a y^* has to continue as a sliding in $\mathcal{R}^{\mathbf{k}}$.

Theorem 4 *Consider the equation* (10) *for the hidden dynamics in codimension* 3*. If the vector field is nodally attractive, the hidden dynamics generically has* 1, 3 *or* 5 *stationary points, of which* 1, 2, *or* 3 *are asymptotically stable, respectively.*

Proof We consider the intersection $\Gamma_1 \cap \Gamma_2$. Nodal attractivity implies that horizontal cross sections of Γ_1 are parts of a hyperbola connecting the front and back faces of the cube, and those of Γ_2 connect the left and right faces of the cube. Consequently, for every fixed $u_3 \in [-1, 1]$ there is a unique $(u_1, u_2, u_3) \in \Gamma_1 \cap \Gamma_2$.

Along the curve $\Gamma_1 \cap \Gamma_2$, the vector field points vertically upwards or downwards according to $g_3(u_1, u_2, u_3) > 0$ or $g_3(u_1, u_2, u_3) < 0$, respectively. It changes the direction when crossing the surface Γ_3, which happens at stationary points.[1] By nodal attractivity we have $g_3(u_1, u_2, u_3) > 0$ at the bottom face $u_3 = -1$, so that the lowest intersection with Γ_3 is an asymptotically stable stationary point. Continuing along the curve $\Gamma_1 \cap \Gamma_2$, the stability at stationary points alternates between asymptotically stable and unstable. Since (again by nodal attractivity) the last intersection with Γ_3 in the cube is asymptotically stable, we must have an odd number of stationary points in the cube. □

Corollary 1 *Assume nodal attractivity and central symmetry.*
- *If the origin is the only stationary point in the cube, then it is asymptotically stable.*
- *If there are 3 stationary points in the cube, then the origin is unstable, and the other two stationary points are asymptotically stable.*

Proof This is an immediate consequence of Theorems 2 and 4. □

Intuitively it is not obvious that nodally attractive vector fields can have more than one stationary point. The constructive examples of Section 5.1 show that there exist nodally attractive vector fields having 3 stationary points. We are not aware of the existence of nodally attractive vector fields with 5 stationary points in the unit cube.

5.1 Construction of nodally attractive vector fields with 3 equilibria

The first example of a nodally attractive vector field with 3 equilibria has been communicated in 2010 by Douglas Ulmer to L. Dieci and N. Guglielmi. Here, we present a systematic way of constructing such examples. We investigate the possibility of achieving nodal attractivity among the centrally symmetric vector fields with three equilibria. We let

$$\alpha_1 = \det(a, c_2, c_3), \qquad \alpha_2 = \det(a, c_1, c_3),$$
$$\alpha_3 = \det(a, c_1, c_2), \qquad \beta = \det(c_1, c_2, c_3),$$

[1] Note that "generically" the curve $\Gamma_1 \cap \Gamma_2$ crosses transversally the surface Γ_3.

so that the expression D of (20) becomes $D = \alpha_3\beta/\alpha_1\alpha_2$. Assuming $D > 0$ we have 3 equilibria by Theorem 2, which are

$$u_1 = \pm\sqrt{\frac{\alpha_1\beta}{\alpha_2\alpha_3}}, \quad u_2 = \pm\sqrt{\frac{\alpha_2\beta}{\alpha_1\alpha_3}}, \quad u_3 = \pm\sqrt{\frac{\alpha_3\beta}{\alpha_1\alpha_2}}$$

in addition to the origin. If β is small compared to the α_j, the equilibria are inside the unit cube. This is achieved by choosing (c_1, c_2, c_3) close to a singular matrix. We arbitrarily put

$$c_{ij} = \begin{cases} -2+\delta & \text{for} \quad i = j \\ \quad 1 & \text{for} \quad i \neq j \end{cases}$$

so that $\det(c_1, c_2, c_3) = 0$ for $\delta = 0$. Moreover, we put $a_j = a$ for all j, where a is another parameter. For this choice of coefficients we have

$$\beta = \delta(\delta-3)^2, \quad \alpha_1 = -\alpha_2 = \alpha_3 = a(\delta-3)^2.$$

This yields $D = -\delta/a$, so that the problem has 3 equilibria if a and δ have opposite sign. They are inside the unit cube if $|\delta| < |a|$. Inspecting the values of $g(u_1, u_2, u_3)$ (with components given by (13)) at the corners of the unit cube, one can see that the corresponding vector field is nodally attractive iff $-2+\delta < a < -\delta$.

In summary we see that the problem is centrally symmetric and nodally attractive with three equilibria in the unit cube iff

$$0 < \delta < 1 \quad \text{and} \quad -2+\delta < a < -\delta. \tag{25}$$

E. g., for $\delta = 1/2$ and $a = -1$, we have equilibria at $u_1 = u_2 = u_3 = \pm 1/\sqrt{2}$, in addition to the one at the origin.

6 Limit behaviour of the solution of the hidden dynamics

The behaviour of the solution $u(\tau)$ of the hidden dynamics for $\tau \to \infty$, corresponding to initial values $u(0) \in \partial\mathcal{C}$ from the incoming solution, gives much insight into the discontinuous system (3). It is of high interest to study this limit. The system of Example 1 illustrates that in general it is very challenging or even impossible to predict the limit behaviour. We restrict the following analysis to simple (but important) situations in codimension 3.

When a codimension 2 sliding solution along $\Sigma_i \cap \Sigma_j$ enters the intersection $\Sigma_i \cap \Sigma_j \cap \Sigma_k$, the initial value for the hidden dynamics (12) lies on the intersection of $\Gamma_i \cap \Gamma_j$ with one of the faces $u_k = \pm 1$, depending on the side from where the solution enters. We denote by $\mathcal{E}$ the set of all such initial values. In the case of a nodally attractive vector field, each face of the cube has exactly one element in $\mathcal{E}$.

6.1 Nodal attractivity with cooperative hidden dynamics

We consider nodally attractive vector fields and, in addition, we assume that the system (12) of the hidden dynamics is *strictly cooperative* on the cube $\mathcal{C}$, i.e.

$$\partial_i g_j(u_1,u_2,u_3) > 0 \qquad \text{for} \quad i \neq j \;\; \text{and} \;\; (u_1,u_2,u_3) \in \mathcal{C}. \tag{26}$$

The dynamics of cooperative systems has been thoroughly studied by Hirsch [11]. It is shown that limit sets are invariant sets of systems in one dimension lower.

Theorem 5 *Assume that the dynamical system* (12) *is strictly cooperative and corresponds to a nodally attractive vector field.*

Then, the solution corresponding to an initial value $u^0 \in \mathcal{E}$ converges to the stationary point of (12) *that is closest to the face on which the initial value lies.*

The proof of this theorem is postponed to Section 6.3, because it is closely related to the monotonicity of solutions that we shall study next.

Remark 1 The example of Section 5.1 is nodally attractive, if the parameters satisfy (25). It is strictly cooperative on the whole cube $\mathcal{C}$, if $|a| < 1$. Hence, for $0 < \delta < 1$ and $-1 < a < -\delta$, the solution of (12) corresponding to an initial value $u^0 \in \mathcal{E}$ converges to the stationary point that is closest to the face on which the initial value lies.

6.2 Monotone convergence

We let $(u_{10},u_{20},u_{30}) \in \mathcal{E}$ be an initial value of the hidden dynamics (12). Without loss of generality we assume that

$$g_1(u_{10},u_{20},u_{30}) = g_2(u_{10},u_{20},u_{30}) = 0, \quad u_{30} \in \{+1,-1\}. \tag{27}$$

We define the signs $\sigma_1,\sigma_2,\sigma_3 \in \{+1,-1\}$ in such a way that

$$\sigma_1\partial_3 g_1(u_{10},u_{20},u_{30}) > 0, \quad \sigma_2\partial_3 g_2(u_{10},u_{20},u_{30}) > 0, \quad u_{30} = -\sigma_3,$$

so that the solution of (12) with initial value (u_{10},u_{20},u_{30}) enters the cube $\mathcal{C}$ in the set

$$\mathcal{D} = \Big\{ (u_1,u_2,u_3) \in \mathcal{C} \;\Big|\; \sigma_i g_i(u_1,u_2,u_3) > 0,\ i=1,2,3 \Big\}. \tag{28}$$

We further denote by $\mathcal{D}_0$ the connected component of $\mathcal{D}$ for which (u_{10},u_{20},u_{30}) lies on the closure of $\mathcal{D}_0$. Recall that $\mathcal{S}$ denotes the set of stationary points in $\mathcal{C}$.

Theorem 6 *Assume that $\partial\mathcal{D}_0$ has an empty intersection with the faces $u_1 = \sigma_1$, $u_2 = \sigma_2$, $u_3 = \sigma_3$ of the cube and that (for $i = 1, 2, 3$)*

$$\sigma_i g_i'(u_1, u_2, u_3)\, g(u_1, u_2, u_3) > 0 \quad \textit{for} \quad (u_1, u_2, u_3) \in (\partial\mathcal{D}_0 \setminus \mathcal{S}) \cap \Gamma_i.$$

Then, the solution of (12) *with initial value* (27) *stays in $\mathcal{D}_0$ and converges to a unique stationary point in the closure of $\mathcal{D}_0$.*

Proof By definition of $\sigma_1, \sigma_2, \sigma_3$ the solution $\big(u_1(\tau), u_2(\tau), u_3(\tau)\big)$ enters the set $\mathcal{D}_0$. As long as it stays there, each component is a monotonic function (increasing for $\sigma_i = +1$ and decreasing for $\sigma_i = -1$).

The border of $\mathcal{D}_0$ is composed of parts of the faces $u_1 = -\sigma_1$, $u_2 = -\sigma_2$, $u_3 = -\sigma_3$ of the cube $\mathcal{C}$, and of subsets of the surfaces $\Gamma_1, \Gamma_2, \Gamma_3$. We show that, with the exception of stationary points, the vector field (12) points on $\partial\mathcal{D}_0$ into the interior of $\mathcal{D}_0$. At the faces of the cube this follows from the definition of $\mathcal{D}$. At a point $(u_1, u_2, u_3) \in (\partial\mathcal{D}_0 \setminus \mathcal{S}) \cap \Gamma_i$ this is a consequence of the fact that the inner product of $\sigma_i g(u_1, u_2, u_3)$ with the gradient of g_i is positive. Therefore, the solution cannot leave $\mathcal{D}_0$.

Moreover, the assumption that $\partial\mathcal{D}_0$ has an empty intersection with the faces $u_1 = \sigma_1$, $u_2 = \sigma_2$, $u_3 = \sigma_3$ implies that the solution cannot leave the cube $\mathcal{C}$. Due to the monotonicity of its components, we have convergence for $\tau \to \infty$ to a stationary point of (12) that lies on the border of $\mathcal{D}_0$.

We still have to prove that there is only one stationary point on $\partial\mathcal{D}_0$. Let u^* be such a stationary point. We denote by B_{u^*} the subset of $\mathcal{D}_0$, which consists of those points that are mapped to u^* by the flow of (12), i.e., $\varphi_\tau(u) \to u^*$ for $\tau \to \infty$. We shall show that B_{u^*} is an open set. Take $u \in B_{u^*}$ and let $T > 0$ be the time such that $\varphi_T(u)$ is sufficiently close to u^*. By asymptotic stability of u^* there is a neighbourhood $U \subset \mathcal{D}_0$ of $\varphi_T(u)$ such that $\varphi_\tau(w) \to u^*$ for $\tau \to \infty$ for all $w \in U$. The set $\varphi_T^{-1}(U) \cap \mathcal{D}_0$ is a neigbourhood of u that is entirely in B_{u^*}. Consequently, B_{u^*} is an open set.

If there were more than one stationary points in $\mathcal{D}_0$, the set $\mathcal{D}_0$ could be written as the distinct union of open sets B_{u^*}, which is in contradiction with the connectedness of $\mathcal{D}_0$. □

6.3 Proof of Theorem 5

We shall apply Theorem 6. Strict cooperativity of the system (12) implies the assumption on the gradient of g_i, and nodal attractivity implies an empty intersection of $\partial\mathcal{D}_0$ with the faces $u_1 = \sigma_1$, $u_2 = \sigma_2$, $u_3 = \sigma_3$ of the cube $\mathcal{C}$ (see a few lines below). An application of

Theorem 6 thus yields convergence of the solution of (12) to the unique stationary point in $\mathcal{D}_0$. This stationary point is the one that is closest to the face, where the initial value lies.

It follows from the proof of Theorem 4 that $\Gamma_i \cap \Gamma_j$ is a curve that connects the faces $u_k = -1$ and $u_k = 1$ (k is the index different from i and j) of the cube. By definition of $\mathcal{D}_0$ the segment of the curve between the initial value $u^0 \in \mathcal{E}$ and the first stationary point is entirely in $\partial\mathcal{D}_0$. This completes the proof of Theorem 5. □

7 Further remarks

The regularization (7) depends on the transition function $\pi(u)$, and so does the differential equation (9) for the hidden dynamics which, in codimension 3 becomes

$$u_i' = g_i\big(\pi(u_1), \pi(u_2), \pi(u_3)\big)$$

instead of (12). The results of the present article are restricted to the multilinear regularization (7), which implies that the vector field of the hidden dynamics is also multilinear.

Assuming that $\pi(u)$ is strictly monotone on $[-1, 1]$, so that $u \leftrightarrow \pi(u)$ is a bijection on $[-1, 1]$, Theorems 1, 2, 4, and Corollary 1 remain true. One only has to replace u_i by $\pi(u_i)$ in the proofs and in the formulas of (21) in Theorem 2. If $\pi(u)$ is, in addition, continuously differentiable on $[-1, 1]$, also Theorems 5 and 6 remain true.

Declarations

Ethical Approval Not applicable.

Availability of supporting data Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests The authors declare no competing interests.

Funding Nicola Guglielmi acknowledges that his research was supported by funds from the Italian MUR (Ministero dell'Università e della Ricerca) within the PRIN-17 Project "Discontinuous dynamical systems: theory, numerics and applications". Ernst Hairer acknowledges the support from the Fonds National Suisse, Project No. 200020-159856.

Authors contributions This is a joint work of both authors.

Acknowledgements We thank the anonymous reviewer for his/her careful reading of the manuscript and for the helpful comments.

References

1. V. Acary and B. Brogliato. *Numerical methods for nonsmooth dynamical systems: applications in mechanics and electronics*. Lecture Notes in Applied and Computational Mechanics, Vol. 35. Springer-Verlag, 2008.
2. M. di Bernardo, C. J. Budd, A. R. Champneys, and P. Kowalczyk. *Piecewise-smooth dynamical systems*, volume 163 of *Applied Mathematical Sciences*. Springer-Verlag London Ltd., London, 2008. Theory and applications.
3. L. Dieci and F. Difonzo. A comparison of Filippov sliding vector fields in codimension 2. *J. Comput. Appl. Math.*, 262:161–179, 2014.
4. L. Dieci and F. Difonzo. Minimum variation solutions for sliding vector fields on the intersection of two surfaces in $\mathbb{R}^3$. *J. Comput. Appl. Math.*, 292:732–745, 2016.
5. A. F. Filippov. Differential equations with discontinuous right-hand side. *Mat. Sb. (N.S.)*, 51 (93):99–128, 1960. Amer. Math. Soc. Transl. Ser. 2, Vol. 42, pp. 199–231.
6. A. F. Filippov. *Differential equations with discontinuous righthand sides*, volume 18 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1988. Translated from the Russian.
7. N. Guglielmi and E. Hairer. Asymptotic expansions for regularized state-dependent neutral delay equations. *SIAM J. Math. Anal.*, 44(4):2428–2458, 2012.
8. N. Guglielmi and E. Hairer. Classification of hidden dynamics in discontinuous dynamical systems. *SIAM J. Appl. Dyn. Syst.*, 14(3):1454–1477, 2015.
9. N. Guglielmi and E. Hairer. Solutions leaving a codimension-2 sliding. *Nonlinear Dynamics*, 88(2):1427–1439, 2017.
10. N. Guglielmi and E. Hairer. An efficient algorithm for solving piecewise-smooth dynamical systems. *Numer. Algorithms*, 89(3):1311–1334, 2022.
11. M. W. Hirsch. Systems of differential equations that are competitive or cooperative. V. Convergence in 3-dimensional systems. *J. Differential Equations*, 80(1):94–106, 1989.
12. M. R. Jeffrey. Hidden dynamics in models of discontinuity and switching. *Phys. D*, 274–274:34–45, 2014.
13. M. R. Jeffrey. *Hidden dynamics*. Springer, Cham, 2018. The mathematics of switches, decisions and other discontinuous behaviour.
14. J. Llibre, P. R. da Silva, and M. A. Teixeira. Study of singularities in non-smooth dynamical systems via singular perturbation. *SIAM J. Appl. Dyn. Syst.*, 8(1):508–526, 2009.
15. J. Llibre, P. R. da Silva, and M. A. Teixeira. Sliding vector fields for non-smooth dynamical systems having intersecting switching manifolds. *Nonlinearity*, 28(2):493–507, 2015.
16. A. Machina and A. Ponosov. Filippov solutions in the analysis of piecewise linear models describing gene regulatory networks. *Nonlinear Anal.*, 74(3):882–900, 2011.
17. A. Morgan and A. Sommese. A homotopy for solving general polynomial systems that respects m-homogeneous structures. *Appl. Math. Comput.*, 24(2):101–113, 1987.
18. Alessandro Pisano and Elio Usai. Sliding mode control: a survey with applications in math. *Math. Comput. Simulation*, 81(5):954–979, 2011.
19. E. Plahte and S. Kjøglum. Analysis and generic properties of gene regulatory networks with graded response functions. *Phys. D*, 201(1-2):150–176, 2005.

20. E. Plahte, T. Mestl, and S. W. Omholt. A methodological basis for description and analysis of systems with complex switch-like interactions. *J. Math. Biol.*, 36(4):321–348, 1998.
21. I. R. Shafarevich. *Basic algebraic geometry.* Springer Study Edition. Springer-Verlag, Berlin-New York, 1977. Translated from the Russian by K. A. Hirsch, Revised printing of Grundlehren der mathematischen Wissenschaften, Vol. 213, 1974.
22. J. Sotomayor and M. A. Teixeira. Regularization of discontinuous vector fields. In *International Conference on Differential Equations (Lisboa, 1995)*, pages 207–223. World Sci. Publ., River Edge, NJ, 1998.
23. V. I. Utkin. *Sliding modes in control and optimization.* Communications and Control Engineering Series. Springer-Verlag, Berlin, 1992. Translated and revised from the 1981 Russian original.