SIAM J. APPLIED DYNAMICAL SYSTEMS
Vol. xx, pp. x

14 june 2015

CLASSIFICATION OF HIDDEN DYNAMICS IN DISCONTINUOUS DYNAMICAL SYSTEMS*

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Abstract. Ordinary differential equations with discontinuous right-hand side, where the discontinuity of the vector field arises on smooth surfaces of the phase space, are the topic of this work. The main emphasis is the study of solutions close to the intersection of two discontinuity surfaces. There, the so-called hidden dynamics describes the smooth transition from ingoing to outgoing solution directions, which occurs instantaneously in the jump discontinuity of the vector field. This article presents a complete classification of such transitions (assuming the vector fields surrounding the intersection are transversal to it).

Since the hidden dynamics is realized by standard space regularizations, much insight is obtained for them. One can predict, in the case of multiple solutions of the discontinuous problem, which solution (classical or sliding mode) will be approximated after entering the intersection of two discontinuity surfaces. A novel modification of space regularizations is presented that permits to avoid (unphysical) high oscillations and makes a numerical treatment more efficient.

Key words. Discontinuous vector fields, regularization, asymptotic expansions, hidden dynamics, stabilization.

AMS subject classifications. 34A36, 65L04

1. Introduction.

In the n-dimensional phase space we consider the hypersurfaces

$$\Sigma_\alpha = \{y \in \mathbb{R}^n \,;\, \alpha(y) = 0\} \qquad \Sigma_\beta = \{y \in \mathbb{R}^n \,;\, \beta(y) = 0\}, \tag{1.1}$$

where $\alpha(y)$ and $\beta(y)$ are scalar smooth functions. We assume that both hypersurfaces intersect transversally, which means that $\nabla\alpha(y)$ and $\nabla\beta(y)$ are linearly independent for $y \in \Sigma = \Sigma_\alpha \cap \Sigma_\beta$. In a neighbourhood of $y \in \Sigma$, these hypersurfaces divide the phase space into four regions which we denote by $\mathcal{R}^{++} = \{y\,;\, \alpha(y) > 0,\, \beta(y) > 0\}$, $\mathcal{R}^{+-} = \{y\,;\, \alpha(y) > 0,\, \beta(y) < 0\}$, and similarly $\mathcal{R}^{-+}$ and $\mathcal{R}^{--}$. We consider discontinuous ordinary differential equations

$$\dot y = \begin{cases} f^{++}(y) & y \in \mathcal{R}^{++} \\ f^{+-}(y) & y \in \mathcal{R}^{+-} \\ f^{-+}(y) & y \in \mathcal{R}^{-+} \\ f^{--}(y) & y \in \mathcal{R}^{--} \end{cases} \tag{1.2}$$

where we assume that $f^{++}(y)$ is sufficiently differentiable and can be smoothly extended to a neighbourhood of the closure of $\mathcal{R}^{++}$. An analogous property is assumed for the other three vector fields. For $y \in \Sigma_\alpha \cup \Sigma_\beta$ we follow the approach of Filippov [1, 2] and consider solutions, for which $\dot y$ is in the convex hull for the adjacent vector fields. Solutions that evolve

*This work was supported by the Italian INdAM - G.N.C.S., and the Fonds National Suisse, project No. 200020-144313/1.

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in a discontinuity surface (or in an intersection of them) are called 'sliding modes'. Recent monographs on discontinuous differential equations, including interesting applications, are [3] and [4].

The situation, where a solution of (1.2) enters one of the hypersurfaces (e.g., Σ_α) is well understood. Either the vector field of the opposite side permits to continue the solution in a classical way or the vector fields point from both sides towards the hypersurface. In the second case there is a unique convex combination of both vector fields lying in the tangent space of Σ_α, so that the solution continues as a sliding mode.

The situation is more complicated when the solution enters the intersection of two hypersurfaces, because a convex combination of four vector fields has three degrees of freedom, but there are only two conditions for such a convex combination to lie in the tangent space of $\Sigma_\alpha \cap \Sigma_\beta$. This ambiguity can be avoided by considering only special convex combinations with two degrees of freedom. Nevertheless, there can be more than one solution and we are confronted with the following question:

- If the discontinuous differential equation (1.2) has more than one solution (classical or sliding mode) after entering the intersection $\Sigma_\alpha \cap \Sigma_\beta$, which is the correct one?

If we define the *correct solution* as a solution of (1.2) that can be realized as the limit of a regularization (based on bilinear interpolation), then the classification of Sections 6 and 7 gives an answer to this question. Such a definition of 'correct solution' is justified, because the discontinuity of problems arising in applications can often be considered as the limit of a sharp but continuous transition from one vector field to another. It is interesting to note that in the case where classical solutions and sliding modes co-exist, it is more likely that the correct solution is a sliding mode.

This article considers a regularization, where a linear (bilinear) interpolation of the adjacent vector fields is considered in an ε-neighbourhood of the hypersurfaces. Such a regularization has been introduced in [5], and is much used [6, 7, 8]. In the present work we address the following questions:

- If more than one solution (classical or sliding mode) exists, which one will be approximated by the regularization?
- If there exists an outgoing classical solution, does the solution of the regularized differential equation always approximate a classical solution?
- If there is neither a classical solution nor a codimension-1 sliding mode, does there exist a codimension-2 sliding mode? Is it unique?

The hypersurfaces (1.1) and the differential equation (1.2) do not change if we replace $\alpha(y)$ by $\kappa_\alpha \alpha(y)$ with $\kappa_\alpha > 0$ and $\beta(y)$ by $\kappa_\beta \beta(y)$ with $\kappa_\beta > 0$. However, the regularization is modified under such a transformation. We study the following questions:

- If the problem (1.2) has more than one stable solution, is it possible to influence which solution will be approximated by choosing suitably κ_α and κ_β?
- In the case, where a codimension-2 sliding mode of (1.2) is approximated by a highly oscillatory solution of the regularized differential equation, can these oscillations be damped or eliminated by suitably choosing the κ's?

The classification of the present work gives answers to these questions.

This paper is organized as follows. Section 2 recalls the definition of classical solutions and that of sliding modes. The connection between sliding modes and the solution of differential-algebraic equations of index 2 is discussed. The bilinear regularization (called 'blending' in [5]) is introduced in Section 3, and some illustrating numerical experiments are presented. To better understand the solution of the regularized differential equation, the technique of asymptotic expansions for singularly perturbed problems is applied in Section 4. The transient part in the solution is responsible for the transition between the ingoing and outgoing solutions at the intersection of the discontinuity surfaces. This transition is characterised by a 2-dimensional dynamical system, which describes the hidden dynamics (a term coined by [9]). Section 5 discusses initial values for this system, gives an interpretation of its solution when time tends to infinity, and introduces a geometric representation. The main part of the article is a classification of the 'correct' outgoing solution in terms of the four vector fields around $\Sigma = \Sigma_\alpha \cap \Sigma_\beta$. This is presented in Section 6, if the solution enters Σ through a codimension-1 sliding mode, and in Section 7, if the solution enters in a spiraling way. The final Section 8 explains how unphysical oscillations in the solution of the regularized differential equation can be suppressed.

2. Classical solutions and sliding modes. If the initial value $y(0) = y_0$ lies in $\mathcal{R}^{--}$, then the solution of (1.2) is that of $\dot y = f^{--}(y)$ as long as it remains in $\mathcal{R}^{--}$. Since $f^{--}(y)$ is assumed to be sufficiently differentiable in a neighbourhood of $\mathcal{R}^{--}$, this solution is unique and continuously differentiable. We call it a *classical solution*. Suppose that at some time $t_\alpha > 0$ with $y_\alpha = y(t_\alpha)$ it reaches the surface Σ_α in the domain $\beta(y) < 0$. Generically there are then two possibilities: (a) either the values $\alpha'(y_\alpha) f^{--}(y_\alpha)$ and $\alpha'(y_\alpha) f^{+-}(y_\alpha)$ have the same sign and a classical solution (with a jump in its first derivative) continues in the domain $\mathcal{R}^{+-}$, or (b) these values have opposite sign and the initial value $y(t_\alpha) = y_\alpha \in \Sigma_\alpha$ neither admits a classical solution in $\mathcal{R}^{--}$ nor one in $\mathcal{R}^{+-}$. Following Filippov [1, 2], we consider as weak solution of (1.2) a function $y(t)$ that stays in the surface Σ_α and for which $\dot y(t)$ lies in the convex combination of $f^{--}(y(t))$ and $f^{+-}(y(t))$. Such a solution is called a *sliding mode* or, more precisely, a codimension-1 sliding mode. The Filippov approach leads to the system $(-1 \le \lambda_\alpha \le 1)$

$$
\begin{aligned}
\dot y &= \Big(\big(1+\lambda_\alpha\big) f^{+-}(y) + \big(1-\lambda_\alpha\big) f^{--}(y) \Big) \Big/ 2 \\
0 &= \alpha(y),
\end{aligned}
\tag{2.1}
$$

which is a differential-algebraic equation (DAE) of index 2 under the generic assumption $\alpha'(y_\alpha)(f^{+-}(y_\alpha) - f^{--}(y_\alpha))/2 \neq 0$ [10, Section VII.1]. This DAE is solved as follows: differentiating the algebraic constraint yields $0 = \alpha'(y)\dot y$, and inserting the differential equation for $\dot y$, permits to express the Lagrange multiplier λ_α as a function of y. Substituting the resulting expression $\lambda_\alpha(y)$ into (2.1) yields an ordinary differential equation for y, for which classical theory can be applied. The case, where a classical solution enters the surface Σ_β is treated similarly.

Consider next the situation, when a classical solution or a codimension-1 sliding mode enters the intersection $\Sigma = \Sigma_\alpha \cap \Sigma_\beta$. The Filippov approach is ambiguous, because a convex combination of four vector fields contains three parameters, but we have only two conditions for staying in Σ. We restrict ourselves to a vector field (introduced in [5]; called bilinear interpolation in [8] and convex canopy in [11]) which leads to the system $(-1 \le \lambda_\alpha, \lambda_\beta \le 1)$

$$
\begin{aligned}
\dot y &= \Big(\big(1+\lambda_\alpha\big)\big(1+\lambda_\beta\big)\, f^{++}(y) + \big(1+\lambda_\alpha\big)\big(1-\lambda_\beta\big)\, f^{+-}(y) \\
&\qquad + \big(1-\lambda_\alpha\big)\big(1+\lambda_\beta\big)\, f^{-+}(y) + \big(1-\lambda_\alpha\big)\big(1-\lambda_\beta\big)\, f^{--}(y)\Big)\Big/ 4 \\
0 &= \alpha(y) \\
0 &= \beta(y).
\end{aligned} \tag{2.2}
$$

This is again a DAE of index 2, if a certain 2-dimensional matrix is invertible. Consistent initial values satisfy $\alpha(y_2) = \beta(y_2) = 0$ with Lagrange parameters determined by $0 = \alpha'(y)\dot y$ and $0 = \beta'(y)\dot y$. These are two quadratic equations (hyperbolas with horizontal and vertical asymptotes) for λ_α and λ_β. They can have zero, one or two intersections in the unit square $[-1,1] \times [-1,1]$. This shows that there can exist more than one codimension-2 sliding modes. In addition to them, there can co-exist codimension-1 sliding modes and even classical solutions. The study of all possible solutions, their stability, and their relation to space regularizations is one of the aims of the present article.

2.1. Definitions. The terms "classical solution" and "sliding modes" are important concepts in this work. We therefore collect here their precise definition:

classical solution is a continuously differentiable function $y(t)$ defined on an intervall I which, except of the end points, stays in one of the four regions $\mathcal{R}^{++}, \mathcal{R}^{+-}, \mathcal{R}^{-+}, \mathcal{R}^{--}$ and satisfies there the differential equation (1.2);

codimension-1 sliding mode along Σ_α in the region $\beta(y) < 0$ is a continuously differentiable function $y(t)$ defined on an intervall I which, except of the end points, stays in $\Sigma_\alpha \cap \{y\,;\, \beta(y) < 0\}$ and satisfies there the differential-algebraic system (2.1) with a continuous function $\lambda_\alpha(t)$. The sliding modes in the region $\beta(y) > 0$ and along Σ_β are defined analogously;

codimension-2 sliding mode is a continuously differentiable function $y(t)$ defined on an intervall I which stays in the intersection $\Sigma_\alpha \cap \Sigma_\beta$ and satisfies there the differential-algebraic system (2.2) with continuous functions $\lambda_\alpha(t)$ and $\lambda_\beta(t)$.

Since the right-hand sides of (2.1) and (2.2) represent a convex combination of the adjacent vector fields, our definition of sliding modes corresponds to Filippov solutions. Note, however, that in (2.2) we consider only a special 2-parameter family of convex combinations. We thus avoid the continuum of Filippov solutions, and generically we have locally unique sliding mode solutions.

2.2. Some recent literature. In the recent literature there has been a significant research on the analysis and numerical approximation of discontinuous differential equations. A seminal reference is the monograph by Acary and Brogliato [3], which extensively discusses event-driven methods and considers several interesting applications, like non-smooth mechanical systems, frictional contact problems, and electronic circuits (see also [12]). Filippov solutions are considered also in the case of codimension-2 sliding manifolds, in which case a certain convex combination of the vector fields is selected a priori, in order to bypass the non-uniqueness of a Filippov solution. This is indeed the bilinear interpolation we consider in this article (see e.g. [8] and [11]).

A number of interesting articles addresses gene regulatory networks which is a very important class of discontinuous systems, describing the interactions of genes and proteins (see e.g. [13], [14], [12]). Quoting the interesting article by Plahte and Kjøglum [15], in these models the rates of change of gene product concentrations are expressed as a sum of regulatory switches in terms of sums of products of sigmoid functions, turning gene activity on and off. These equations are usually transformed by replacing the sigmoid functions by step functions, i.e. leading to discontinuous differential equations. Interestingly this is indeed the opposite idea to that of regularization which is explored in this paper, which in fact would produce the reverse transition from the discontinuous system to the original one with sigmoid switching functions. In [15] the singularly perturbed problem associated with the use of sigmoidal functions is addressed. A fast (stretched) time and a slow time scale (the standard time) is introduced and a so-called $\mathbb{Z}$-cube is considered, where the fast scale can be studied. This is in analogy to the unit square Q we introduce in the present article. Our analysis of the hidden dynamics, which is the main objective, is new and represents an achievement in this line of research. In [16], the authors present a general methodology to analyse models involving switchings (modeled by sigmoid functions) and give a significant insight into qualitative and quantitative aspects of the dynamics, in particular, for stationary points and their stability. Our research is complementary to this work.

In [17], according to the common replacement of sigmoid functions by step functions, the authors study some properties of piecewise linear discontinuous differential systems describing gene regulatory networks. The global existence and uniqueness of Filippov solutions are studied, and the concept of Filippov stationary point is extensively exploited. The analysis focuses on stationary points so that the obtained results have little overlap with those obtained in the present work where the emphasis is on general discontinuous systems and on general solutions. In [18] the authors compare the Filippov theory of differential inclusions and the singular perturbation techniques, still on gene regulatory networks, and follow the approach presented in [15]. The results presented by the authors, in particular Lemma 5.1, make use of tools which are similar to the one we consider although they do not provide a complete classification of the hidden dynamics that we present here. An extension to gene regulatory systems with delays is studied in [19], where the authors use a technique to remove delays which determines a system of singularly perturbed differential equations.

Finally let us mention the recent publication [7], where the authors model gene regulatory networks by a system of ordinary differential equations for the concentrations where the right-hand side is piecewise linear. They show that the classical Utkin's approach (see e.g. [20]) and the sigmoidal regularization are equivalent in the case of codimension-2 manifolds which are locally attractive (this means that all vector fields are directed towards the sliding manifold).

3. Numerical approaches.

There are several possibilities to solve numerically discontinuous differential equations.

- An interesting survey on direct difference methods for differential inclusions, naturally associated to discontinuous differential equations, is given in [21] and [22]. The authors study convergence, the use of implicit methods, and localisation procedures that permit to get higher-order methods. The application of implicit Runge–Kutta methods to differential inclusions satisfying a one-sided Lipschitz condition is studied in [23].

- It is also possible to localize the time instants when the solution enters one of the surfaces (event detection, see for example [24]). Then one investigates if classical solutions and/or sliding modes exist, one decides which solution should be computed and, for the case that one wants to follow a sliding mode, one switches to a different solver that is able to treat differential-algebraic equations of index 2.
- Another possibility is to regularize the discontinuous system, and to solve the resulting stiff differential equation by a suitable code (for example, by Radau5 of [10]). The present work is devoted to this approach which is appealing, because like direct methods it does not require a switching between codes and it automatically selects a solution if more than one solution co-exists. The use of a non-differentiable interpolation, like that of (3.2), does not make any difficulties for codes with step size strategies. They propose small step sizes close to a point where the solution has a jump in the derivative, and keep the local error everywhere below the user defined tolerance.

3.1. Regularization. The idea is to smooth out the discontinuities of the vector field (1.2). We assume that the functions $f^{++}, f^{+-}, \ldots$ are defined and smooth on an ε-neighbourhood of $\mathcal{R}^{++}, \mathcal{R}^{+-}, \ldots$. Motivated by the choice (2.2) for a convex combination of the four vector fields, we consider the differential equation

$$\begin{aligned} \dot y \;=\; \Big(&\big(1+\pi(u)\big)\big(1+\pi(v)\big)\, f^{++}(y) + \big(1+\pi(u)\big)\big(1-\pi(v)\big)\, f^{+-}(y) \\ +\; &\big(1-\pi(u)\big)\big(1+\pi(v)\big)\, f^{-+}(y) + \big(1-\pi(u)\big)\big(1-\pi(v)\big)\, f^{--}(y)\Big)\Big/ 4 \end{aligned} \tag{3.1}$$

where $u = \alpha(y)/\varepsilon$, $v = \beta(y)/\varepsilon$, and $\varepsilon > 0$ is a small regularization parameter. Here, $\pi(u)$ is a (scalar, continuous) sigmoid function, which takes the value -1 for an argument ≤ -1, the value $+1$ for an argument $\ge +1$ and which interpolates the values monotonically on the interval $[-1, 1]$. Throughout this article we use

$$\pi(u) = \begin{cases} -1 & u \le -1 \\ u & |u| \le 1 \\ 1 & u \ge 1 \end{cases} \tag{3.2}$$

but other interpolation functions can be considered as well.

We note that outside the stripes $|\alpha(y)| \ge \varepsilon$ and $|\beta(y)| \ge \varepsilon$, the system (3.1) coincides with (1.2). Within these stripes the vector field of (3.1) is a convex combination of the two (or four in the intersection of the two stripes) vector fields of the adjacent regions of (1.2). Due to the factor ε in the denominator of u and v the differential equation is of singular perturbation type. For a numerical treatment one has to apply codes that are suitable for stiff differential equations.

3.2. Numerical experiments. We present three numerical experiments to study the effect of regularization on a solution of (1.2) that has entered the intersection $\Sigma = \Sigma_\alpha \cap \Sigma_\beta$. The first example illustrates that the solution of (3.1) can follow a codimension-2 sliding mode even if (1.2) possesses a classical solution leaving Σ. The other two examples show the effect of replacing $\alpha(y)$ by $\kappa_\alpha \alpha(y)$ with $\kappa_\alpha \ge 1$ and $\beta(y)$ by $\kappa_\beta \beta(y)$ with $\kappa_\beta \ge 1$, $\min(\kappa_\alpha, \kappa_\beta) = 1$. Note that such a transformation does not change the surfaces Σ_α and Σ_β, and has therefore

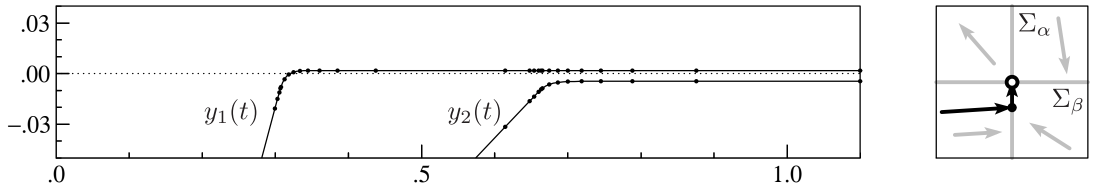

Figure 1. *Example 1 of Section 3.2. The small picture shows the vector fields (grey) and the solution in the phase space. The big picture shows the solution of the regularized differential equation as a function of time.*

no influence on the solutions of (1.2). Since we are interested in the behaviour of solutions close to the codimension-2 manifold Σ, it is not restrictive to assume the four vector fields in (1.2) to be constant.

Example 1: Codimension-2 sliding mode is stronger than a classical solution. In the 2-dimensional phase space we consider the surfaces (1.1) with $\alpha(y) = y_1$ and $\beta(y) = y_2$, and the differential equation (1.2) with constant vector fields

$$f^{++} = \begin{pmatrix} 0.4 \\ -2.6 \end{pmatrix}, \quad f^{+-} = \begin{pmatrix} -1.1 \\ 0.7 \end{pmatrix}, \quad f^{-+} = \begin{pmatrix} -0.7 \\ 0.8 \end{pmatrix}, \quad f^{--} = \begin{pmatrix} 1.6 \\ 0.1 \end{pmatrix}. \tag{3.3}$$

These vector fields are indicated as grey arrows in the small picture of Figure 1. With initial values $y_1(0) = -0.5$, $y_2(0) = -0.2$ the solution of (1.2) starts in $\mathcal{R}^{--}$ and enters the surface Σ_α at $y_1(t_1) = 0$, $y_2(t_1) = -0.16875$ with $t_1 = 0.0625$. The solution continues as a sliding mode along Σ_α until it reaches the manifold $\Sigma = \Sigma_\alpha \cap \Sigma_\beta$. Inspecting the vector fields one sees that there is an outgoing classical solution into the region $\mathcal{R}^{-+}$. There is no codimension-1 sliding mode, but there are two codimension-2 sliding modes: one for $\lambda_\alpha \approx 0.1692$ and $\lambda_\beta \approx -0.4497$, the other for $\lambda_\alpha \approx -0.4579$ and $\lambda_\beta \approx 0.3672$.

For the solution of the regularized differential equation (3.1) one could expect that the classical solution is stronger than the sliding modes, so that the solution of (3.1) approaches the classical solution. That this is not the case is shown in the left picture of Figure 1 for a computation with $\varepsilon = 0.01$. Due to the regularization the solutions are not exactly on the surfaces Σ_α and Σ_β but $\mathcal{O}(\varepsilon)$ close to them. This is explained by the asymptotic expansion analysis of Section 4. A theoretical explanation of the fact that the codimension-2 sliding mode and not the classical solution is approximated by the regularization, is given in Section 6.4 below.

Example 2: Stabilization of codimension-2 sliding modes. With the same surfaces as in the previous example we consider the differential equation (1.2) with (see Figure 2)

$$f^{++} = \begin{pmatrix} 1 \\ -1.5 \end{pmatrix}, \quad f^{+-} = \begin{pmatrix} -3 \\ -10.5 \end{pmatrix}, \quad f^{-+} = \begin{pmatrix} 1 \\ 4.5 \end{pmatrix}, \quad f^{--} = \begin{pmatrix} 1 \\ 7.5 \end{pmatrix}. \tag{3.4}$$

With initial values $y_1(0) = -0.1$, $y_2(0) = -2$ the solution of (1.2) starts in $\mathcal{R}^{--}$, enters the surface Σ_α at $y_1(0.1) = 0$, $y_2(0.1) = -1.25$, and continues as a sliding mode along Σ_α until it reaches $\Sigma = \Sigma_\alpha \cap \Sigma_\beta$. Since there is no classical solution and no codimension-1 sliding mode that starts in Σ, the solution stays there as a codimension-2 sliding mode.

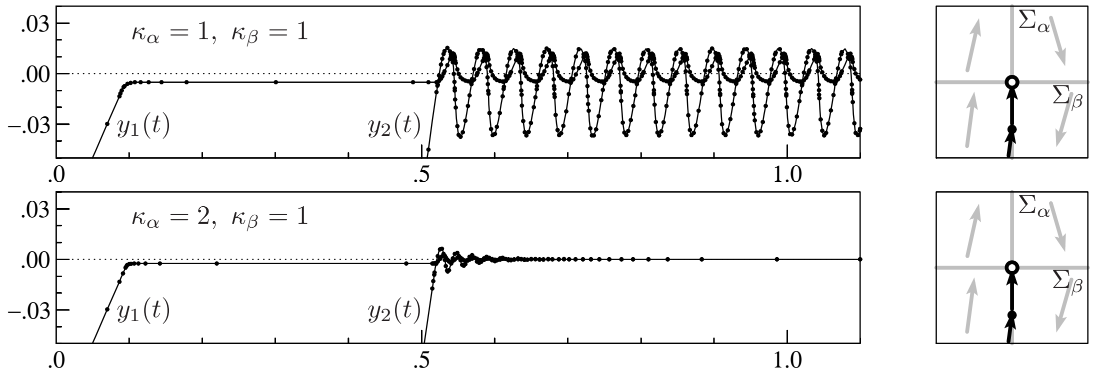

Figure 2. *Example 2 of Section 3.2. The small pictures show the vector fields (grey) and the solution in the phase space. The big pictures show the solution of the regularized differential equation as a function of time.*

For the regularized differential equation (3.1) we again choose $\varepsilon = 0.01$. We study the influence of scaling for the functions $\alpha(y)$ and $\beta(y)$ with κ_α and κ_β, respectively. The upper picture of Figure 2 shows the solution with the standard choice $\kappa_\alpha = \kappa_\beta = 1$. Unphysical oscillations of amplitude $\mathcal{O}(\varepsilon)$ and of frequency $\mathcal{O}(\varepsilon^{-1})$ can be observed. These oscillations force a numerical method to take small steps of size $\mathcal{O}(\varepsilon)$ (natural output points by the code Radau5 are indicated by small circles) and makes the integration inefficient. For the choice $\kappa_\beta = 1$ and $\kappa_\alpha > 1.5$ (lower picture of Figure 2) one has only a few oscillations that are rapidly damped. The numerical integrator can soon take large step sizes.

Example 3: Switching between a classical solution and a codimension-2 sliding mode. With the surfaces of Example 1 we consider the differential equation (1.2) with

$$f^{++} = \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}, \quad f^{+-} = \begin{pmatrix} -3 \\ -2.5 \end{pmatrix}, \quad f^{-+} = \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}, \quad f^{--} = \begin{pmatrix} 1 \\ 1.5 \end{pmatrix} \tag{3.5}$$

(see the grey arrows in the small pictures of Figure 3). The solution starts at $y_1(0) = -0.5$, $y_2(0) = -1$, enters the surface Σ_α at $y_1(0.5) = 0$, $y_2(0.5) = -0.25$, and finally enters $\Sigma = \Sigma_\alpha \cap \Sigma_\beta$. From the picture we can see that there exists a classical solution in the region $\mathcal{R}^{++}$, and an investigation of the DAE (2.2) shows that there is also a codimension-2 sliding mode.

Which solution will be selected by the regularization? As before we put $\varepsilon = 0.01$ and consider scaled functions $\alpha(y)$ and $\beta(y)$. For the standard choice $\kappa_\alpha = \kappa_\beta = 1$ (upper picture of Figure 3) the solution follows the codimension-2 sliding mode. Similar as in Example 1, it comes as a surprise that in the presence of a classical solution, a sliding mode is selected by the regularization. Numerical experiments with $\kappa_\alpha = 1$ and different values of κ_β have shown that there exists $\kappa^* \approx 1.703$ such that for $\kappa_\beta < \kappa^*$ the solution of (3.1) follows the codimension-2 sliding mode, whereas for $\kappa_\beta > \kappa^*$ it follows the classical solution in $\mathcal{R}^{++}$ (compare both pictures of Figure 3).

The theoretical analysis of Section 6 permits us to explain the behaviour of all three examples (see Section 6.4).

4. Asymptotic expansion of the solution of the regularized problem. To understand the behaviour of the solution of the regularized differential equation (3.1), the study of asymptotic

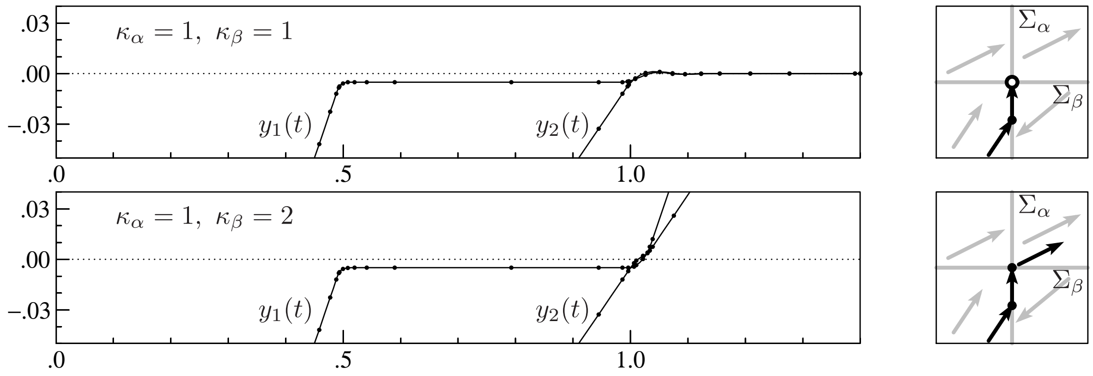

Figure 3. *Example 3 of Section 3.2. The small pictures show the vector fields (grey) and the solution in the phase space. The big pictures show the solution of the regularized differential equation as a function of time.*

expansions gives much insight. We present the main ideas without going into details. The aim of this section is to motivate the differential equations describing the hidden dynamics, and not to present rigorous error estimates. For the codimension-1 case the techniques are closely related to that of [25], where state-dependent neutral delay equations are considered.

4.1. Entering a codimension-1 surface. We first consider the situation where the solution of (1.2) enters transversally the hypersurface Σ_α sufficiently far away from Σ_β. Without loss of generality we assume that it enters from $\alpha(y) < 0$ at a point $y_\alpha \in \Sigma_\alpha$ for which $\alpha'(y_\alpha) f^-(y_\alpha) > 0$. In this situation the regularized differential equation takes the form

$$\dot y = \Big(\big(1+\pi(u)\big) f^+(y) + \big(1-\pi(u)\big) f^-(y) \Big) \Big/ 2, \qquad u = \alpha(y)/\varepsilon, \tag{4.1}$$

where we suppress the index corresponding to β, so that $f^+(y)$ is the vector field in the region $\alpha(y) > 0$ and $f^-(y)$ in $\alpha(y) < 0$. This equation coincides with (3.1) as long as $|\beta(y)| \ge \varepsilon$. In the stripe $|\alpha(y)| \le \varepsilon$ we are concerned with a singularly perturbed differential equation. We denote by $t_0(\varepsilon)$ the time instant when the solution enters from $\alpha(y) < -\varepsilon$ this stripe, i.e.,

$$\alpha\big(y(t_0(\varepsilon))\big) = -\varepsilon. \tag{4.2}$$

Since $\alpha'(y_\alpha) f^-(y_\alpha) > 0$, the Implicit Function Theorem guarantees that $t_0(\varepsilon)$ can be expanded into a series in powers of ε. We therefore also have an expansion $y\big(t_0(\varepsilon)\big) = y_\alpha + \varepsilon a_1 + \mathcal{O}(\varepsilon^2)$.

To study the behaviour of the solution of (4.1) close to Σ_α, we consider in addition to the *slow* time t also a *fast* time $\tau = t/\varepsilon$, and we make the ansatz

$$y\big(t_0(\varepsilon) + t\big) = y_0(t) + \varepsilon\big(y_1(t) + \eta_0(t/\varepsilon)\big) + \mathcal{O}(\varepsilon^2), \tag{4.3}$$

where $y_0(t), y_1(t)$ describe the smooth part of the solution and the function $\eta_0(\tau)$ captures the transient part. For $t = 0$, this expansion has to match the expansion for $y\big(t_0(\varepsilon)\big)$, which means that $y_0(0) = y_\alpha$ and $y_1(0) + \eta_0(0) = a_1$. In the differential equation (4.1) the division by ε occurs only in the definition of u. We expand this expression around the ε-independent term and obtain (with $\tau = t/\varepsilon$)

$$\frac{1}{\varepsilon}\,\alpha\big(y(t_0(\varepsilon)+t)\big) = \frac{1}{\varepsilon}\,\alpha\big(y_0(t)\big) + \alpha'\big(y_0(t)\big)\big(y_1(t) + \eta_0(\tau)\big) + \mathcal{O}(\varepsilon). \tag{4.4}$$

To avoid the singularity for $\varepsilon \to 0$, we have to determine a function $y_0(t)$ that satisfies

$$\alpha\big(y_0(t)\big) = 0. \tag{4.5}$$

We aim in getting a small defect for the solution approximation (4.3). For this we insert (4.3) and (4.4) into (4.1), we replace all appearances of t with $\varepsilon\tau$, and then we put $\varepsilon = 0$. This yields the differential equation

$$\eta_0'(\tau) = \Big(\big(1+\pi\big(u(\tau)\big)\big)f^+(y_\alpha) + \big(1-\pi\big(u(\tau)\big)\big)f^-(y_\alpha)\Big)\Big/2 \; - \; \dot y_0(0) \tag{4.6}$$

with the scalar function $u(\tau) = \alpha'(y_\alpha)\big(y_1(0) + \eta_0(\tau)\big)$. To obtain a relation for the function $u(\tau)$, we multiply (4.6) with $\alpha'(y_\alpha)$ and notice that, by differentiation of (4.5), we have $\alpha'(y_\alpha)\dot y_0(0) = 0$. This yields the differential equation for $u(\tau)$

$$u' = \Big(\big(1+\pi(u)\big)f_\alpha^+ + \big(1-\pi(u)\big)f_\alpha^-\Big)\Big/2, \tag{4.7}$$

where $f_\alpha^+ = \alpha'(y_\alpha)f^+(y_\alpha)$ and $f_\alpha^- = \alpha'(y_\alpha)f^-(y_\alpha)$. The initial value $u(0) = -1$ is obtained from (4.2), because $\alpha\big(y(t_0(\varepsilon))\big) = \alpha(y_\alpha + \varepsilon a_1 + \ldots) = \varepsilon\alpha'(y_\alpha)a_1 + \mathcal{O}(\varepsilon^2) = \varepsilon u(0) + \mathcal{O}(\varepsilon^2)$. For this initial value we have $u'(0) = f_\alpha^- > 0$, so that the solution is monotonically increasing. It is either bounded and converges to a stationary point, or it tends to infinity. We thus distinguish between the following two (generic) situations:

Codimension-1 sliding mode. Suppose that the solution of (4.7) converges to a stationary point $u^* \in (-1,1)$. In this case we consider the differential equation

$$\dot y_0(t) = \Big(\big(1+\lambda(t)\big)f^+\big(y_0(t)\big) + \big(1-\lambda(t)\big)f^-\big(y_0(t)\big)\Big)\Big/2 \tag{4.8}$$

together with $\alpha\big(y_0(t)\big) = 0$ from (4.5). This is a DAE, for which $y_0(0) = y_\alpha$ and $\lambda(0) = u^*$ are consistent initial values, i.e, $\alpha(y_\alpha) = 0$ and $\alpha'(y_\alpha)\big((1+u^*)f^+(y_\alpha) + (1-u^*)f^-(y_\alpha)\big)\big/2 = 0$. We therefore have a unique function $y_0(t)$ on an ε-independent non-empty interval which, as discussed in Section 2, is a codimension-1 sliding mode of (1.2).

The function $\eta_0(\tau)$ is obtained by simple integration from (4.6), where the integration constant is chosen such that $\eta_0(\tau)$ converges (exponentially fast) to zero for $\tau \to \infty$. Consequently, the transient term in (4.3) is rapidly damped out and visible only on a small interval of length $\mathcal{O}(\varepsilon)$. Beyond this interval the solution (4.3) of the regularized equation (3.1) approximates (for $\varepsilon \to 0$) a codimension-1 sliding mode of (1.2).

Classical solution. Suppose that the solution of (4.7) satisfies $u(\tau) \to \infty$ for $\tau \to \infty$. In this case it is not possible to define $y_0(t)$ as a codimension-1 sliding mode, because there is no consistent initial value $\lambda(0) \in (-1,1)$ for (4.8). We thus define $y_0(t) \equiv y_\alpha$, so that (4.5) is satisfied, and we arbitrarily put $y_1(t) \equiv 0$. The monotonic function $u(\tau)$ will reach the value $+1$ at some (fast) time τ_α, and then it is linear with slope $f_\alpha^+ > 0$. For $u(\tau) \geq 1$ also the vector field of (4.6) is constant, so that $\eta_0(\tau) = \eta_0(\tau_\alpha) + (\tau - \tau_\alpha)f^+(y_\alpha)$ for $\tau \geq \tau_\alpha$. Inserted into (4.3) and using the known ε-expansion for $t = 0$ we obtain

$$y\big(t_0(\varepsilon) + t\big) = y_\alpha + \varepsilon a_1 + t f^+(y_\alpha) + \mathcal{O}(\varepsilon^2)$$

on intervals of length $\mathcal{O}(\varepsilon)$. On such intervals, this solution of (3.1) approximates the classical solution defined by $\dot{y} = f^{+}(y)$ passing through $y(0) = y_\alpha$.

We insist that it is not our intention to give rigorous error estimates for the asymptotic expansions. This can be done by estimating the defect of the approximation, when inserted into (3.1), and by using Gronwall-type estimates (see for example [26] and [27]). Our aim is to highlight the importance of the differential equation (4.7). In the situation, where the solution enters the intersection of two hypersurfaces, this equation is the so-called 'hidden dynamics' which is the central theme of the present work.

4.2. Entering a codimension-2 manifold. An extension of the approach of asymptotic expansions to the situation, where the solution of (1.2) enters a codimension-2 manifold, is nearly straight-forward. We denote the entering point by y_0, and we let $t_0(\varepsilon)$ be the time instant, when the solution enters the region $\{y\,;\,|\alpha(y)| \le \varepsilon,\, |\beta(y)| \le \varepsilon\}$. Under generic assumptions, $t_0(\varepsilon)$ and also $y\big(t_0(\varepsilon)\big) = y_0 + \varepsilon a_1 + \mathcal{O}(\varepsilon^2)$ can be expanded into powers of ε. Following the approach of Section 4.1 we make the ansatz (4.3) and compute the critical expressions $\alpha(y)/\varepsilon$ and $\beta(y)/\varepsilon$ as in (4.4). To avoid the singularity for $\varepsilon \to 0$ the function $y_0(t)$ has to satisfy

$$\alpha\big(y_0(t)\big) = 0, \qquad \beta\big(y_0(t)\big) = 0. \tag{4.9}$$

As before we insert (4.3) into (3.1), we replace all appearances of t with $\varepsilon\tau$, and then we put $\varepsilon = 0$. This yields the differential equation

$$\begin{aligned} \eta_0'(\tau) \;=\; & \Big(\big(1+\pi\big(u(\tau)\big)\big)\big(1+\pi\big(v(\tau)\big)\big)\, f^{++}(y_0) + \big(1+\pi\big(u(\tau)\big)\big)\big(1-\pi\big(v(\tau)\big)\big)\, f^{+-}(y_0) \\ & + \big(1-\pi\big(u(\tau)\big)\big)\big(1+\pi\big(v(\tau)\big)\big)\, f^{-+}(y_0) + \big(1-\pi\big(u(\tau)\big)\big)\big(1-\pi\big(v(\tau)\big)\big)\, f^{--}(y_0)\Big)\Big/4 \\ & - \dot{y}_0(0) \end{aligned}$$

with the scalar functions $u(\tau) = \alpha'(y_0)\big(y_1(0) + \eta_0(\tau)\big)$ and $v(\tau) = \beta'(y_0)\big(y_1(0) + \eta_0(\tau)\big)$ which, when multiplied with ε, represent the distance of the solution to the manifolds Σ_α and Σ_β, respectively. To obtain relations for the functions $u(\tau)$ and $v(\tau)$, we multiply the differential equation for $\eta_0(\tau)$ once with $\alpha'(y_0)$ and a second time with $\beta'(y_0)$, and notice that by differentiation of (4.9) we have $\alpha'(y_0)\dot{y}_0(0) = 0$ as well as $\beta'(y_0)\dot{y}_0(0) = 0$. This yields the 2-dimensional dynamical system

$$\begin{aligned} u' \;=\; & \Big(\big(1+\pi(u)\big)\big(1+\pi(v)\big)\, f_\alpha^{++} + \big(1+\pi(u)\big)\big(1-\pi(v)\big)\, f_\alpha^{+-} \\ & + \big(1-\pi(u)\big)\big(1+\pi(v)\big)\, f_\alpha^{-+} + \big(1-\pi(u)\big)\big(1-\pi(v)\big)\, f_\alpha^{--}\Big)\Big/4 \\ v' \;=\; & \Big(\big(1+\pi(u)\big)\big(1+\pi(v)\big)\, f_\beta^{++} + \big(1+\pi(u)\big)\big(1-\pi(v)\big)\, f_\beta^{+-} \\ & + \big(1-\pi(u)\big)\big(1+\pi(v)\big)\, f_\beta^{-+} + \big(1-\pi(u)\big)\big(1-\pi(v)\big)\, f_\beta^{--}\Big)\Big/4, \end{aligned} \tag{4.10}$$

where we use the abbreviation $f_\alpha^{++} = \alpha'(y_0)f^{++}(y_0), \ldots, f_\beta^{--} = \beta'(y_0)f^{--}(y_0)$. Initial values are determined by the incoming solution, and the behaviour for $\tau \to \infty$ tells us which solution (classical or sliding mode) is approximated by the regularization (3.1). This will be explained in detail in Section 5. This system can already be found in the publication [5]. The name

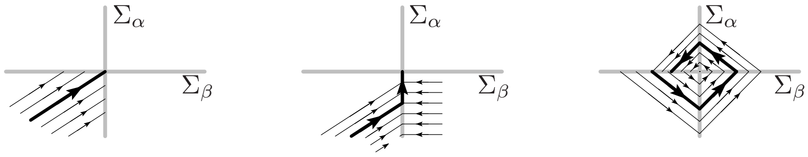

Figure 4. *Different possibilities of entering the codimension-2 manifold* $\Sigma = \Sigma_\alpha \cap \Sigma_\beta$.

'hidden dynamics' has recently been proposed in [9]. In fact, this dynamical system is not visible in the original equation (1.2), but it is determined by the choice of regularization. It informs us, how the incoming derivative vector $\dot y(0-)$ changes into the outgoing vector $\dot y(0+)$, when the solution crosses or enters a discontinuity manifold.

5. Hidden dynamics.

The solution of (1.2) is continuous, but its derivative has jump discontinuities when crossing or entering one of the manifolds Σ_α and Σ_β. For the regularized problem this discontinuity is smoothed out by a rapid transition from one state to the other. In the limit, where the regularization parameter tends to zero, this transition is described by the hidden dynamics (4.10), which takes place instantaneously.

In the following, we abbreviate the right-hand side of (4.10) by $g(u,v) = \big(g_\alpha(u,v), g_\beta(u,v)\big)$, so that the dynamical system becomes

$$\begin{aligned} \dot u &= g_\alpha(u,v) \\ \dot v &= g_\beta(u,v). \end{aligned} \tag{5.1}$$

Note that by definition of the function $\pi(u)$ we have $g_\alpha(u,v) = g_\alpha(-1,v)$ for $u \le -1$, $g_\alpha(u,v) = g_\alpha(1,v)$ for $u \ge 1$, $g_\alpha(u,v) = g_\alpha(u,-1)$ for $v \le -1$, $g_\alpha(u,v) = g_\alpha(u,1)$ for $v \ge 1$, and similarly for g_β.

5.1. Initial values determined by incoming solution.

The initial values for the dynamical system (5.1) are obtained from the solution that enters the codimension-2 manifold Σ. We distinguish between three different situations (see Figure 4):

Classical solution enters Σ. If it enters from the region $\mathcal{R}^{--}$, then we have $u(0) = -1$ and $v(0) = -1$ (left picture of Figure 4); if it enters from $\mathcal{R}^{-+}$ we have $u(0) = -1$ and $v(0) = +1$, etc. The initial values are on the corners of the square $Q = [-1,1] \times [-1,1]$, which we call unit square.

Codimension-1 sliding mode enters Σ. If it enters along Σ_α from the region where $\beta(y) < 0$, then there exists $u_0 \in (-1,1)$ such that $g_\alpha(u_0,-1) = 0$ and $g_\beta(u_0,-1) > 0$, and the initial values are $u(0) = u_0$ and $v(0) = -1$; if it enters from the region $\beta(y) > 0$, all values -1 have to be replaced by $+1$. If the sliding is along Σ_β, the roles of u and v have to be interchanged.

Approaching Σ *by spiraling around it.* If there is neither a classical solution nor a codimension-1 sliding mode that enters Σ, the solution can spiral around Σ and finally enter it at finite time. In this situation all values on the border of the unit square Q are potential initial values, and none of them is highlighted.

We don't elaborate the first situation, because it is not generic for a given initial condition and can be studied similar to the second one. The other two situations will be studied in detail in the following sections.

5.2. Different outgoing solutions. The behaviour, for $\tau \to \infty$, of the solution of (5.1) with initial values from Section 5.1 determine the kind of solution of (1.2) which will be followed by the regularization (3.1) after the codimension-2 manifold Σ has been entered. We distinguish again between three different situations:

Classical solution. If $u(\tau) \to +\infty$ and $v(\tau) \to +\infty$, which requires that $g_\alpha(1,1) > 0$ and $g_\beta(1,1) > 0$, a classical solution in the region $\mathcal{R}^{++}$ will be followed; if $u(\tau) \to +\infty$ and $v(\tau) \to -\infty$ there will be a classical solution in $\mathcal{R}^{+-}$, etc.

Codimension-1 sliding mode. If $u(\tau) \to u^* \in (-1,1)$ and $v(\tau) \to +\infty$, which requires that $g_\alpha(u^*,1) = 0$ and $g_\beta(u^*,1) > 0$, a codimension-1 sliding mode along Σ_α into the region $\beta(y) > 0$ will be followed. If $u(\tau) \to u^*$ but $v(\tau) \to -\infty$, the codimension-1 sliding will be along Σ_α into the region $\beta(y) < 0$. For sliding modes along Σ_β, the roles of u and v have to be interchanged.

Codimension-2 sliding mode. If $u(\tau) \to u^* \in (-1,1)$ and $v(\tau) \to v^* \in (-1,1)$, which requires that $g_\alpha(u^*,v^*) = g_\beta(u^*,v^*) = 0$, the solution of (1.2) will follow a codimension-2 sliding mode. It can also happen that the stationary point (u^*,v^*) is unstable, and that the hidden dynamics (5.1) has a limit cycle around it. This situation translates into high frequency oscillations of small amplitude in the approximation of a codimension-2 sliding mode of (1.2).

To investigate the asymptotic stability of the stationary solution of (5.1) we consider the Jacobi matrix

$$G(u,v) = \begin{pmatrix} \partial_u g_\alpha & \partial_v g_\alpha \\ \partial_u g_\beta & \partial_v g_\beta \end{pmatrix} (u,v). \tag{5.2}$$

Asymptotic stability of a stationary point is equivalent to $\det G > 0$ and $\operatorname{trace} G < 0$.

5.3. Geometric study. The differential equation (5.1) is a 2-dimensional dynamical system and much insight can be obtained by geometric considerations. The nontrivial dynamics takes place on the unit square $Q = [-1,1] \times [-1,1]$. We collect here the most important properties. They are illustrated in the figures of Section 6.

- The vectors $g(-1,-1), g(1,-1), g(1,1), g(-1,1)$ are indicated by arrows attached to the corners of the unit square Q. They take the values $(f_\alpha^{--}, f_\beta^{--})$, $(f_\alpha^{+-}, f_\beta^{+-})$, etc., corresponding to the four vector fields surrounding Σ.
- Since we work with bilinear interpolation, the curve defined by $g_\alpha(u,v) = 0$ and restricted to Q represents a hyperbola with vertical and horizontal asymptotes. It is plotted in black with an arrow indicating that the region $g_\alpha(u,v) > 0$ lies to the left of the curve. On this curve the vector $g(u,v)$ is vertical.
- The curve defined by $g_\beta(u,v) = 0$ is also a hyperbola (plotted in grey) and the arrow has the meaning as before. On this curve the vector $g(u,v)$ is horizontal.
- Stationary points of (5.1) are the intersections of the hyperbolas $g_\alpha(u,v) = 0$ and $g_\beta(u,v) = 0$. There can be two, one, or zero intersections in the unit square.
- The initial value corresponding to an incoming codimension-1 sliding mode is indicated by a thick arrow pointing to an element of the border of Q. If it comes through sliding along Σ_α from $\beta(y) < 0$ it is characterised by $g_\alpha(u_0,-1) = 0$.

Lemma 5.1. *If, at a stationary point (u^*,v^*) of (5.1), the arrow for $g_\beta(u,v) = 0$ points into the region where $g_\alpha(u,v) > 0$, then we have* $\det G(u^*,v^*) > 0$.

Proof. The row vectors of the matrix G are the gradients of the functions g_α and g_β. They are orthogonal to the level curves $g_\alpha(u,v)=0$ and $g_\beta(u,v)=0$, respectively, and point into the region where the function is positive. If the arrow for $g_\beta(u,v)=0$ points into the region where $g_\alpha(u,v)>0$, then the direction of the gradient $\nabla g_\beta(u^*,v^*)$ is obtained from that of $\nabla g_\alpha(u^*,v^*)$ by a rotation of an angle between 0 and 180°. This proves the statement. ■

The question that we shall address in the following sections is to find simple criteria that permit to decide in advance which solution will be followed. The surprising fact is that, even in the case where classical solutions exist it is more likely for the solution of the regularized equation (3.1) to follow a codimension-2 sliding mode.

6. Analysis of hidden dynamics: solution enters as a codimension-1 sliding mode.

Probably the most important situation is, where the solution enters the codimension-2 manifold Σ through a codimension-1 sliding. Without loss of generality we can assume that the sliding is along Σ_α from the region $\beta(y)<0$, and that

$$\begin{aligned}&g_\alpha(-1,-1)>0,\qquad g_\beta(-1,-1)>0,\qquad g_\alpha(1,-1)<0,\\&\text{there exists } u_0\in(-1,1) \text{ such that } g_\alpha(u_0,-1)=0 \text{ and } g_\beta(u_0,-1)>0\end{aligned}\tag{6.1}$$

(this corresponds to the picture in the middle of Figure 4). In the following we treat the situations $\partial_v g_\alpha(u_0,-1)<0$ (left turning) and $\partial_v g_\alpha(u_0,-1)>0$ (right turning) separately. We do not consider the non-generic situation, where $\partial_v g_\alpha(u_0,-1)$ vanishes.

6.1. Left turning situation.

In this section we assume $\partial_v g_\alpha(u_0,-1)<0$. This means that the branch of the hyperbola $g_\alpha(u,v)=0$ starting at $(u_0,-1)$ lies in the region $u\le u_0$ (hence "left turning"). We shall see that in this case the solution either stays in Σ_α or continues in the region $\alpha(y)<0$. In the following we use the notation $\gamma_\beta=\{(u,v)\in Q\,;\,g_\beta(u,v)=0\}$ for the hyperbola defined by g_β restricted to the unit square Q, and

$$\gamma_\alpha^-=\{(u,v)\in Q\,;\,g_\alpha(u,v)=0,\ u\le u_0\}$$

for the branch of the hyperbola $g_\alpha(u,v)=0$ starting at $(u_0,-1)$.

Theorem 6.1. *In addition to assumption (6.1) let $\partial_v g_\alpha(u_0,-1)<0$.*

- (a) *Assume that γ_α^- intersects transversally γ_β inside the unit square, and denote (u^*,v^*) the intersection point closest to $(u_0,-1)$.*
  - (a1) *If $g_\beta(1,u^*)<0$, then (u^*,v^*) is an asymptotically stable equilibrium of (5.1), and we have $u(\tau)\to u^*$ and $v(\tau)\to v^*$ for $\tau\to\infty$.*
  - (a2) *If $g_\beta(1,u^*)>0$, then (u^*,v^*) is also asymptotically stable, but the solution either converges to (u^*,v^*) or it approaches a limit cycle.*
- (b) *Assume that $\gamma_\alpha^-\cap\gamma_\beta=\emptyset$.*
  - (b1) *If $g_\alpha(-1,1)>0$, then there exists $u^*\in(-1,1)$ such that $g_\alpha(u^*,1)=0$ and the solution of (5.1) is such that $u(\tau)\to u^*$ and $v(\tau)\to+\infty$ for $\tau\to\infty$.*
  - (b2) *If $g_\alpha(-1,1)<0$ and $g_\beta(-1,1)<0$, then there exists $v^*\in(-1,1)$ such that $g_\beta(-1,v^*)=0$ and the solution of (5.1) is such that $u(\tau)\to-\infty$ and $v(\tau)\to v^*$ for $\tau\to\infty$.*
- (c) *If $\gamma_\alpha^-\cap\gamma_\beta=\emptyset$, $g_\alpha(-1,1)<0$, and $g_\beta(-1,1)>0$, then the solution of (5.1) is such that $u(\tau)\to-\infty$ and $v(\tau)\to+\infty$ for $\tau\to\infty$.*

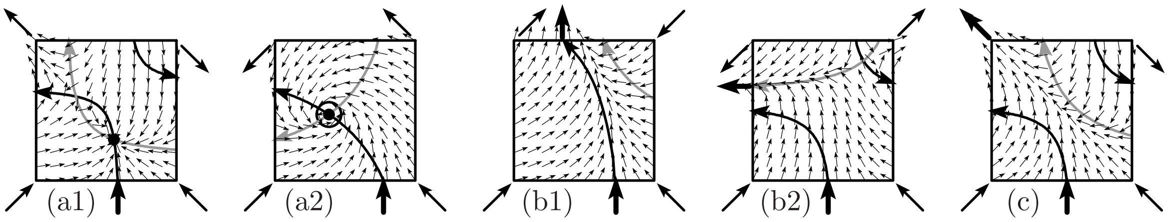

Figure 5. *Vector fields for the situations of Theorem 6.1; the hyperbola $g_\alpha(u,v)=0$ (black curve) is oriented such that the region $g_\alpha(u,v)>0$ is to the left; the gray curve corresponds to the hyperbola $g_\beta(u,v)=0$; the vector field (5.1) at the corners of the unit square is indicated by arrows.*

Remark. This theorem covers all possible generic situations. In the case (a), the solution of the system (3.1) follows the codimension-2 sliding mode corresponding to (u^*,v^*). In the case (b1), it follows the codimension-1 sliding mode along Σ_α into the region $\beta(y)>0$, whereas in the case (b2) it follows the codimension-1 sliding mode along Σ_β into the region $\alpha(y)<0$. In the case (c), it follows the classical solution into the region $\mathcal{R}^{-+}$.

Proof. The assumption $\partial_v g_\alpha(u_0,-1)<0$ implies that the branch γ_α^- of the hyperbola γ_α, which passes through $(u_0,-1)$, turns left (see Figure 5). Since $g_\alpha(u_0,v)\le 0$ for all v (also outside the unit square), the solution of (5.1) with initial value $(u_0,-1)$ remains in the half plane $u\le u_0$. Moreover, we have $g_\beta(u,-1)>0$ for $u\le u_0$, so that the solution also stays in the half-plane $v\ge -1$.

(a) Let (u^*,v^*) be the intersection of γ_α^- with γ_β which is closest to $(u_0,-1)$. Since $g_\alpha(u,v)>0$ to the left of (u^*,v^*) and $g_\beta(u,v)>0$ below it, the diagonal elements of $G(u^*,v^*)$ are strictly negative. Moreover, $\det G(u^*,v^*)>0$ by Lemma 5.1, so that this equilibrium is asymptotically stable.

(a1) If $g_\beta(1,v^*)<0$, an (oriented) branch of γ_β starts on the border of Q in the region $u>u_0$ and $v\le v^*$ and passes through the equilibrium (u^*,v^*) (first picture of Figure 5). It can have one or two intersection points with γ_α^-. The solution of (5.1), starting at $(u_0,-1)$, has to stay to the right of γ_α and below γ_β, so that it converges straight ahead (without spiraling) to the equilibrium (u^*,v^*).

(a2) If $g_\beta(1,v^*)>0$, an (oriented) branch of γ_β starts on the border of Q in the region $u>u^*$ and $v>v^*$, and this branch has only one intersection point with γ_α (second picture of Figure 5). Since the solution with initial value $(u_0,-1)$ remains bounded, which follows by an inspection of the vector field, the Theorem of Poincaré–Bendixson (see, e.g., [28]) implies that the solution either converges to the equilibrium or to a limit cycle around it.

(b1) The assumptions $g_\alpha(-1,-1)>0$ and $g_\alpha(-1,1)>0$ imply that the branch γ_α^- cannot intersect the left side of the unit square Q, and therefore it intersects the upper side at some point $(u^*,1)$, where $-1<u^*<u_0$ (third picture of Figure 5). Since $\gamma_\alpha^-\cap\gamma_\beta=\emptyset$, the solution of (5.1) cannot cross the branch γ_α^-, and therefore it increases monotonically from $(u_0,-1)$ to some point $(u_1,1)$, where $u^*<u_1<u_0$ is such that $g_\beta(u_1,1)>0$. From there, the solution remains outside Q and tends to infinity like $u(\tau)\to u^*$ and $v(\tau)\to+\infty$ for $\tau\to\infty$.

(b2) Assume that $\gamma_\alpha^-\cap\gamma_\beta=\emptyset$, $g_\alpha(-1,1)<0$, and $g_\beta(-1,1)<0$. Consequently, the hyperbola γ_α^- intersects the left side of the unit square at some point $(-1,v_\alpha)$, and the hyperbola γ_β intersects it at a point $(-1,v^*)$ with $v_\alpha<v^*<1$ (fourth picture of Figure 5).

By inspecting the vector field, one sees that the solution of (5.1) has to end up in the region $u < -1$ and $v_\alpha < v < 1$, where it tends to infinity like $u(\tau) \to -\infty$ and $v(\tau) \to v^*$ for $\tau \to \infty$.

(c) Finally, assume that $\gamma_\alpha^- \cap \gamma_\beta = \emptyset$, $g_\alpha(-1,1) < 0$, and $g_\beta(-1,1) > 0$. This implies that γ_α^- intersects the left side of the unit square at some point $(-1, v_\alpha)$, and γ_β neither intersects γ_α^- nor the left side of Q (last picture of Figure 5). Since the vector field is vertical (upwards) on γ_α^- and horizontal (left directed) on γ_β (provided that γ_β intersects Q), the solution of (5.1) has to stay between them and it has to end up in the region $u < -1$, $v > 1$. There the vector field is constant and the solution tends to infinity like $u(\tau) \to -\infty$ and $v(\tau) \to +\infty$ for $\tau \to \infty$. ■

Remark. The situation (a2) of the previous theorem needs a comment. We believe that in this situation the solution of (5.1) always converges to the equilibrium (u^*, v^*). This is motivated by numerical experiments. Under additional assumptions, the existence of a limit cycle can be excluded as follows. If (5.1) has a periodic solution $\delta(t) = \big(\delta_\alpha(t), \delta_\beta(t)\big)$, then Green's Theorem (divergence theorem) implies

$$\iint_D \Big(\partial_u g_\alpha(u,v) + \partial_v g_\beta(u,v)\Big)\, \mathrm{d}u\, \mathrm{d}v \;=\; \oint_\delta \Big(g_\alpha(u,v)\, \mathrm{d}v - g_\beta(u,v)\, \mathrm{d}u\Big), \tag{6.2}$$

where D is the interior of the closed curve δ. The right-hand side of (6.2) vanishes, because $\mathrm{d}u = \dot\delta_\alpha(t)\mathrm{d}t = g_\alpha(u,v)\mathrm{d}t$ and $\mathrm{d}v = \dot\delta_\beta(t)\mathrm{d}t = g_\beta(u,v)\mathrm{d}t$. If one can prove that the left-hand side of (6.2) is non zero, we get a contradiction to the assumption of the existence of a limit cycle. At the equilibrium (u^*, v^*), which is in the interior of δ, the expressions $\partial_u g_\alpha(u,v)$ and $\partial_v g_\beta(u,v)$ are both negative. Note that $\partial_u g_\alpha$ changes sign only at the horizontal asymptote of the hyperbola γ_α, and $\partial_v g_\beta$ changes sign only at the vertical asymptote of the hyperbola γ_β. Therefore, the left-hand side of (6.2) is strictly negative, if the horizontal asymptote of γ_α is outside the stripe $-1 \le v \le 1$, and the vertical asymptote of γ_β is outside of $-1 \le u \le u_0$.

6.2. Right turning situation. We next consider the situation, where $\partial_v g_\alpha(u_0, -1) > 0$, so that the branch of the hyperbola $g_\alpha(u,v) = 0$ starting at $(u_0, -1)$ lies in the region $u \ge u_0$ (hence "right turning"). We shall see that the solution then stays either in Σ_α or continues in the region $\alpha(y) > 0$. If $g_\beta(1,-1) > 0$, then a simple symmetry argument (exchanging $u \leftrightarrow -u$ and $g_\alpha \leftrightarrow -g_\alpha$) brings us back to the situation covered by Theorem 6.1. We thus assume $g_\beta(1,-1) < 0$ in the following theorem. Furthermore, we use the notation

$$\gamma_\alpha^+ = \{(u,v) \in Q\, ;\, g_\alpha(u,v) = 0,\, u \ge u_0\}$$

for the branch of the hyperbola $g_\alpha(u,v) = 0$ starting at $(u_0, -1)$.

Theorem 6.2. *In addition to assumption (6.1) let $\partial_v g_\alpha(u_0,-1) > 0$ and $g_\beta(1,-1) < 0$.*

(a) *Assume that γ_α^+ intersects transversally γ_β inside the unit square, and denote (u^*, v^*) the intersection point closest to $(u_0, -1)$. We further denote by γ_β^* the oriented branch of γ_β that passes through (u^*, v^*).*

  (a1) *If γ_β^* ends up at the left side of Q below v^*, then (u^*, v^*) is an asymptotically stable equilibrium of (5.1), and $u(\tau) \to u^*$ and $v(\tau) \to v^*$ for $\tau \to \infty$.*

  (a2) *If γ_β^* intersects γ_α^+ only in (u^*, v^*) and ends up at the left side of Q above v^* or at the upper side of Q, then the solution of (5.1) either converges to (u^*, v^*) or it approaches a limit cycle around it.*

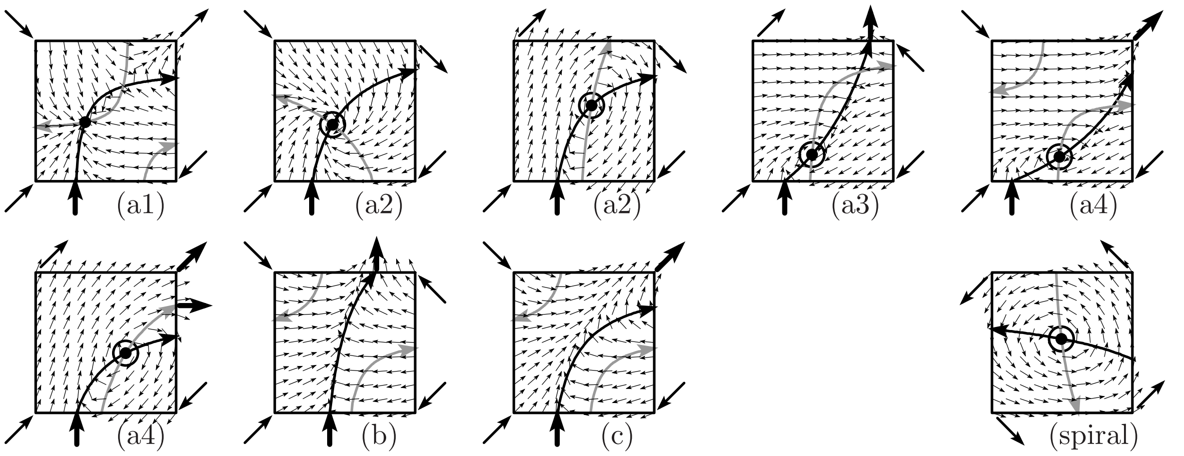

Figure 6. *Vector fields for the situations of Theorem 6.2 and of Theorem 7.1 (spiral); the interpretation of curves and arrows is the same as in Figure 5.*

(a3) *If γ_β^* intersects γ_α^+ twice and if $g_\alpha(1,1) < 0$ so that there exists $\hat{u} \in (u_0, 1)$ with $g_\alpha(\hat{u}, 1) = 0$, then the solution of (5.1) either converges to (u^*, v^*) or it approaches a limit cycle around it, or it satisfies $u(\tau) \to \hat{u}$ and $v(\tau) \to +\infty$ for $\tau \to \infty$.*

(a4) *If γ_β^* ends up at the right side of Q above v^* and if $g_\alpha(1,1) > 0$, then the solution of (5.1) either converges to (u^*, v^*) or it approaches a limit cycle around it, or it satisfies $u(\tau) \to +\infty$ and $v(\tau) \to +\infty$ for $\tau \to \infty$.*

(b) *If $\gamma_\alpha^+ \cap \gamma_\beta = \emptyset$ and $g_\alpha(1,1) < 0$, then there exists $u^* \in (u_0, 1)$ such that $g_\alpha(u^*, 1) = 0$ and the solution of (5.1) is such that $u(\tau) \to u^*$ and $v(\tau) \to +\infty$ for $\tau \to \infty$.*

(c) *If $\gamma_\alpha^+ \cap \gamma_\beta = \emptyset$ and $g_\alpha(1,1) > 0$, then the solution of (5.1) is such that $u(\tau) \to +\infty$ and $v(\tau) \to +\infty$ for $\tau \to \infty$.*

Remark. Again, all possible generic situations are covered by this theorem. Note that for the cases (a2)–(a4) the branch γ_β^* necessarily starts at the bottom line of Q. As in Theorem 6.1 the different cases characterize the type of solutions of the regularized differential equation (3.1) that is followed after entering the codimension-2 manifold Σ. A particular situation arises in the case (a4), where a codimension-1 sliding mode exists along Σ_β into $\alpha(y) > 0$, but this solution is unstable and separates the basins of attraction for a classical solution and for a codimension-2 sliding mode.

Proof. The proof is similar to that of Theorem 6.1. The assumption $\partial_v g_\alpha(u_0, -1) > 0$ implies that the branch γ_α^+ of the hyperbola γ_α, which passes through $(u_0, -1)$, turns right (see Figure 6). Since $g_\alpha(u_0, v) \geq 0$ for all v (also outside the unit square), the solution of (5.1) with initial value $(u_0, -1)$ remains in the half plane $u \geq u_0$. The assumptions $g_\beta(u_0, -1) > 0$ and $g_\beta(1, -1) < 0$ imply that a branch of the hyperbola γ_β starts at the bottom side of the unit square at $(u_1, -1)$ with $u_0 < u_1 < 1$.

(a1) By assumption, the branch of γ_β passing through (u^*, v^*) hits the left side of the square at some point $(-1, v_1)$ with $-1 < v_1 < v^*$ (first picture of Figure 6). The same argument as in the situation (a1) of Theorem 6.1, after exchanging left and right, shows that the solution of (5.1) converges straight ahead to the equilibrium (u^*, v^*).

(a2) For the cases (a2)–(a4) the branch γ_β^* starts at the bottom side of Q at $(u_1,-1)$. In the situation (a2), we have $\det G(u^*,v^*)>0$ by Lemma 5.1 and $\partial_u g_\alpha(u^*,v^*)<0$, but the trace of $G(u^*,v^*)$ can be positive or negative (second and third pictures of Figure 6). Therefore, the stationary point (u^*,v^*) can be asymptotically stable or unstable. Since γ_β^* does not traverse the right side of the unit square, an inspection of the vector field shows that the solution of (5.1) with initial value $(u_0,-1)$ is bounded. As a consequence, the Poincaré–Bendixson Theorem implies that the solution either converges to (u^*,v^*) or to a limit cycle surrounding this equilibrium.

(a3) Since $g_\alpha(u_0,1)>0$ and $g_\alpha(1,1)<0$, there exists $\hat u\in(u_0,1)$ with $g_\alpha(\hat u,1)=0$ (fourth picture of Figure 6). The solution of (5.1) either remains bounded and turns around (u^*,v^*) or it hits the upper side of Q at some point between u_0 and $\hat u$. In the first case the Poincaré–Bendixson Theorem implies that the solution either converges to the equilibrium or to a limit cycle around it. In the second case we have $u(\tau)\to\hat u$ and $v(\tau)\to\infty$ for $\tau\to\infty$.

(a4) By assumption, the branch γ_β^* turns right and intersects the right side of the unit square at a point $(1,v_1)$. Since $g_\alpha(1,1)>0$, also γ_α^+ ends up at the right side of the unit square Q. There can be two or one intersections with γ_α^+ (fifth and sixth pictures of Figure 6). If the solution of (5.1) starting at $(u_0,-1)$ touches the border of Q at some point with $v>v_1$, then it leaves the unit square and enters the region, where $u>1$ and $v>1$. Otherwise, the solution is bounded and the Poincaré–Bendixson Theorem completes the proof.

(b) The proof for this situation (seventh picture of Figure 6) is essentially the same as that for the situation (b1) of Theorem 6.1.

(c) The last situation (eigth picture of Figure 6) is treated as the situation (c) of Theorem 6.1. ■

Remark. In the situation (a2) of Theorem 6.2 the derivative $\partial_v g_\beta(u^*,v^*)$ can be negative (second picture of Figure 6) or positive (third picture of Figure 6). If it is negative, the equilibrium is asymptotically stable and the remark after the proof of Theorem 6.1 applies.

6.3. Solution of the regularized differential equation. With the interpretation of Section 5.2 all results of the previous two subsections can be reinterpreted as statements for the solution of the regularized differential equation (3.1). Since the results of Theorems 6.1 and 6.2 give a complete classification, we get even more information. Let us start with interpreting the condition $\partial_v g_\alpha(u_0,-1)<0$ in terms of the vector fields of (1.2), see also (4.10).

Lemma 6.3. *Assume that $g_\alpha(-1,-1)>0$ and that there exists $u_0\in(-1,1)$ such that $g_\alpha(u_0,-1)=0$. Then, we have*

- $\partial_v g_\alpha(u_0,-1)<0$ *if and only if* $f_\alpha^{-+}f_\alpha^{+-}-f_\alpha^{++}f_\alpha^{--}>0$.
- $\partial_v g_\alpha(u_0,-1)>0$ *if and only if* $f_\alpha^{-+}f_\alpha^{+-}-f_\alpha^{++}f_\alpha^{--}<0$.

The statement remains true if the argument $(u_0,-1)$ of $\partial_v g_\alpha$ is replaced by an arbitrary point of the branch of the hyperbola $g_\alpha(u,v)=0$ passing through $(u_0,-1)$.

Proof. If $\partial_v g_\alpha(u_0,-1)\neq 0$, we can write the function $g_\alpha(u,v)$ as

$$g_\alpha(u,v)=c_\alpha(u-u_\alpha)(v-v_\alpha)-d_\alpha \tag{6.3}$$

with $c_\alpha\neq 0$, so that $u=u_\alpha$ and $v=v_\alpha$ represent the asymptotes of the hyperbola. A simple algebraic argument shows that the expression $f_\alpha^{-+}f_\alpha^{+-}-f_\alpha^{++}f_\alpha^{--}$, written in terms of the function g_α as $g_\alpha(-1,1)g_\alpha(1,-1)-g_\alpha(1,1)g_\alpha(-1,-1)$, is equal to $4c_\alpha d_\alpha$.

For the case, where $c_\alpha d_\alpha > 0$, the hyperbola $g_\alpha(u,v) = 0$ consists of a left lower and a right upper branch. By our assumptions $g_\alpha(-1,-1) > 0$ and $g_\alpha(u_0,-1) = 0$, we see that g_α takes positive values left to the point $(u_0,-1)$. Consequently, it takes negative values above $(u_0,-1)$ implying that $\partial_v g_\alpha(u_0,-1) < 0$. On the other hand, for a point $(u_0,-1)$ on the hyperbola, we can have positive values to the left and negative values above only if $c_\alpha d_\alpha > 0$.

For $c_\alpha d_\alpha < 0$, the hyperbola $g_\alpha(u,v) = 0$ consists of a left upper and a right lower branch. Since g_α takes positive values left to the point $(u_0,-1)$, it takes in this case also positive values above $(u_0,-1)$ which is possible only if $\partial_v g_\alpha(u_0,-1) > 0$.

Since $\partial_v g_\alpha(u,v)$ does not change sign on a branch of the hyperbola, the argument $(u_0,-1)$ can be replaced by any point of the branch passing through $(u_0,-1)$. ■

Theorem 6.4. *Assume that the solution enters Σ as sliding mode along Σ_α from $\beta(y) < 0$. If there is no outgoing codimension-1 and no codimension-2 sliding mode, then we have:*

- *if $f_\alpha^{-+} f_\alpha^{+-} - f_\alpha^{++} f_\alpha^{--} > 0$, there exists a classical solution of (1.2) in $\mathcal{R}^{-+}$ and the solution of the regularized equation (3.1) follows this classical solution.*
- *if $f_\alpha^{-+} f_\alpha^{+-} - f_\alpha^{++} f_\alpha^{--} < 0$, there exists a classical solution of (1.2) in $\mathcal{R}^{++}$ and the solution of the regularized equation (3.1) follows this classical solution.*

Proof. The assumption on how the solution enters the intersection $\Sigma = \Sigma_\alpha \cap \Sigma_\beta$ implies that $g_\alpha(-1,-1) > 0$, $g_\alpha(1,-1) < 0$, and that there exists $u_0 \in (-1,1)$ such that $g_\alpha(u_0,-1) = 0$ and $g_\beta(u_0,-1) > 0$.

Consider first the situation $f_\alpha^{-+} f_\alpha^{+-} - f_\alpha^{++} f_\alpha^{--} > 0$ which, by Lemma 6.3, is equivalent to $\partial_v g_\alpha(u_0,-1) < 0$. If $g_\beta(-1,-1) > 0$, condition (6.1) is satisfied, so that Theorem 6.1 can be applied. Only in the situation (c) there is neither an outgoing codimension-1 sliding mode nor a codimension-2 sliding mode. In this case we have a classical solution in $\mathcal{R}^{-+}$ and the solution of the regularized problem follows this solution.

If $f_\alpha^{-+} f_\alpha^{+-} - f_\alpha^{++} f_\alpha^{--} > 0$, but $g_\beta(-1,-1) < 0$, the assumption $g_\beta(u_0,-1) > 0$ implies $g_\beta(1,-1) > 0$. Consequently, the transformation $u \leftrightarrow -u$ together with $\alpha(y) \leftrightarrow -\alpha(y)$, which induces the transformation $g_\alpha(u,v) \leftrightarrow -g_\alpha(-u,v)$, implies that all assumptions of Theorem 6.2 are satisfied. Only the situation (c) has neither a codimension-1 nor a codimension-2 sliding mode, but there is a classical solution in the region, where $\beta(y) > 0$ and $-\alpha(y) > 0$. This proves the existence of a classical solution in $\mathcal{R}^{-+}$.

The statement for $f_\alpha^{-+} f_\alpha^{+-} - f_\alpha^{++} f_\alpha^{--} < 0$ follows from the first statement by exchanging $u \leftrightarrow -u$ and $\alpha(y) \leftrightarrow -\alpha(y)$. ■

Theorem 6.5. *Assume that the solution enters Σ as a sliding mode along Σ_α from $\beta(y) < 0$. If there is no outgoing classical solution and no codimension-2 sliding mode, then we have:*

- *if $f_\alpha^{-+} f_\alpha^{+-} - f_\alpha^{++} f_\alpha^{--} > 0$ and $f_\alpha^{-+} < 0$, there exists a codimension-1 sliding mode of (1.2) along Σ_β in the region $\alpha(y) < 0$.*
- *if $f_\alpha^{-+} f_\alpha^{+-} - f_\alpha^{++} f_\alpha^{--} < 0$ and $f_\alpha^{++} > 0$, there exists a codimension-1 sliding mode of (1.2) along Σ_β in the region $\alpha(y) > 0$.*
- *if $f_\alpha^{-+} f_\alpha^{+-} - f_\alpha^{++} f_\alpha^{--} > 0$ and $f_\alpha^{-+} > 0$ or if $f_\alpha^{-+} f_\alpha^{+-} - f_\alpha^{++} f_\alpha^{--} < 0$ and $f_\alpha^{++} < 0$, there exists a codimension-1 sliding mode of (1.2) along Σ_α in the region $\beta(y) > 0$.*

In all cases the solution of the regularized differential equation (3.1) follows the corresponding codimension-1 sliding mode.

Proof. The assumption on the incoming solution implies $g_\alpha(-1,-1) > 0$, $g_\alpha(1,-1) < 0$, and that there exists $u_0 \in (-1,1)$ such that $g_\alpha(u_0,-1) = 0$ and $g_\beta(u_0,-1) > 0$.

If $f_\alpha^{-+} f_\alpha^{+-} - f_\alpha^{++} f_\alpha^{--} > 0$, it follows from Lemma 6.3 that $\partial_v g_\alpha(u_0,-1) < 0$. The assumption that there is no outgoing classical solution together with $f_\alpha^{-+} < 0$ imply that $g_\beta(-1,1) < 0$. The nonexistence of codimension-2 sliding modes implies $\gamma_\alpha^- \cap \gamma_\beta = \emptyset$, so that the sign of $g_\beta(u,v)$ does not change on γ_α^-. Consequently, we have $g_\beta(-1,v_1) > 0$, where $v_1 \in (-1,1)$ is defined by $g_\alpha(-1,v_1) = 0$, and therefore also $g_\beta(-1,-1) > 0$. We are now in the position to apply Theorem 6.1. Only the situation (b2) is possible. This proves the first statement of the theorem. The second statement follows from the first one by exchanging $u \leftrightarrow -u$ and $\alpha(y) \leftrightarrow -\alpha(y)$.

The assumptions for the last statement imply that the hyperbola for $g_\alpha(u,v) = 0$, starting at $(u_0,-1)$, traverses the upper side of the unit square. Since there is no codimension-2 sliding mode, the only possible situations are (b1) of Theorem 6.1 and (b) of Theorem 6.2. The discussion of Section 5.2 concludes the proof. ■

6.4. Explanation of the examples of Section 3.2. Theorem 6.1 and Theorem 6.2 permit us to understand and to explain the behaviour of the regularized solution for the examples of Section 3.2.

Example 1. For the vector field (3.3) of Example 1 we have

$$\begin{aligned} g_\alpha(u,v) &= 0.95\,uv - 0.4\,u - 0.2\,v + 0.05 \\ g_\beta(u,v) &= -uv - 0.7\,u - 0.65\,v - 0.25 \end{aligned}$$

Condition (6.1) is satisfied with $u_0 = 5/27$, and we have $\partial_v g_\alpha(u_0,-1) < 0$, so that Theorem 6.1 applies. There are two intersection points of the hyperbolas γ_α and γ_β in the unit square, exactly as drawn in the first picture of Figure 5, situation (a1). This explains, why the solution of Figure 1 approaches a codimension-2 sliding mode without any oscillations.

Example 2. For the vector field (3.4) of Example 2 we have (note that $\alpha(y)$ and $\beta(y)$ are scaled by κ_α and $\kappa_\beta = 1$, respectively)

$$\begin{aligned} g_\alpha(u,v) &= \kappa_\alpha(4uv - 4u + 4v) \\ g_\beta(u,v) &= 6(2uv - 4u + v) \end{aligned} \qquad G(0,0) = \begin{pmatrix} -4\,\kappa_\alpha & 4\,\kappa_\alpha \\ -24 & 6 \end{pmatrix}.$$

Condition (6.1) is satisfied with $u_0 = -1/2$. The only intersection of the hyperbolas γ_α and γ_β is $(u^*,v^*) = (0,0)$. Moreover, we have $\partial_v g_\alpha(u_0,-1) > 0$, $g_\beta(1,-1) < 0$, and $g_\beta(1,1) < 0$, so that γ_β^* does not traverse the right side of the unit square. We are therefore in the situation (a2) of Theorem 6.2. The determinant of $G(0,0)$ is always positive, but its trace is negative only for $\kappa_\alpha > 1.5$. Consequently, we have a limit cycle for the hidden dynamics for $1 \le \kappa_\alpha < 1.5$ resulting in the high frequency oscillations observed in Figure 2. For $\kappa_\alpha > 1.5$ we observe asymptotic convergence to a codimension-2 sliding mode.

Example 3. For the vector field (3.5) of Example 3 we have (with $\kappa_\alpha = 1$)

$$\begin{aligned} g_\alpha(u,v) &= 4uv - 4u + 4v \\ g_\beta(u,v) &= \kappa_\beta(4uv - 4u + 2v) \end{aligned} \qquad G(0,0) = \begin{pmatrix} -4 & 4 \\ -4\,\kappa_\beta & 2\,\kappa_\beta \end{pmatrix}$$

which corresponds to the situation (a4) of Theorem 6.2. There exists a value $\kappa^* \approx 1.703$, such that for $\kappa_\beta = \kappa^*$ the solution follows a codimension-1 sliding mode along Σ_β into the region $\alpha(y) > 0$. However, this situation is not generic (unstable), because for $\kappa_\beta < \kappa^*$ the solution approaches a codimension-2 sliding mode and for $\kappa_\beta > \kappa^*$ a classical solution in $\mathcal{R}^{++}$.

7. Analysis of hidden dynamics: solution enters in spiral mode. It remains to consider the situation where the solution spirals inwards to approach the codimension-2 manifold Σ. Without loss of generality we assume that it spirals counterclockwise as illustrated in the picture to the right of Figure 4. This fixes uniquely the sign pattern at the corners as

$$\begin{array}{llll} g_\alpha(-1,-1) > 0, & g_\alpha(1,-1) > 0, & g_\alpha(1,1) < 0, & g_\alpha(-1,1) < 0, \\ g_\beta(-1,-1) < 0, & g_\beta(1,-1) > 0, & g_\beta(1,1) > 0, & g_\beta(-1,1) < 0. \end{array} \tag{7.1}$$

Let us investigate, what it means that the solution spirals inwards. Starting in the region $\mathcal{R}^{--}$ at a value which satisfies $\beta(y) \approx 0$ and $\alpha(y) = -\varepsilon$ with small $\varepsilon > 0$, the solution satisfies $\alpha(y(t)) \approx -\varepsilon + t g_\alpha(-1,-1)$ and $\beta(y(t)) \approx t g_\beta(-1,-1)$ until $\alpha(y(t))$ vanishes. This happens for $t \approx \varepsilon / g_\alpha(-1,-1)$, where $\alpha(y) \approx 0$ and $\beta(y) = \varepsilon g_\beta(-1,-1)/g_\alpha(-1,-1)$. Proceeding similarly in the remaining three regions, we obtain that after a complete round the solution satisfies $\beta(y) \approx 0$ and $\alpha(y) = -\gamma\varepsilon$ where, in the case of inwards spiraling solutions, the contraction factor γ is asymptotically given by

$$0 < \gamma < 1 \qquad \text{with} \qquad \gamma = \frac{g_\beta(-1,-1)}{g_\alpha(-1,-1)} \cdot \frac{g_\alpha(1,-1)}{g_\beta(1,-1)} \cdot \frac{g_\beta(1,1)}{g_\alpha(1,1)} \cdot \frac{g_\alpha(-1,1)}{g_\beta(-1,1)}. \tag{7.2}$$

Theorem 7.1. *Under assumptions (7.1) and (7.2) there exists a unique equilibrium (u^*, v^*) of the dynamical system (5.1). At this equilibrium point we have that $\det G(u^*, v^*) > 0$ and at least one among the derivatives $\partial_u g_\alpha(u^*, v^*)$ and $\partial_v g_\beta(u^*, v^*)$ is negative. The solution of (5.1) either converges to (u^*, v^*) or it approaches a limit cycle around this equilibrium.*

Proof. It follows from the sign pattern (7.1) that the hyperbola for $g_\alpha(u,v) = 0$ goes from the right to the left side of the unit square, and the hyperbola for $g_\beta(u,v) = 0$ from the top to the bottom (see the picture with label "spiral" of Figure 6). Lemma 5.1 thus implies $\det G(u^*, v^*) > 0$ at the unique stationary point.

The condition (7.2) implies that at least one of the expressions

$$\left| \frac{g_\alpha(1,-1) g_\alpha(-1,1)}{g_\alpha(-1,-1) g_\alpha(1,1)} \right|, \qquad \left| \frac{g_\beta(1,1) g_\beta(-1,-1)}{g_\beta(-1,1) g_\beta(1,-1)} \right| \tag{7.3}$$

is strictly smaller than 1. Assume first that this is true for the first expression of (7.3). From assumption (7.1) on the sign pattern at the corners of the unit square we see that this is equivalent to

$$g_\alpha(1,-1)\, g_\alpha(-1,1) > g_\alpha(-1,-1)\, g_\alpha(1,1). \tag{7.4}$$

As in the proof of Lemma 6.3 we write the function $g_\alpha(u,v)$ as $g_\alpha(u,v) = c_\alpha (u - u_\alpha)(v - v_\alpha) - d_\alpha$ with $c_\alpha \neq 0$, so that the inequality (7.4) becomes equivalent to $c_\alpha\, d_\alpha > 0$. In this situation the hyperbola $g_\alpha(u,v) = 0$ consists of a left lower and a right upper branch. If $c_\alpha > 0$ (and hence also $d_\alpha > 0$), the sign pattern of g_α at the corners of the unit square and the fact that $g_\alpha < 0$ on the asymptotes imply that the stationary point (u^*, v^*) lies on the left lower branch of the hyperbola. Consequently, $v^* < v_\alpha$ and $\partial_u g_\alpha(u^*, v^*) = c_\alpha(v^* - v_\alpha) < 0$. For $c_\alpha < 0$ (and $d_\alpha < 0$), the stationary point (u^*, v^*) has to be on the right upper branch of the hyperbola, implying $v^* > v_\alpha$ and again $\partial_u g_\alpha(u^*, v^*) = c_\alpha(v^* - v_\alpha) < 0$.

If the second expression of (7.3) is strictly smaller than 1, we have $c_\beta\, d_\beta < 0$ (the constants c_β and d_β are defined by $g_\beta(u, v) = c_\beta(u - u_\beta)(v - v_\beta) - d_\alpha$), so that the hyperbola $g_\beta(u, v) = 0$ consists of a left upper and a right lower branch. The same arguments as in the first case then show that $\partial_v g_\beta(u^*, v^*) = c_\beta(u^* - u_\beta) < 0$.

Since the solutions of (5.1) are bounded and (u^*, v^*) is the only stationary point, an application of the Poincaré–Bendixson Theorem completes the proof. ■

Remark. The statement of Theorem 7.1 remains true under milder assumptions. It is not necessary to require that the solution enters in spiral mode. It is sufficient to assume that there is no outgoing classical solution and no outgoing codimension-1 sliding mode. This is a consequence of situation (a) of Theorem 6.1 (second picture of Figure 5) and of situations (a1) and (a2) of Theorem 6.2 (second and third pictures of Figure 6).

8. Stabilization. As an interesting outcome of our theoretical investigation we obtain a modification of the standard regularization that permits us to avoid unphysical oscillations around codimension-2 sliding modes.

Corollary 8.1. *Whenever the solution of the hidden dynamics (5.1) tends to a limit cycle around an unstable equilibrium, then there exists a scaling $\alpha(y) \to \kappa_\alpha \alpha(y)$, $\beta(y) \to \kappa_\beta \beta(y)$ with $\min(\kappa_\alpha, \kappa_\beta) = 1$, which makes the equilibrium (u^*, v^*) asymptotically stable.*

Proof. By our classification the only possible situations are (a2), (a3), and (a4) of Theorem 6.2, for which we have $\partial_u g_\alpha(u^*, v^*) < 0$, and the situation of Theorem 7.1, for which at least one among the derivatives $\partial_u g_\alpha(u^*, v^*)$ and $\partial_v g_\beta(u^*, v^*)$ is negative. It is therefore always possible to choose $\kappa_\alpha \geq 1$ and $\kappa_\beta \geq 1$ such that the trace $\kappa_\alpha \partial_u g_\alpha(u^*, v^*) + \kappa_\beta \partial_v g_\beta(u^*, v^*)$ of G for the scaled problem is negative. Since the scaling does not change the sign of $\det G(u^*, v^*)$ (in fact, this determinant is positive for all occurring situations), we get an asymptotically stable equilibrium. ■

Corollary 8.1 tells us that the situation of Example 2 of Section 3.2 is generic. Whenever the solution of the regularized equation (3.1) oscillates around a codimension-2 sliding mode, it is possible to make the equilibrium (u^*, v^*) asymptotically stable with a suitable choice of κ_α and κ_β. In all our numerical experiments (see also the remark after the proof of Theorem 6.1) we observed that such a scaling rapidly damps the oscillations and makes a numerical treatment efficient. Note, however, that the matrix $G(u^*, v^*)$ depends on the point $y_0 \in \Sigma_\alpha \cap \Sigma_\beta$, so that also the parameters κ_α and κ_β depend on $y_0 \in \Sigma$. Theoretically, this does not cause any problems, because our analysis is local.

Ideally, one would like to fix in advance the parameters $\kappa_\alpha(y)$ and $\kappa_\beta(y)$ (for $y \in \Sigma$), which avoid unphysical oscillations in the solution of the regularized differential equation (3.1). For realistic problems this seems to be a challenging question, which is not addressed in the present work.

9. Conclusions. This article considers ordinary differential equations with discontinuous vector field. Our main interest is the study of solutions close to the intersection of two discontinuity surfaces. To avoid the continuum of solutions in Filippov's approach, we restrict our considerations to a 2-parameter family of convex combinations of the adjacent vector fields. Using singular perturbation techniques, we interpret the solution of the discontinuous differential equations as the limit of the solutions of a regularized problem. This leads to a 2-dimensional dynamical system (the so-called hidden dynamics) which determines the transition between the solution entering the intersection of the two surfaces and its continuation.

In the present work we study the hidden dynamics and we give a complete characterisation of such transitions. In the presence of more than one solution (classical or sliding modes) this provides a natural selection procedure of the most meaningful solution. The surprising new insight is that in the co-existence of classical and codimension-2 sliding mode solutions, in most cases the codimension-2 sliding mode is the correct solution. A byproduct of our analysis is a modification of space regularizations that permits to avoid artificial high oscillations and makes the numerical treatment more efficient.

Understanding escape conditions, when a codimension-2 sliding mode leaves the intersection of the discontinuity surfaces, is an interesting open research problem.

Acknowledgments. We are grateful to both anonymous referees for valuable suggestions and remarks, and for pointing us to relevant literature on gene regulatory networks. Part of this work has been done during the stay of the first author at the University of Geneva.

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