Numerical approaches for state-dependent neutral delay equations with discontinuities

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Abstract

This article presents two regularization techniques for systems of state-dependent neutral delay differential equations which have a discontinuity in the derivative of the solution at the initial point. Such problems have a rich dynamics and besides classical solutions can have weak solutions in the sense of Utkin. Both of the presented techniques permit the numerical solution of such problems with the code RADAR5, which is designed to compute classical solutions of stiff and differential-algebraic (state-dependent) delay equations.

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1. Introduction

Phenomena with memory often lead to delay differential equations, and when the derivative at a time instant also depends on the derivative in the past we are concerned with neutral delay equations. In this article we are interested in systems of state-dependent neutral delay equations of the form

$$\begin{aligned} \dot y(t) &= f\Big(y(t), \dot y(\alpha(y(t)))\Big) \qquad \text{for} \quad t > 0 \\ y(t) &= \varphi(t) \qquad \text{for} \quad t \le 0 \end{aligned} \tag{1}$$

with smooth vector functions $f(y,z)$, $\varphi(t)$ and scalar deviating argument $\alpha(y)$ satisfying $\alpha(y(t)) < t$ (non-vanishing delay). More general equations (e.g., dependence of f on time t and on $y(\alpha(y(t))))$ could be treated as well without presenting additional difficulties. In the present article we focus on the situation, where the derivative of the solution has a jump discontinuity at the starting point, i.e.,

$$\dot{\varphi}(0) \neq f(\varphi(0), \dot{\varphi}(\alpha(\varphi(0)))). \tag{2}$$

Such a system has the following particularities:

- since it is of neutral type, this discontinuity is in general propagated to further breaking points;
- since the deviating argument is state-dependent, it may occur that at breaking points a classical solution ceases to exist.

Let us discuss the second item in some more detail. At the first breaking point t_0, where we have $\alpha(y(t_0)) = 0$ and $\alpha(y(t)) < 0$ for $t < t_0$, the left-hand derivative of $\alpha(y(t))$ is generically positive, i.e., $\alpha'(y(t_0)f(y(t_0), \dot{y}(0^-)) > 0$, and we expect that the solution enters the region $\alpha(y) > 0$. However, if the right-hand derivative of $\alpha(y(t))$ is negative, i.e., $\alpha'(y(t_0)f(y(t_0), \dot{y}(0^+)) < 0$, it cannot enter this region, and a classical solution ceases to exist.

Such a situation is closely related to ordinary differential equations having a discontinuous vector field. In this situation it is possible to consider weak solutions (in the sense of Filippov [3] and/or Utkin [13]), where one looks for solutions staying in the manifold $\alpha(y) = 0$ (sliding mode) and one permits the derivative $\dot{y}(0)$ to be multi-valued.

To our knowledge, codes for delay equations cannot handle such a situation in an efficient way. Typically, the code will stop the integration at such a breaking point with the message that too small step sizes are needed. The aim of the present article is to discuss regularizations of the neutral delay equations (1), which permit the use of standard software packages for an efficient computation of classical and weak solutions.

In the present article we study two regularization techniques for the problem (1). The first one (Section 2.1) consists in changing the derivative of the initial function $\varphi(t)$ on the interval $(-\varepsilon, 0]$ in such a way that the discontinuity of $\dot{y}(t)$ is suppressed at the origin. The second one (Section 2.2) is based on turning the problem into the $\varepsilon \to 0$ limit of a singularly perturbed delay equation (as proposed in [7]). We shall show in Sections 3 and 4 that the solutions of the regularized problems (which are classical solutions) remain close to a solution of (1) independent of whether it is a classical or a weak one. Numerical experiments (Section 5) demonstrate the applicability of the code RADAR5 to the regularized problems, and they confirm the theoretical results of this paper.

2. Regularization techniques

The functions $f(y, z)$, $\alpha(y)$, and $\varphi(t)$ of the system (1) are assumed to be sufficiently differentiable. The discontinuity of the solution is generated by the fact that the derivative $\dot\varphi(0)$ does not match the right-hand side of the delay equation at $t = 0$. We consider two approaches (Sections 2.1 and 2.2) of regularizing this discontinuity. Other regularizations have been considered in [4], where the right-hand side is replaced by its average on a small interval, and in [1], where the problem is regularized by its numerical discretization based on the Euler method (an idea also used for other classes of differential equations as discussed in [2]).

The analysis of singularly perturbed state-dependent delay equations is an interesting subject in itself and has received the attention of many researchers in recent years (see e.g. [11]) both from the theoretical and the numerical point of view.

2.1. Regularization of the initial function

By introducing a new variable for the derivative, a neutral delay equation can be transformed into a differential-algebraic delay equation. In our situation the equation (1) becomes

$$\begin{aligned} \dot y(t) &= z(t) \\ 0 &= f(y(t), z(\alpha(y(t)))) - z(t), \end{aligned} \tag{3}$$

where $y(t) = \varphi(t)$ and $z(t) = \dot\varphi(t)$ for $t \le 0$. This permits us to treat the functions $y(t)$ and $z(t)$ independently of each other. We do not touch the condition $y(t) = \varphi(t)$ for $t \le 0$, but we replace the condition $z(t) = \dot\varphi(t)$ on the interval $-\varepsilon \le t \le 0$ by

$$z(t) = \dot\varphi(-\varepsilon) + \chi(t/\varepsilon)(\dot y_0^+ - \dot\varphi(-\varepsilon)), \qquad \dot y_0^+ = f(\varphi(0), \dot\varphi(\alpha(\varphi(0)))), \tag{4}$$

where $\chi : \mathbb{R} \to \mathbb{R}$ is a sufficiently differentiable function satisfying $\chi(-1) = 0$, $\chi(0) = 1$, and $\chi'(\tau) > 0$ for $\tau \in [0, 1]$, e.g., the linear interpolation polynomial $\chi(\tau) = \tau + 1$. In this way, the function $z(t)$ is continuous at $t = 0$ and the problem will have a (classical) solution, where the original problem did not. Consequently, codes for differential-algebraic (index 1), state-dependent delay equations can be applied to solve the problem.

We remark that this regularization presents similarities with the approach considered in [12] and in general to regularizations which replace the discontinuity in the right-hand side by a continuous link.

2.2. Regularization to a non-neutral delay equation

Another type of regularization is by turning the algebraic relation of (3) into a singularly perturbed differential equation as follows ($0 < \varepsilon \ll 1$)

$$\begin{aligned} \dot{y}(t) &= z(t) \\ \varepsilon\, \dot{z}(t) &= f(y(t), z(\alpha(y(t)))) - z(t). \end{aligned} \tag{5}$$

In this situation we can keep the initial functions $y(t) = \varphi(t)$ and $z(t) = \dot{\varphi}(t)$ for $t \leq 0$. Continuity in $z(t)$ at $t = 0$ is guaranteed by the fact that we now have a differential equation also for this variable. This regularization is proposed in [7], where the $\varepsilon \to 0$ behavior is analyzed at the breaking point, where the classical solution ceases to exist for the first time.

Any code for stiff, state-dependent delay equations (like RADAR5 by Guglielmi and Hairer [5, 6]) can be applied to solve the singularly perturbed problem (5).

3. Analysis of the regularization of Section 2.1

Let t_0 be the first breaking point of the problem (1), i.e., $a_0 = y(t_0)$ satisfies $\alpha(a_0) = 0$. Assuming that the solution enters transversally the manifold $\alpha(y) = 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha(y(t))\Big|_{t=t_0} = \alpha'(a_0)\, f(a_0, \dot{y}_0^-) > 0, \tag{6}$$

where $\dot{y}_0^- = \dot{\varphi}(0)$. The regularization of Section 2.1 does not affect the solution as long as $\alpha(y(t)) \leq -\varepsilon$, and we denote the first time instant with equality by $t_0(\varepsilon) < t_0$, i.e.,

$$\alpha(y(t_0(\varepsilon))) = -\varepsilon. \tag{7}$$

Because of (6), the implicit function theorem guarantees that $t_0(\varepsilon)$ can be expanded into a series of powers of ε. Consequently, this is also true for the solution at $t_0(\varepsilon)$ and we have an expansion

$$y(t_0(\varepsilon)) = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \dots\ . \tag{8}$$

3.1. Asymptotic expansion after the first breaking point

Beyond the point $t_0(\varepsilon)$ and as long as $-\varepsilon \le \alpha(y(t)) \le 0$, the solution of the regularized problem satisfies

$$\dot{y}(t) = f\Big(y(t), \dot{\varphi}(-\varepsilon) + \chi\Big(\frac{\alpha(y(t))}{\varepsilon}\Big)(\dot{y}_0^+ - \dot{\varphi}(-\varepsilon))\Big). \tag{9}$$

To cope with the singularity at $\varepsilon = 0$, we separate the solution into a smooth and a transient part by an asymptotic expansion of the form

$$y(t_0(\varepsilon) + t) = \sum_{j=0}^{N} \varepsilon^j y_j(t) + \varepsilon \sum_{j=0}^{N-1} \varepsilon^j \eta_j(t/\varepsilon) + \mathcal{O}(\varepsilon^{N+1}), \tag{10}$$

which has to match the expansion (8) for $t = 0$ i.e.,

$$y_0(0) = a_0, \qquad y_j(0) + \eta_{j-1}(0) = a_j \quad \text{for} \quad j \ge 1. \tag{11}$$

There is some freedom in choosing the coefficient functions, e.g., an expression t^k can be considered as part of $y_0(t)$ or of $\eta_{k-1}(\tau)$, because $t^k = \varepsilon^k \tau^k$ for $\tau = t/\varepsilon$. If we are interested in intervals of length $\mathcal{O}(\varepsilon)$, we can exploit this freedom to get simple formulas. If intervals of length $\mathcal{O}(1)$ are an issue, we assume that $\eta_j(\tau)$ converges exponentially fast to zero for $\tau \to \infty$.

The argument of χ in equation (9) can be written as

$$\frac{1}{\varepsilon}\,\alpha(y(t_0(\varepsilon) + t)) = \frac{1}{\varepsilon}\,\alpha(y_0(t)) + \alpha'(y_0(t))(y_1(t) + \eta_0(\tau)) + \mathcal{O}(\varepsilon). \tag{12}$$

To avoid the division by ε we assume

$$\alpha(y_0(t)) = 0. \tag{13}$$

We then distinguish the following two situations:

Weak solution. There exists $\theta_0 \in (0, 1]$ such that

$$\alpha'(a_0) f(a_0, \dot{y}_0^- + \theta_0(\dot{y}_0^+ - \dot{y}_0^-)) = 0. \tag{14}$$

If $0 < \theta_0 < 1$, the original problem (1) possesses a weak solution evolving in the manifold $\alpha(y) = 0$. If $\theta_0 = 1$, the (weak or classical) solution is tangential to the manifold.

Classical solution. We have

$$\alpha'(a_0) f(a_0, \dot{y}_0^- + \theta(\dot{y}_0^+ - \dot{y}_0^-)) > 0 \qquad \text{for} \qquad 0 \le \theta \le 1. \tag{15}$$

This implies $\alpha'(a_0) f(a_0, \dot{y}_0^+) > 0$, and consequently the existence of a classical solution beyond the breaking point t_0.

3.2. Construction of the coefficient functions (weak solution)

Inserting (12) together with (13) into (9) and expanding into powers of ε, the smooth (τ-independent) part of the ε^0 coefficient becomes

$$\dot{y}_0(t) = f(y_0(t), \dot{y}_0^- + u_0(t)(\dot{y}_0^+ - \dot{y}_0^-)), \qquad u_0(t) = \chi(\alpha'(y_0(t))y_1(t)), \quad (16)$$

where $\dot{y}_0^- = \dot{\varphi}(0)$. Differentiating the assumption (13) with respect to time and using the relation (16) yields

$$\alpha'(y_0(t))f(y_0(t), \dot{y}_0^- + u_0(t)(\dot{y}_0^+ - \dot{y}_0^-)) = 0. \quad (17)$$

Assumption (14) implies that for $t = 0$ this relation holds with $u_0(0) = \theta_0$. If we assume in addition that[1]

$$\alpha'(a_0)f_z(a_0, \dot{y}_0^- + \theta_0(\dot{y}_0^+ - \dot{y}_0^-))(\dot{y}_0^+ - \dot{y}_0^-) < 0, \quad (18)$$

the implicit function theorem guarantees that $u_0(t)$ can be expressed in terms of $y_0(t)$ in an ε-independent neighborhood of (a_0, θ_0). Consequently, (16) is a differential equation on the manifold $\alpha(y) = 0$, which has a unique solution for the initial value $y_0(0) = a_0$.

The leading term of the non-smooth part is obtained by subtracting the smooth part (16) from (9), then substituting $\varepsilon\tau$ for t, and finally putting $\varepsilon = 0$. This leads to

$$\begin{aligned} \eta_0'(\tau) &= f(a_0, \dot{y}_0^- + \chi(\alpha'(a_0)(y_1(0) + \eta_0(\tau)))(\dot{y}_0^+ - \dot{y}_0^-)) \\ &- f(a_0, \dot{y}_0^- + \theta_0(\dot{y}_0^+ - \dot{y}_0^-)). \end{aligned} \quad (19)$$

Under the assumption (14) the scalar function

$$\widehat{\eta}_0(\tau) = \alpha'(a_0)(y_1(0) + \eta_0(\tau))$$

therefore solves the differential equation

$$\widehat{\eta}_0'(\tau) = \alpha'(a_0)f(a_0, \dot{y}_0^- + \chi(\widehat{\eta}_0(\tau))(\dot{y}_0^+ - \dot{y}_0^-)). \quad (20)$$

The initial value $\widehat{\eta}_0(0) = -1$ is obtained from (7), (12) and (13). Since this differential equation is scalar and autonomous, and $\widehat{\eta}_0'(0) > 0$ by (6), its

[1]It would be sufficient to assume that the expression is different from zero. However, the sign is important for the estimation of the remainder in the asymptotic expansion.

solution converges monotonically and exponentially fast to the value ν_0 for which $\chi(\nu_0) = \theta_0$ is the smallest positive root of the equation (14). Once $\widehat{\eta}_0(\tau)$ is known, the function $\eta_0(\tau)$ is obtained by integration from (19). The integration constant is determined by assuming $\eta_0(\tau) \to 0$ for $\tau \to \infty$. The matching condition (11) then fixes the initial value $y_1(0)$.

The construction of the functions $y_1(t)$ and $\eta_1(\tau)$ is very similar to the above procedure. The ε^1 coefficient of the smooth part yields

$$\dot{y}_1(t) = g_1(y_0(t), y_1(t)) + f_z(y_0(t), \dot{y}_0^- + u_0(t)(\dot{y}_0^+ - \dot{y}_0^-))u_1(t)(\dot{y}_0^+ - \dot{y}_0^-), \quad (21)$$

where $u_1(t) = \chi'(\alpha'(y_0(t))y_1(t))\alpha'(y_0(t))y_2(t)$, and g_1 is a smooth function. In the first step we have expressed $u_0(t) = \chi(\alpha'(y_0(t))y_1(t))$ as a function of $y_0(t)$. Differentiating this relation with respect to time shows that $\chi'(\alpha'(y_0(t))y_1(t))\alpha'(y_0(t))\dot{y}_1(t)$ can be expressed in terms of $y_1(t)$ and the known functions $y_0(t)$, $\dot{y}_0(t)$. The same is true for $\alpha'(y_0(t))\dot{y}_1(t)$, because $\chi'(\tau)$ is uniformly bounded away from zero (see the assumptions on $\chi(\tau)$ in Section 2.1). Multiplying (21) with $\alpha'(y_0(t))$, we see that under the assumption (18) the value $u_1(t)$ can also be expressed in terms of $y_1(t)$ and the known functions $y_0(t)$, $\dot{y}_0(t)$. As in the first step, we obtain one after another the functions $y_1(t)$, then $\widehat{\eta}_1(\tau) = \alpha'(a_0)(y_2(0) + \eta_1(\tau))$, $\eta_1(\tau)$, and the initial value $y_2(0)$. This procedure is extended straightforwardly to obtain further coefficient functions of the asymptotic expansion (10).

3.3. Construction of the coefficient functions (classical solution)

Under the assumption (15) we expect the solution of the regularized problem to satisfy $-\varepsilon \le \alpha(y(t)) \le 0$ only on an interval of length $\mathcal{O}(\varepsilon)$. We therefore exploit the freedom in the asymptotic expansion and assume $y_0(t) = a_0$ to be constant, and $y_j(t) = 0$ for $j \ge 1$.

Differential equations for the remaining coefficient functions $\eta_j(\tau)$ are obtained by inserting the expansion (10) into (9) and comparing like powers of ε. In this way we get

$$\eta_0'(\tau) = f(a_0, \dot{y}_0^- + \chi(\alpha'(a_0)\,\eta_0(\tau))(\dot{y}_0^+ - \dot{y}_0^-)), \quad (22)$$

and linear differential equations for $\eta_j(\tau)$ with coefficients depending on $\eta_0(\tau), \ldots, \eta_{j-1}(\tau)$. Initial values $\eta_j(0)$ are given by (11).

It follows from (7), (12) and (13) that the initial value $\eta_0(0) = a_1$ satisfies $\alpha'(a_0)\eta_0(0) = -1$. As in Section 3.2 the scalar function $\alpha'(a_0)\eta_0(\tau)$ is monotonically increasing. However, its slope is at least c, where

$$c = \min_{0 \le \theta \le 1} \alpha'(a_0) f(a_0, \dot y_0^- + \theta\,(\dot y_0^+ - \dot y_0^-)) > 0,$$

as long as $\alpha'(a_0)\eta_0(\tau) \le 0$. Consequently, $\alpha'(a_0)\eta_0(\tau) \ge -1 + c\tau$, which implies that

$$\alpha(y(t_0(\varepsilon)+t)) \ge -\varepsilon + ct + \mathcal{O}(\varepsilon^2).$$

The solution satisfies $-\varepsilon \le \alpha(y(t)) \le 0$ on an interval $0 \le t \le \Delta t$ with $\Delta t \le \varepsilon/c + \mathcal{O}(\varepsilon^2)$.

3.4. Estimation of the remainder

For the estimation of the remainder in (10) it is convenient to introduce a new variable $v = \alpha(y)/\varepsilon$, so that after differentiation of $v(t)$ the problem (9) becomes equivalent to the singularly perturbed differential equation

$$\begin{aligned} \dot y(t) &= f(y(t), \dot\varphi(-\varepsilon) + \chi(v(t))(\dot y_0^+ - \dot\varphi(-\varepsilon))) \\ \varepsilon\,\dot v(t) &= \alpha'(y(t)) f(y(t), \dot\varphi(-\varepsilon) + \chi(v(t))(\dot y_0^+ - \dot\varphi(-\varepsilon))), \end{aligned} \tag{23}$$

provided that the initial value for $v(t)$ satisfies $v = \alpha(y)/\varepsilon$. The coefficient functions of Sections 3.2 and 3.3 have been constructed in such a way that the truncated asymptotic expansion, when inserted into the differential equation (23), has a defect of size $\mathcal{O}(\varepsilon^{N+1})$.

In the situation of Section 3.2 we assume (cf. condition (18))

$$\max_{0 \le \theta \le \theta_0} \alpha'(a_0) f_z(a_0, \dot y_0^- + \theta(\dot y_0^+ - \dot y_0^-))(\dot y_0^+ - \dot y_0^-) < 0 \tag{24}$$

This together with the monotonicity of $\chi(\tau)$ permits us to follow the proof of Theorem VI.3.2 of [8], and to conclude that the remainder in (10) is bounded by $\mathcal{O}(\varepsilon^{N+1})$. In the situation of Section 3.3 it is sufficient to work with a classical Gronwall inequality, because only intervals of length $\mathcal{O}(\varepsilon)$ have to be considered. We thus have proved the following result.

Theorem 1 (weak solution). *Consider the regularization of Section 2.1, and assume the existence of* $\theta_0 \in (0,1)$ *such that (14) and (24) hold. Furthermore, assume that the solution* $y_0(t)$, $u_0(t)$ *of the reduced problem (16)-(17) exists on an* ε*-independent interval* $[0,T]$, *and satisfies*

$$\max_{0 \le t \le T} \alpha'(y_0(t)) f_z(y_0(t), \dot y_0^- + u_0(t)(\dot y_0^+ - \dot y_0^-))(\dot y_0^+ - \dot y_0^-) < 0.$$

Then, the problem (9) has, for sufficiently small $\varepsilon > 0$, a unique solution on the interval $[t_0(\varepsilon), t_0(\varepsilon) + T]$, which is of the form (10). The coefficient functions are those constructed in Section 3.2. $\square$

Theorem 2 (transition to classical solution). *Consider the regularization of Section 2.1, and assume that (15) holds. Then, the problem (9) has, for sufficiently small $\varepsilon > 0$, a unique solution of the form*

$$y(t_0(\varepsilon) + t) = y_0 + \varepsilon \sum_{j=0}^{N-1} \varepsilon^j \eta_j(t/\varepsilon) + \mathcal{O}(\varepsilon^{N+1}), \qquad 0 \le t \le T(\varepsilon),$$

where $T(\varepsilon) = \mathcal{O}(\varepsilon)$ is such that $\alpha(y(t_0(\varepsilon) + T(\varepsilon))) = 0$, the constant value y_0 is given by (11), and the coefficient functions are those constructed in Section 3.3. The function $\eta_0(\tau)$ is monotonically increasing. $\square$

Example 1. We consider the neutral delay equation

$$\dot{y}(t) = f(\dot{y}(y(t) - 1)), \qquad f(z) = \gamma(1 - \beta_1 z)(1 - \beta_2 z), \tag{25}$$

with $y(t) = 0$ for $t \le 0$. The constants γ, $\beta_1 < \beta_2$ are assumed to be positive. The solution of the regularized problem (in the sense of Section 2.1) is $y(t) = \gamma t$ for $0 \le t \le t_0(\varepsilon) = \gamma^{-1}(1 - \varepsilon)$. Beyond this breaking point we have the following situations:

- $\beta_2 \gamma < 1$: there is only a classical solution;
- $\beta_1 \gamma < 1 < \beta_2 \gamma$: there is only a weak solution;
- $1 < \beta_1 \gamma$: there is a classical and a weak solution.

In the first two cases the solution of the regularized problem approximates correctly the unique solution. In the third case, it always approximates a weak solution. Moreover, it approaches monotonically the manifold defined by $\alpha(y) = y - 1 = 0$ (see Figure 1). This is in contrast to the regularization of Section 2.2, where the regularized solution turns out to approximate, depending on the choice of the parameters γ, β_1 and β_2, either the classical or the weak solution, and in the latter case it typically oscillates around a limit manifold (see [7]).

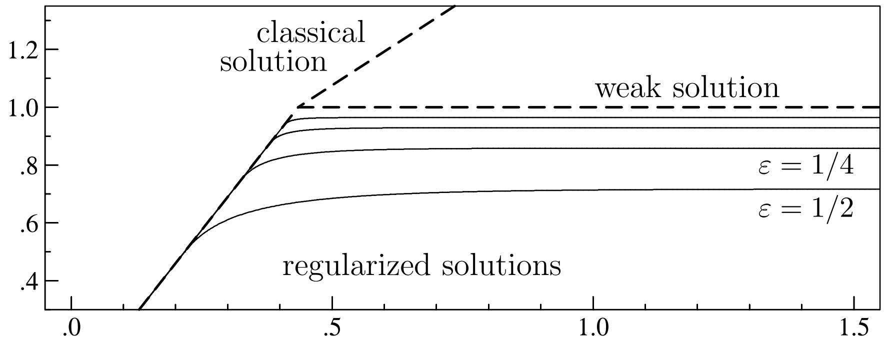

Figure 1: Solution of the regularized problem (in the sense of Section 2.1) with $\varepsilon = 1/2, 1/4, 1/8, 1/16$ for example 1 with parameters $\gamma = 2.3$, $\beta_1 = 0.6$, $\beta_2 = 1$.

3.5. Continuation of the solution

The expansions of Theorems 1 and 2 are valid on compact intervals as long as $\alpha(y(t))$ remains between $-\varepsilon$ and 0. We let $t_1(\varepsilon)$ be the first breaking point after $t_0(\varepsilon)$, and we distinguish between the following situations:

- $\alpha(y(t_1(\varepsilon))) = -\varepsilon$ and the solution continues in the region $\alpha(y) < -\varepsilon$. Here, the problem (1) is the ordinary differential equation, where $\dot{y}$ is replaced by $\dot{\varphi}$ in the right-hand side. Standard existence and uniqueness theory can be applied. The depencence on ε comes only through the initial value at $t_1(\varepsilon)$. For the next breaking point $t_2(\varepsilon)$ we are exactly in the same situation as at $t_0(\varepsilon)$.

- $\alpha(y(t_1(\varepsilon))) = 0$ and the solution continues in the region $0 < \alpha(y) < t_0(\varepsilon)$. Also here the expression $\dot{y}(\alpha(y(t)))$ is independent of ε, and the standard theory for ordinary differential equations is applicable. Denoting the next breaking point by $t_2(\varepsilon)$, we have the two subcases:

  $\alpha(y(t_2(\varepsilon))) = 0$ and the solution continues in the region $-\varepsilon < \alpha(y) < 0$. This case is identical to the one discussed in Sections 3.2 and 3.3. The only difference is that the initial value for $\widehat{\eta}_0$ is zero, and the function $\widehat{\eta}_0(\tau)$ is monotonically decreasing.

  $\alpha(y(t_2(\varepsilon))) = t_0(\varepsilon)$ and the solution continues in the region $t_0(\varepsilon) < \alpha(y) < t_1(\varepsilon)$. The difference to the former analysis is that the function $\dot{\varphi}(-\varepsilon) + \chi(t/\varepsilon)(\dot{y}_0^+ - \dot{\varphi}(-\varepsilon))$ in the argument of (9) is replaced by the asymptotic expansion (10), which is a smooth function of t/ε. Its dominating term is monotonically increasing as it is assumed for $\chi(\tau)$.

The analysis at further breaking points is similar and thus omitted.

4. Analysis of the regularization of Section 2.2

The regularization of Section 2.2 was introduced and analyzed in [7]. One of the aims of the present work is to compare this regularization with that of Section 2.1, and to extend the analysis of [7] to subsequent breaking points.

The main difference between the two regularizations is the treatment of the discontinuity in the derivative at the starting point. Whereas the regularization of Section 2.1 modifies the derivative $\dot\varphi(t)$ of the starting function on an interval of length ε just before the starting point 0 and leaves the equation unchanged right after it, the regularization of Section 2.2 does not touch the initial function nor its derivative, but replaces the differential-algebraic system (3) having inconsistent initial values by the singularly perturbed problem (5). This removes the discontinuity in the derivative by introducing a transient layer immediately after the initial point.

4.1. Asymptotic expansion after the first breaking point

The first breaking point $t_0(\varepsilon)$ appears when $\alpha(y(t))$ crosses zero, i.e., when $\alpha(t_0(\varepsilon) = 0$. At this point the solution possesses an asymptotic expansion

$$y(t_0(\varepsilon)) = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \dots, \qquad \dot y(t_0(\varepsilon)) = b_0 + \varepsilon b_1 + \varepsilon^2 b_2 + \dots . \tag{26}$$

The analysis of [7] shows that on an ε-independent non-empty interval the solution has the form

$$y(t_0(\varepsilon)+t) = \sum_{j=0}^{N} \varepsilon^j y_j(t) + \varepsilon \sum_{j=0}^{N-1} \varepsilon^j \eta_j(t/\varepsilon) + \mathcal{O}(\varepsilon^{N+1}), \tag{27}$$

as it is the case in the analysis of Section 3. The smooth function $y_0(t)$ is the same as before, but there is a significant difference in the function $\eta_0(\tau)$ which dominates the transient layer after the breaking point. The function $\widehat\eta_0(\tau) = \alpha'(a_0)(y_1(0)+\eta_0(\tau))$ which, apart from a constant, is the component of $\eta_0(\tau)$ that is orthogonal to the manifold $\alpha(y)=0$, is the solution of the scalar second order differential equation[2]

$$\widehat\eta_0{}''(\tau) = -\,\widehat\eta_0{}'(\tau) + g(\max(0,\widehat\eta_0(\tau))) \tag{28}$$

[2]Recall that for the regularization of Section 2.1, $\widehat\eta_0(\tau)$ is a solution of the scalar first order differential equation (20), which can be written as $\widehat\eta_0{}'(\tau) = g(1-\chi(\widehat\eta_0(\tau)))$.

with initial values $\widehat{\eta}_0(0) = 0$, $\widehat{\eta}_0'(0) = \alpha'(a_0)b_0 > 0$, where

$$g(\eta) = \alpha'(a_0) f(a_0, \dot{y}_0^+ + \mathrm{e}^{-\eta}(\dot{y}_0^- - \dot{y}_0^+)). \tag{29}$$

It is proved in [7] that the solution of the initial value problem (28) completely determines the behavior of the regularized solution. There are exactly two possibilities: either the solution $(\widehat{\eta}_0(\tau), \widehat{\eta}_0'(\tau))$ of (28) converges to a stationary point $(c, 0)$ satisfying $g(c) = 0$ with $c > 0$, or it approaches exponentially fast the linear function $\widehat{\eta}_0(\tau) \approx g(+\infty)\tau$ (with $g(+\infty) > 0$) when $\tau \to \infty$. In the first case, the solution of the singularly perturbed problem (5) approaches the weak solution of (1), which satisfies $\dot{y}(t_0) = f(y(t_0), \dot{y}_0^+ + \mathrm{e}^{-c}(\dot{y}_0^- - \dot{y}_0^+))$. In the second case, it approaches the classical solution of (1).

The main differences between the regularization of Section 2.1, which we abbreviate (INI) because it modifies the initial function, and the regularization of Section 2.2, which we denote (SPD) because it is defined via a singularly perturbed differential equation, are the following:

- In the co-existence of weak and classical solutions, the regularization (INI) always approaches a weak solution. More precisely, it approaches the weak solution for which the second argument in the derivative of the solution $\dot{y}(t_0) = f(y(t_0), \dot{y}_0^- + \theta_0(\dot{y}_0^+ - \dot{y}_0^-))$ is closest to $\dot{y}_0^-$.
- In the co-existence of weak and classical solutions, the regularization (SPD) can approach a weak or a classical solution, depending on the solution of the initial value problem (28). This is illustrated in Figure 2 at the neutral equation of Example 1. Both, the weak and the classical

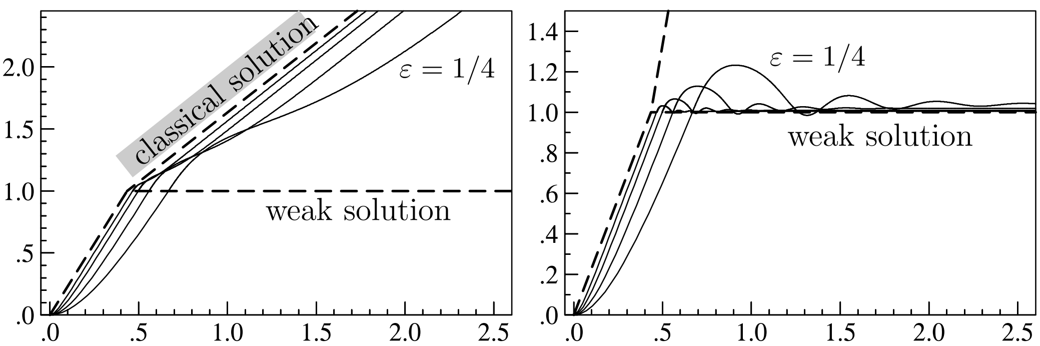

Figure 2: Solution of the regularized problem (in the sense of Section 2.2) with $\varepsilon = 1/4, 1/8, 1/16, 1/32$ of example 1 with parameters $\gamma = 2.3$, $\beta_1 = 0.6$, $\beta_2 = 1$ (left picture) and $\gamma = 2.3$, $\beta_1 = 0.6$, $\beta_2 = 3$ (right picture).

solution are shown by a broken line. With data corresponding to the left picture, the classical solution is approximated, whereas in the situation of the right picture the weak solution is approximated. Notice that the data of the left picture of Figure 2 are the same as those of Figure 1.

- The transition at the first breaking point is monotone for the regularization (INI). When a weak solution is approximated, the regularized solution never crosses the manifold $\alpha(y) = 0$ close to this breaking point.
- The transition at the first breaking point shows in general damped high oscillations of amplitude $\mathcal{O}(\varepsilon)$ for the regularization (SPD). The regularized solution always crosses the manifold $\alpha(y) = 0$ at least once. When a weak solution is approximated, the expression $\alpha(y(t))$ for the regularized solution can change sign several times on an interval of length $\mathcal{O}(\varepsilon)$ before it stays positive on an ε-independent interval.
- With the regularization (INI), breaking points of the neutral problem (1) induce pairs of breaking points in the regularized equation (the first, when $\alpha(y(t))$ crosses $-\varepsilon$, and the second, when it crosses 0). This is not the case for the regularization (SPD).

4.2. Asymptotic expansion at subsequent breaking points

The article [7] gives a detailed analysis for the regularization (SPD) at the first breaking point and, in the case of a weak solution, at the point where the solution leaves the manifold $\alpha(y) = 0$. This is done by the study of various kinds of asymptotic expansions separating smooth and transient parts of the solution. Here, we indicate an extension of this analysis to the situation of subsequent breaking points, i.e., points t_1, for which the solution of the neutral delay equation (1) satisfies $\alpha(y(t_1)) = t_0$.

A breaking point $t_1(\varepsilon)$ that is induced by the previous breaking point $t_0(\varepsilon)$ is defined by (invoking the implicit function theorem)

$$\alpha(y(t_1(\varepsilon))) = t_0(\varepsilon) = t_0 + \varepsilon t_0^1 + \varepsilon^2 t_0^2 + \dots \ .$$

The solution at $t_1(\varepsilon)$ of the regularized problem admits an expansion

$$y(t_1(\varepsilon)) = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \dots , \qquad \dot y(t_1(\varepsilon)) = b_0 + \varepsilon b_1 + \varepsilon^2 b_2 + \dots \ .$$

The coefficients a_j and b_j are different from those in (26), but they will play the same role. The same argumentation as for the first breaking point leads to $\alpha'(a_0)a_1 = t_0^1$ and $\alpha'(a_0)b_0 > 0$. Beyond the next breaking point we consider the asymptotic expansion

$$y(t_1(\varepsilon)+t) = \sum_{j=0}^{N} \varepsilon^j y_j(t) + \varepsilon \sum_{j=0}^{N-1} \varepsilon^j \eta_j(t/\varepsilon) + \mathcal{O}(\varepsilon^{N+1}), \tag{30}$$

where we again do not change the notation of the coefficient functions. Inserting this expansion and its term-by-term derivative into the singularly perturbed equation (5), and comparing like powers of ε in the transient and smooth parts, yields equations that are very much the same as those obtained in Section 5 of [7]. The only difference comes from the fact that the argument $\alpha(y(t))$ of the delay term is no longer close to zero, but it is now close to $t_0(\varepsilon)$. This affects only the function $\widehat{\eta}_0(\tau) = \alpha'(a_0)(y_1(0) + \eta_0(\tau))$, which also satisfies the differential equation (28), but the initial values are $\widehat{\eta}_0(0) = \alpha'(a_0)a_1 = t_0^1$ and $\widehat{\eta}_0'(0) = \alpha'(a_0)b_0 > 0$, and the function $g(\eta)$ is now given by

$$g(\eta) = \alpha'(a_0) f(a_0, \dot{y}_0^+ + \widetilde{\zeta}_0(\eta)), \tag{31}$$

where $\widetilde{\zeta}_0(\tau)$ is the derivative with respect to τ of the known function $\eta_0(\tau)$, which is the leading transient term in the asymptotic expansion (10) after the first breaking point. This function satisfies $\widetilde{\zeta}_0(0) = \dot{y}_0^- - \dot{y}_0^+$ and it converges exponentially fast to zero for $\tau \to +\infty$. Here, the vectors $\dot{y}_0^-$ and $\dot{y}_0^+$ represent the left-hand and right-hand derivative of the solution of the neutral equation (1) at the first breaking point t_0.

In [7] the equation (28) is studied for the case where $g(\eta)$ is given by (29), and it turned out to be advantageous to introduce a new variable $\theta = \mathrm{e}^{-\eta}$ in place of η. This can also be done for the function (31), so that all statements and proofs carry over to the present situation. In particular, there are again exactly two possibilities for the solution of (28) with the new function $g(\eta)$: either it converges to a finite stationary point and the regularization approximates a weak solution after the breaking point $t_1(\varepsilon)$, or it tends to infinity like $g(+\infty)\tau$ and the regularization approximates a classical solution. An extension to subsequent breaking points is straightforward.

It is worth to remark that several phenomena occur over different time scales ($\mathcal{O}(1)$, $\mathcal{O}(\varepsilon)$ and $\mathcal{O}(\sqrt{\varepsilon})$) - see [7] for details - giving rise to a rich multiscale dynamics which presents very interesting analogies to some phenomena discussed in [11] for a singularly perturbed scalar delay differential equation.

5. Numerical experiments with RADAR5

Both regularizations of the present work are useful for a numerical treatment of state-dependent neutral delay equations, because they transform the problem into a (non-neutral) state-dependent delay equation, so that standard software can be applied. Notice, however, that for very small $\varepsilon > 0$ the resulting equations are stiff, which restricts the class of integrators. In this section we present some experiments with the code RADAR5 of [5].

5.1. A system of Lotka–Volterra-like equations

We consider a model similar to that considered in [9, 10] to describe the interaction between two competing species. This kind of model has received the attention of researchers in the last few years since it seems natural to include a memory effect in such a type of interplay. The main modification we include here is that we make use of a state-dependent delay (α_1) in contrast to the fact that usually simply constant delays are considered. The equations we consider (for $t \geq 0$) are the following:

$$\begin{aligned} \dot{y}_1(t) &= r_1\, y_1(t) \Big(1 - \rho\, \dot{y}_1(\alpha_1(y_1(t))) - a\, y_2(\alpha_2(t))\Big) \\ \dot{y}_2(t) &= r_2\, y_2(t) \Big(\frac{1}{1 - b\, y_1(t)} - y_2(t) - c\, y_1(\alpha_2(t))\Big) \end{aligned} \tag{32}$$

where the deviating arguments are

$$\alpha_1(y_1(t)) = y_1(t) - 0.5 \qquad \text{and} \qquad \alpha_2(t) = t - 5,$$

and the positive constants in the model are chosen as $r_1 = r_2 = 1$, $\rho = 3$, $a = 0.2$, $b = 1.8$, and $c = 0.25$.

We also consider as initial functions, for $t \leq 0$, $y_1(t) = 0.33 - t/10$, $y_2(t) = 2.22 + t/10$. It turns out that a classical solution ceases to exist at the breaking point $\xi \approx 0.4366338$. For this reason we consider the regularized system, according to Section 2.2,

$$\begin{aligned} \dot{y}_1(t) &= z_1(t) \\ \varepsilon\, \dot{z}_1(t) &= r_1\, y_1(t) \Big(1 - \rho\, z_1(\alpha_1(y_1(t))) - a\, y_2(\alpha_2(t))\Big) - z_1(t) \\ \dot{y}_2(t) &= r_2\, y_2(t) \Big(\frac{1}{1 - b\, y_1(t)} - y_2(t) - c\, y_1(\alpha_2(t))\Big) \end{aligned} \tag{33}$$

which coincides with (32) if $\varepsilon = 0$.

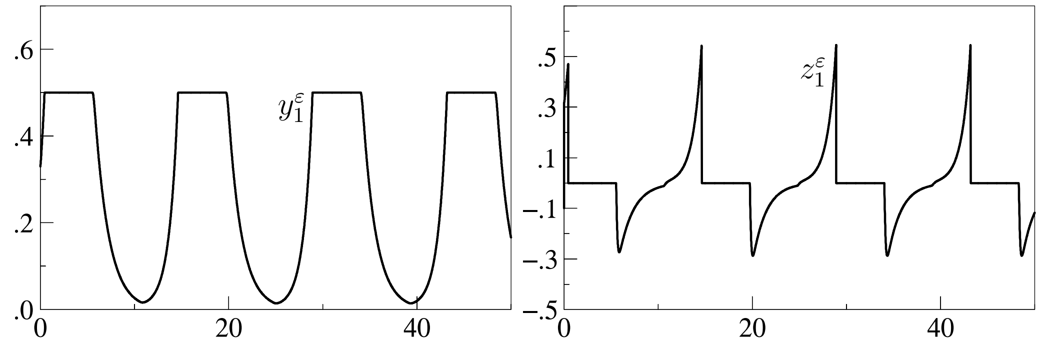

Figure 3: The first solution component of the regularized problem (33) (left) and its derivative (right).

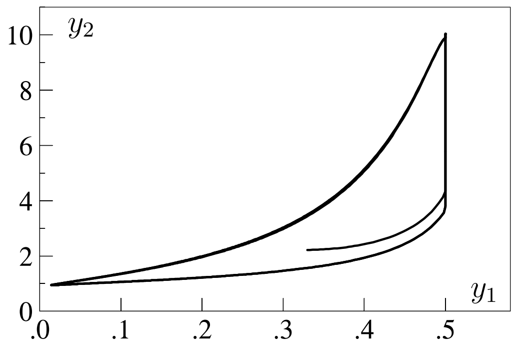

Figure 4: Solution of the regularized problem (33) in the phase space (y_1, y_2).

We denote by $y_1^\varepsilon(t), z_1^\varepsilon(t)$ and $y_2^\varepsilon(t)$ the solution of (33). Figures 3 and 4 show pictures of the solution computed by RADAR5 for $\varepsilon = 10^{-4}$. Note that when $y_1(t)$ is constant it corresponds to the fact that $\alpha_1(y_1(t))$ remains constant which implies that we are in the presence of a weak solution. This appears clearly in Figure 3 and also in Figure 4 where the vertical branch $y_1^\varepsilon(t) \approx 0.5$ identifies a regime of weak solution. The observed dynamics is interesting since it consists of periodic solutions alternating classical and weak branches. In Figure 4 a limit cycle is observed with the typical shape of Lotka–Volterra systems, that is phases where the magnitudes of the populations have alternatively small and large sizes.

Using the regularization of Section 2.1 provides very similar pictures.

We conclude that by means of the considered regularizations we are able to approximate generalized periodic solutions of mixed type (classical/weak) in a simple way, which does not need any modification of the existing code.

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