Is the Additive Schwarz Method with Harmonic Extension Just Parallel Schwarz in Disguise?

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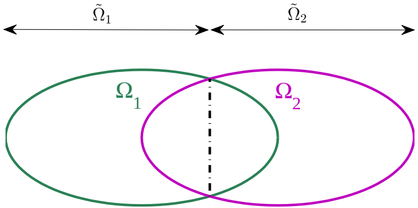
Outline

(Discretized) Parallel Schwarz Method



$$A_{11} u_1^{k+1} = f_1 - A_{12} u_2^k \qquad \text{(on } \Omega_1\text{)}$$
$$A_{22} u_2^{k+1} = f_2 - A_{21} u_1^k \qquad \text{(on } \Omega_2\text{)}$$

- Lions (1988)
- Works on *subdomain solutions* u_1^k, u_2^k
- No built-in notion of global solution

Additive Schwarz (Dryja & Widlund, 1988)

$$U^{k+1} = U^k + \sum_j R_j^T A_j^{-1} R_j (f - AU^k)$$

1. Starts with *global solution* U^k and *global residual* $f - AU^k$

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2. Restricts residual to subdomains
3. Solves subdomain problems in parallel
4. Update U^k by adding all the subdomain solutions

- Does not converge within the overlap as an iterative method (Efstathiou & Gander 2003)
- OK as preconditioner when used with Krylov

Restricted Additive Schwarz (RAS)

$$U^{k+1} = U^k + \sum_j \tilde{R}_j^T A_j^{-1} R_j (f - AU^k)$$

$$\tilde{R}_j = \begin{cases} I & \text{on } \tilde{\Omega}_j \\ 0 & \text{on } \Omega_j \setminus \tilde{\Omega}_j \end{cases}$$

- Cai & Sarkis (1999)
- Removes redundant updates
- Clear interpretation as iteration on subdomains (Gander 2008)

Additive Schwarz with Harmonic Extension (ASH)

$$U^{k+1} = U^k + \sum_j R_j^T A_j^{-1} \tilde{R}_j (f - AU^k)$$

- Cai & Sarkis (1999)
- Removes redundancy on *residual* rather than on the update
- "Transpose" of RAS
- Interpretation as iteration on subdomains?

Previous work

- Convergence of RAS and ASH analyzed for M-matrix case (Frommer & Szyld 2000)
- Related method RASHO (= Restricted Additive Schwarz preconditioner with Harmonic Overlap) analyzed by Cai, Dryja and Sarkis (2003)

Assumption

- No cross points, i.e., every grid point must belong to *at most* two subdomains.
- Algebraically, require

$$(R_i - \tilde{R}_i)(AR_j^T - R_j^T A_j) = 0$$
$$(R_i A - A_i R_i)(R_j^T - \tilde{R}_j^T) = 0$$

for all $i \neq j$.

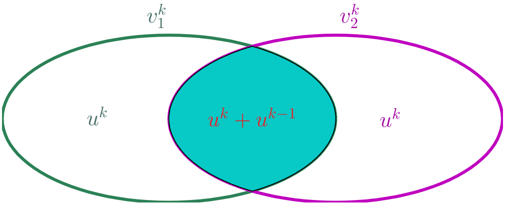
Main Theorem

Suppose the domain Ω is subdivided so that there are no cross points. Then the iterates u^k of the ASH method is related to the iterates of v^k of the discretized parallel Schwarz method

$$\begin{aligned} v_i^0 &= 0, \\ A_i v_i^1 &= \tilde{R}_i f, \\ A_i v_i^k &= R_i f - A_{i\Gamma} v^{k-1} \qquad \text{for } k \geq 2 \end{aligned}$$

via the relation

$$\sum_j R_j^T v_j^k = u^k + \Big[\sum_j R_j^T R_j - I\Big] u^{k-1}.$$



$$\sum_j R_j^T v_j^k = u^k + \left[\sum_j R_j^T R_j - I\right] u^{k-1}$$

- u^k and v_j^k is identical outside the overlap
- Fixes the problem of adding too much in additive Schwarz

- For $k \geq 2$,

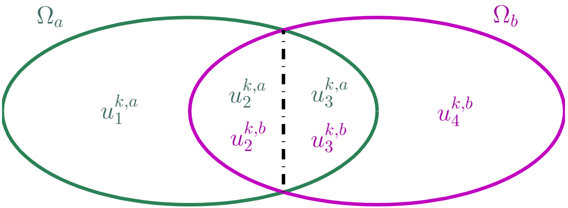
$$A_i v_i^k = R_i f - A_{i\Gamma} v^{k-1}$$

is the discretized version of the subdomain solve

$$\begin{aligned} \mathcal{L} v_i^k &= f, \quad x \in \Omega_j \\ v_i^k &= v_j^{k-1}, \quad x \in \Gamma_{ij}. \end{aligned}$$

- Boundary values well defined since we disallow cross points

Sketch of Proof



- Assume two subdomains (easy to generalize to multiple subdomains with no cross points)
- Main obstacle: boundary values “blanked out” by $\tilde{R}_i$
- Key idea: Add back the missing pieces using solution from the previous iteration

Sketch of Proof

- For $k = 1$:

$$\left.\begin{aligned} A_{11}u_1^{1,a} + A_{12}u_2^{1,a} \phantom{+ A_{23}u_3^{1,a}} &= f_1 \\ A_{21}u_1^{1,a} + A_{22}u_2^{1,a} + A_{23}u_3^{1,a} &= f_2 \\ A_{32}u_2^{1,a} + A_{33}u_3^{1,a} &= 0 \end{aligned}\right\} \text{ in } \Omega_a$$

$$\left.\begin{aligned} A_{22}u_2^{1,b} + A_{23}u_3^{1,b} &= 0 \\ A_{32}u_2^{1,b} + A_{33}u_3^{1,b} + A_{34}u_4^{1,b} &= f_3 \\ A_{43}u_3^{1,b} + A_{44}u_4^{1,b} &= f_4 \end{aligned}\right\} \text{ in } \Omega_b$$

- Global solution U^1 defined as

$$U_1^1 = u_1^{1,a}, \quad U_2^1 = u_2^{1,a} + u_2^{1,b}, \quad U_3^1 = u_3^{1,a} + u_3^{1,b}, \quad U_4^1 = u_4^{1,b}$$

- Residual $R^1 = f - AU^1$ satisfies

$$R_1^1 = f_1 - A_{11} u_1^{1,a} - A_{12}(u_2^{1,a} + u_2^{1,b}) = -A_{12} u_2^{1,b}$$

since

$$\boxed{A_{11} u_1^{1,a} + A_{12} u_2^{1,a} = f_1.}$$

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$$R_2^1 = f_2 - A_{21}u_1^{1,a} - A_{22}(u_2^{1,a} + u_2^{1,b}) - A_{23}(u_3^{1,a} + u_3^{1,b}) = 0$$

since

$$\boxed{A_{21}u_1^{1,a} + A_{22}u_2^{1,a} + A_{23}u_3^{1,a} = f_2, \quad A_{22}u_2^{1,b} + A_{23}u_3^{1,b} = 0}$$

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$$R_3^1 = f_3 - A_{32}(u_2^{1,a} + u_2^{1,b}) - A_{33}(u_3^{1,a} + u_3^{1,b}) - A_{34}u_4^{1,b} = 0$$

since

$$A_{32}u_2^{1,a} + A_{33}u_3^{1,a} = 0, \quad A_{32}u_2^{1,b} + A_{33}u_3^{1,b} + A_{34}u_4^{1,a} = f_3.$$

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- Residual $R^1 = f - AU^1$ satisfies

$$\begin{aligned}
R_1^1 &= f_1 - A_{11}u_1^{1,a} - A_{12}(u_2^{1,a} + u_2^{1,b}) = -A_{12}u_2^{1,b} \\
R_2^1 &= f_2 - A_{21}u_1^{1,a} - A_{22}(u_2^{1,a} + u_2^{1,b}) - A_{23}(u_3^{1,a} + u_3^{1,b}) = 0 \\
R_3^1 &= f_3 - A_{32}(u_2^{1,a} + u_2^{1,b}) - A_{33}(u_3^{1,a} + u_3^{1,b}) - A_{34}u_4^{1,b} = 0 \\
R_4^1 &= f_4 - A_{43}(u_3^{1,a} + u_3^{1,b}) - A_{44}u_4^{1,b} = -A_{43}u_3^{1,a}
\end{aligned}$$

since

$$\boxed{A_{43}u_3^{1,b} + A_{44}u_4^{1,b} = f_4.}$$

- For $k = 2$, the global solution is

$$U^2 = U^1 + \delta u^{2,a} + \delta u^{2,b},$$

where

$$\begin{aligned} A_{11}\delta u_1^{2,a} + A_{12}\delta u_2^{2,a} &= -A_{12} u_2^{1,b} \ (= R_1^1) \\ A_{21}\delta u_1^{2,a} + A_{22}\delta u_2^{2,a} + A_{23}\delta u_3^{2,a} &= 0 \ (= R_2^1) \\ A_{32}\delta u_2^{2,a} + A_{33}\delta u_3^{2,a} &= 0 \end{aligned}$$

- For $k = 2$, the global solution is

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where

$$\begin{aligned}
A_{11}\delta u_1^{2,a} + A_{12}\delta u_2^{2,a} \quad &= -A_{12}u_2^{1,b} \; (= R_1^1) \\
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A_{32}\delta u_2^{2,a} + A_{33}\delta u_3^{2,a} \quad &= 0
\end{aligned}$$

- Recall that $u^{1,a}$ satisfies

$$\begin{aligned}
A_{11}\; u_1^{1,a} + A_{12}\; u_2^{1,a} \quad &= f_1 \\
A_{21}\; u_1^{1,a} + A_{22}\; u_2^{1,a} + A_{23}\; u_3^{1,a} \quad &= f_2 \\
A_{32}\; u_2^{1,a} + A_{33}\; u_3^{1,a} \quad &= 0
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Sketch of Proof

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A_{11} u_1^{2,a} + A_{12}\ u_2^{2,a} & & &= f_1 - A_{12} u_2^{1,b} \\
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\end{aligned}$$

- Add the two sets of equations!

Sketch of Proof

$$\begin{aligned}
A_{11} u_1^{2,a} + A_{12}\, u_2^{2,a} &= f_1 - A_{12} u_2^{1,b} \\
A_{21} u_1^{2,a} + A_{22}(u_2^{2,a} + \color{red}{u_2^{1,b}}) + A_{23}(u_3^{2,a} + \color{red}{u_3^{1,b}}) &= f_2 \\
A_{32}\, u_2^{2,a} + A_{33}\, u_3^{2,a} &= 0
\end{aligned}$$

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Sketch of Proof

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\begin{aligned}
A_{11}u_1^{2,a} + A_{12}(u_2^{2,a} + u_2^{1,b}) &= f_1 \\
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- Add the two sets of equations!
- Define

$$
v_1^{k,a} = u_1^{k,a}, \quad v_2^{k,a} = u_2^{k,a} + u_2^{k-1,b}, \quad v_3^{k,a} = u_3^{k,a} + u_3^{k-1,b}
$$

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Sketch of Proof

- For $k = 2$:

$$\left.\begin{aligned} A_{11} v_1^{2,a} + A_{12} v_2^{2,a} &= f_1 \\ A_{21} v_1^{2,a} + A_{22} v_2^{2,a} + A_{23} u_3^{2,a} &= f_2 \\ A_{32} v_2^{2,a} + A_{33} v_3^{2,a} &= f_3 - A_{34} u_4^{1,b} \end{aligned}\right\} \text{ in } \Omega_a$$

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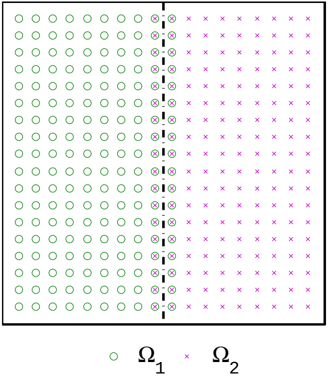
- Parallel Schwarz!
- Use induction to prove result for all k

Consequence

- As an iterative method, ASH and Parallel Schwarz converge at the same rate.

Example

- Solve $-\Delta u = f$ on $[-1, 1] \times [-1, 1]$
- Homogeneous Dirichlet B.C.
- 20×20 grid
- Ω_1 and Ω_2 has a two-row overlap



Example

Its.	Parallel Schwarz				ASH	
	Error(Ω_1)	Ratio	Error(Ω_2)	Ratio	Error	Ratio
1	60.4103		61.2159		69.0920	
2	35.5724	0.5888	35.0937	0.5733	40.1592	0.5812
3	20.4046	0.5736	20.6840	0.5894	23.3519	0.5815
4	12.0281	0.5895	11.8656	0.5737	13.5796	0.5815
5	6.9001	0.5737	6.9947	0.5895	7.8969	0.5815
6	4.0676	0.5895	4.0126	0.5737	4.5923	0.5815
7	2.3335	0.5737	2.3654	0.5895	2.6706	0.5815
8	1.3756	0.5895	1.3570	0.5737	1.5530	0.5815
9	0.7891	0.5737	0.7999	0.5895	0.9031	0.5815
10	0.4652	0.5895	0.4589	0.5737	0.5252	0.5815

- $\sqrt{0.5895 \cdot 0.5737} = 0.5815$ (!)

Conclusion

- As iterative methods, Parallel Schwarz and ASH are closely related and have the same convergence rate.
- Proof is algebraic and does not depend on features of the PDE or boundary conditions.
- Argument can be extended to treat optimized transmission conditions.
- Interpretation for decompositions with cross points?