Potential Ordering Methods for Nonlinear Solution of Three-Phase Flow in Porous Media

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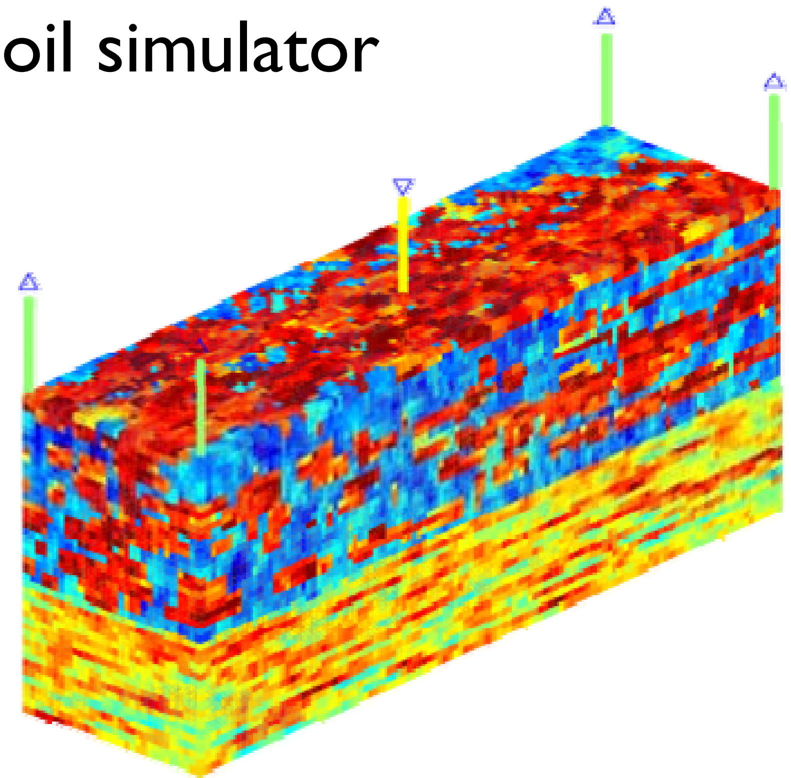
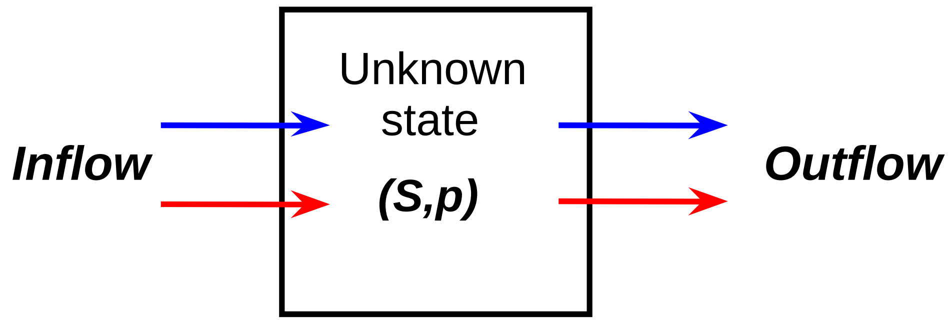
Objective

- Build the fastest possible simulator for fully-implicit black oil simulator
- Exploit physics of Darcy flow
- Algorithms should be easy to implement within existing simulators



Black Oil Equations



$$\frac{\partial(\phi S_p)}{\partial t} - \nabla \cdot \left[K\lambda_p(S)\nabla(P-\gamma_p z)\right] = q_p, \quad p = o, w, g$$

- Mass-balance equations (*In – out = Accum.*)
- Upstream weighting for *S*
- Implicit time discretization
- Must solve nonlinear system to get S^{n+1}, p^{n+1}

Black Oil Equations

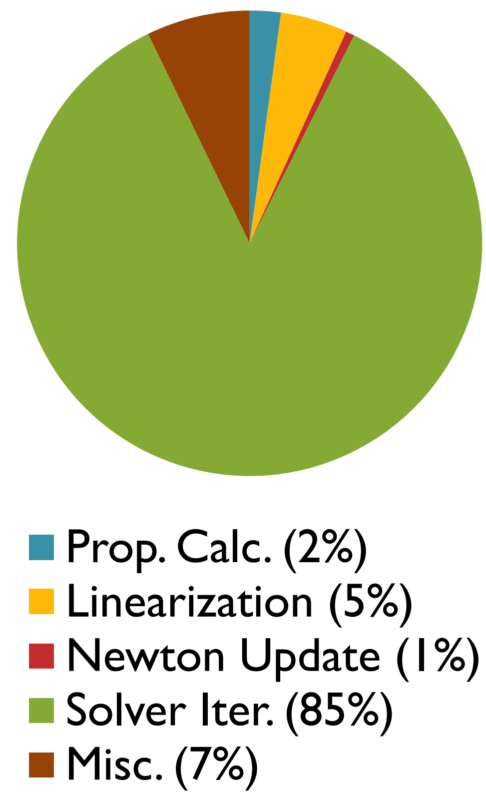
$$\frac{\partial(\phi S_p)}{\partial t} - \nabla \cdot \left[K\lambda_p(S)\nabla\left(P - \gamma_p z\right)\right] = q_p, \quad p = o, w, g$$

- System of PDEs (one equation per phase)
- Pressure-driven flow
- Heterogeneous media
- Nonlinear coupling between flow and transport
- Non-convex (and possibly non-monotonic) flux functions
- External forces (gravity)

Where does all the time go?

- Bottlenecks:
  - Solution of large nonlinear systems due to time-implicit discretization
  - Solution of linear systems (Newton's method)

Simulation time
(2-phase, 1 million grid blocks)



What are the challenges?

- Nonlinear solver:
  - Newton's method is locally quadratically convergent, BUT…
  - May diverge if initial guess is poor.
  - Can use previous time step as initial guess
  - Restriction on Δt
  - Want to choose Δt based on accuracy, not stability of nonlinear solver

What are the challenges?

- Linear solver:
  - Jacobians for fully-implicit discretizations are *highly non-symmetric*, *indefinite* and *ill-conditioned*
  - Operator contains both *elliptic* (pressure-driven flow) and *hyperbolic* (transport) components
  - Standard preconditioning techniques (ILU, multigrid, SPAI) are inadequate

Key Observations

- Two different mechanisms
- Flow:
  - Pressure driven
  - Globally coupled via elliptic PDE
- Transport:
  - Directional, based on pressure field
  - Acyclic: edges always go from high to low pressure
  - Upstream weighting allows partial decoupling of nonlinear system

Outline

- Potential ordering
- Application to nonlinear solver:
  - *Reduced-order Newton method*
- Application to linear solver:
  - *Ordering for CPR preconditioner*

POTENTIAL ORDERING

Jacobian Matrix

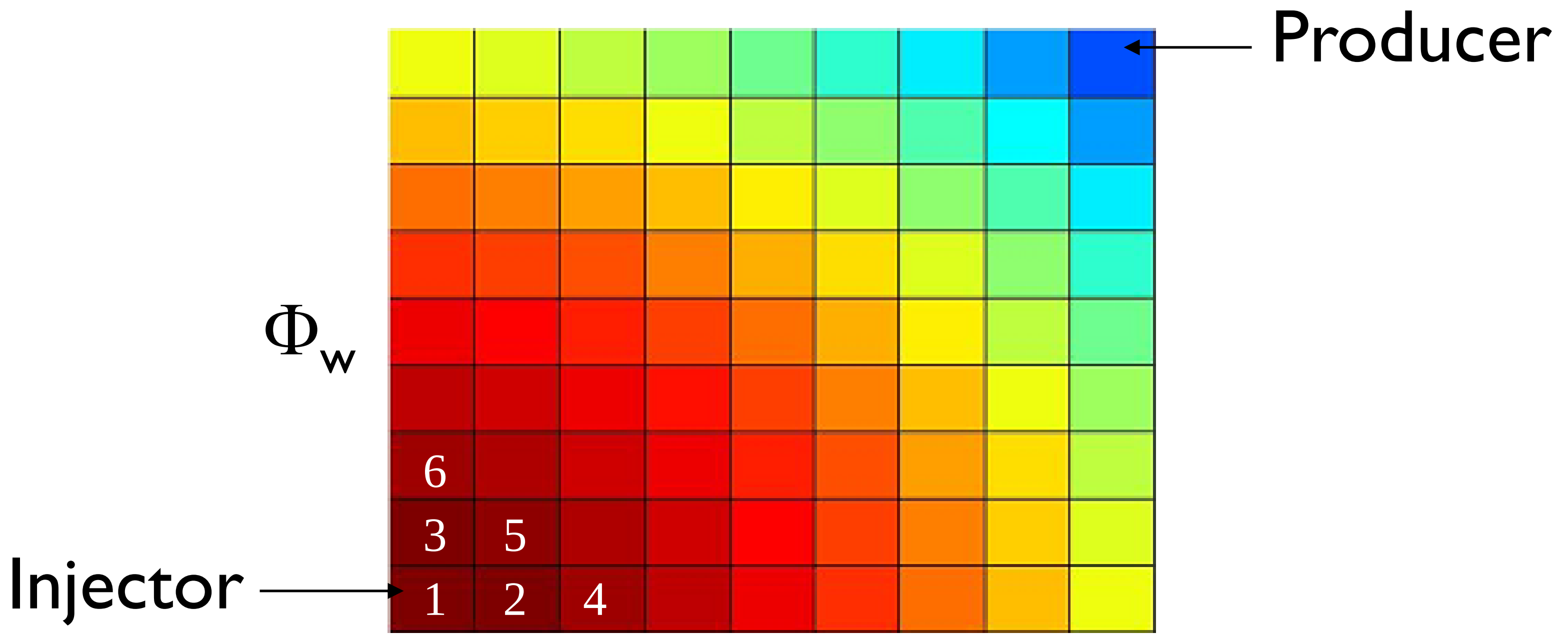
- Three-phase flow:

$$\begin{array}{c} \\ w \\ o \\ g \end{array} \quad \begin{array}{c} \begin{array}{ccc} S_w & S_g & p \end{array} \\ \begin{bmatrix} J_{ww} & 0 & J_{wp} \\ J_{ow} & J_{og} & J_{op} \\ J_{gw} & J_{gg} & J_{gp} \end{bmatrix} \end{array} \begin{bmatrix} \delta S_w \\ \delta S_g \\ \delta p \end{bmatrix} = - \begin{bmatrix} R_w \\ R_o \\ R_g \end{bmatrix}$$

- Water component independent of gas phase
- Reorder to make J_{ww} *and* J_{og} lower triangular (possible because of upstream differencing)
- Ordering for water and oil *can be different*

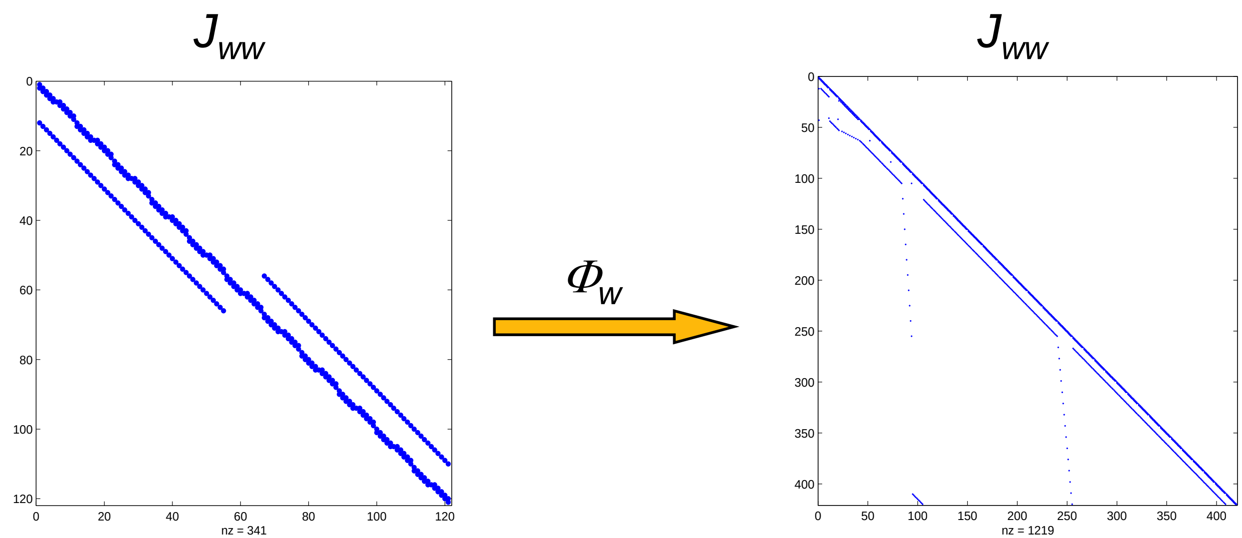
Potential Ordering

- Use flow directions to reorder equations and variables:



Potential Ordering

- Order the water equations and S_w by decreasing Φ_w:



- Use Φ_o to order oil equations and S_g

Potential Ordering

- After reordering:

$$J = \begin{bmatrix} \triangle & & J_{wp} \\ J_{ow} & \triangle & J_{op} \\ J_{gw} & J_{gg} & J_{gp} \end{bmatrix} = \begin{bmatrix} \triangle & J_{ps} \\ J_{sp} & J_{pp} \end{bmatrix}$$

  (Note: J_{ow} may not be triangular!)
- Once pressure is known, *back substitute* to get saturations

Potential Ordering

- Triangulation possible for any flow configuration:
  - Cocurrent flow
  - Countercurrent flow due to buoyancy
  - Countercurrent flow due to capillarity
  - Two- or three-phase flow

- Ordering ideas studied in:
  - Multiphase flow: Appleyard & Cheshire (1980), Natvig et al. (2006)
  - Navier-Stokes: Chin *et al.* (1992), Meister & Vömel (2001)

REDUCED-ORDER NEWTON METHOD

Reduced Newton Iteration

- If pressure is known, J_{ss} lower triangular

$\Rightarrow$ We can solve $F_s(S,p) = 0$ for S *one unknown at a time.*

$$
\begin{aligned}
&F_{s1}(S_{w1}, && ,p) = 0 \\
&F_{s2}(S_{w1}, S_{w2}, && ,p) = 0 \\
&\quad\vdots \\
&\quad\vdots \\
&F_{sN}(S_{w1}, S_{w2}, && S_{wN}, p) = 0, \\
\text{and } &F_{gi}(S_{w1}, S_{w2}, && S_{wN}, p) = 0, \quad i = 1,2,\ldots,N.
\end{aligned}
$$

- In other words, $S = S(p)$.

Reduced Newton Iteration

- Solve the remaining $N \times N$ system

$$F_g\,(S\,(p), p) = 0$$

  for pressure p using Newton's method
- Advantages:
  - Reduces order by a factor of N_p (= no. of phases)
  - Retains quadratic convergence
  - Resolves strongest nonlinearity during back substitution

Reduced Newton – Algorithm

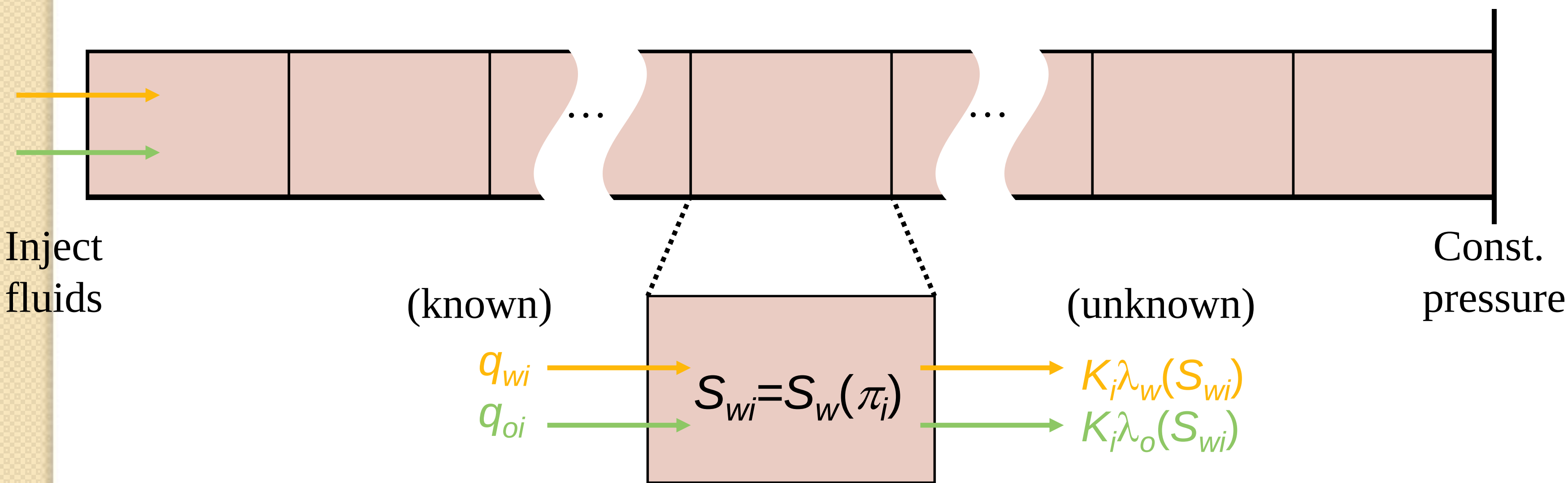
- While not converged, do:
  1. Compute cell ordering for each phase $p = w, o, g$;
  2. Evaluate Jacobian $J = \begin{bmatrix} J_{ss} & J_{sp} \\ J_{ps} & J_{pp} \end{bmatrix}$ at $(S(p^k), p^k)$;
  3. Solve $(J_{pp} - J_{ps} J_{ss}^{-1} J_{sp})\, \delta p^k = -r^k$;
  4. Compute $p^{k+1} = p^k + \delta p^k$;
  5. Update $S^{k+1} = S(p^{k+1})$ *nonlinearly* by solving $F_s(S^{k+1}, p^{k+1})$ *one variable at a time*.

Comparisons

Reduced Newton	Seq. Implicit Method
Full coupling between saturation and pressure	Frozen total velocity field
Exact mass conservation for all phases, for any Δt	Mass-balance errors that grow with Δt for some phases (see Aziz & Settari)
Saturations calculated by single-cell solves, one unknown at a time	Globally coupled saturation solves (when countercurrent flow is present)

Convergence Analysis

- Rigorous analysis possible for two-phase 1-D flow



$$F_{wi} = (S_i - S^0)/\Delta t + K_i\lambda_w(S_i)\pi_i - q_{wi} \equiv 0 \quad \text{(constraint)}$$

$$F_{oi} = -(S_i - S^0)/\Delta t + K_i\lambda_o(S_i)\pi_i - q_{oi} \quad \text{(obj. function)}$$

Convergence Analysis

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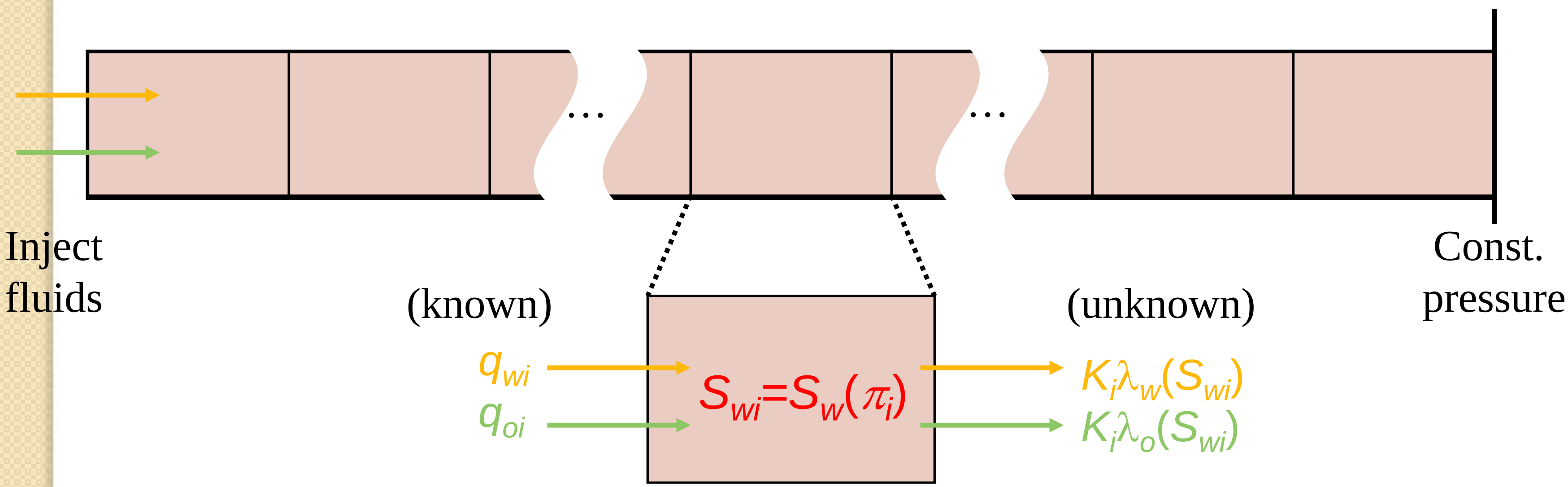


$$F_{oi} = -(S_{wi}(\pi_i) - S^0)/\Delta t + K_i\lambda_o(S_{wi}(\pi_i))\pi_i - q_{oi} \quad \text{(obj. function)}$$

Convergence Analysis

- Analysis based on convexity of reduced objective function
- Cocurrent flow:
  - Global convergence for arbitrary Δt
  - Proof uses convexity of objective function
- Countercurrent flow:
  - Convergence when gravity effects are moderate:

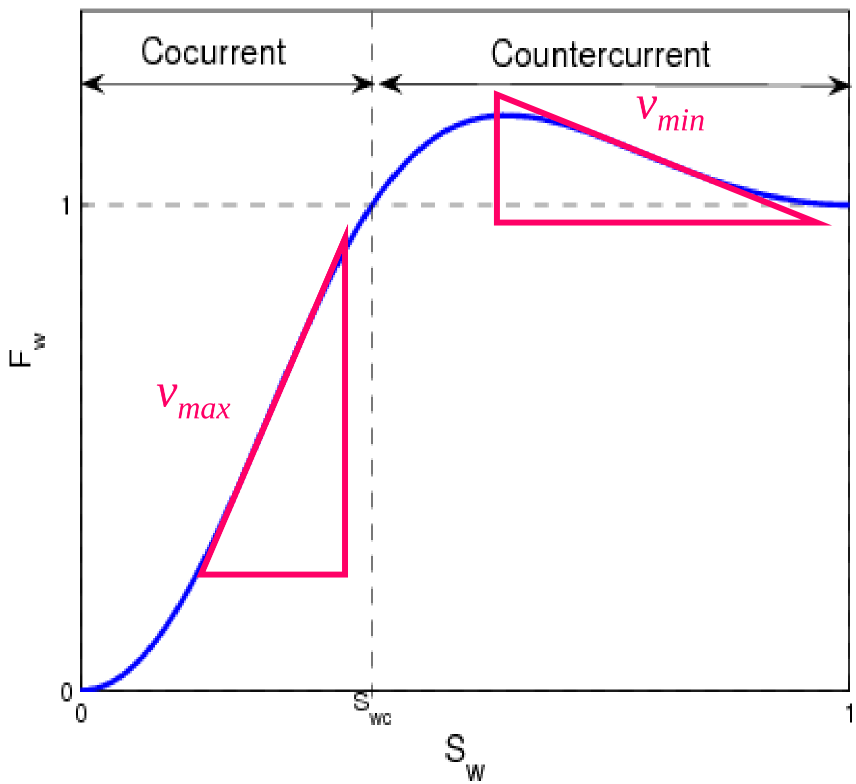
$$\Delta t \cdot |v_{\min}| < \Delta x$$

Convergence Analysis

- Convergence when

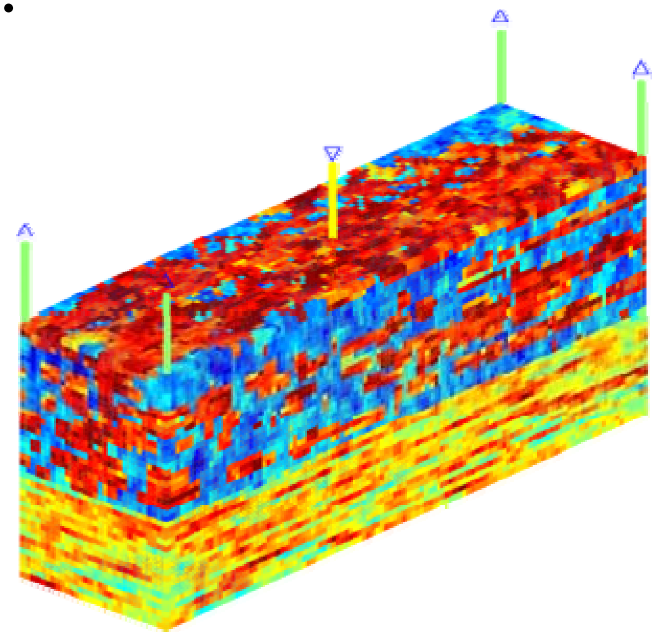
$$\Delta t \cdot |v_{\min}| < \Delta x$$

- Generally less severe than forward CFL
- For strong gravity, hybridize the method



Large Heterogeneous Example

- Fine scale SPE 10 problem
- $60 \times 220 \times 85$ = 1.12 million cells
- Spatially-varying porosity & permeability fields
- Three time-stepping strategies:
  - Δt = 0.0183 PVI (short)
  - Δt = 0.0366 PVI (medium)
  - Δt = 0.183 PVI (long)
- Matching densities: $\rho_o = \rho_w$



Large Heterogeneous Example

	Standard		Reduced		
	Short	Medium	Short	Medium	Long
# Time steps	58	38	53	26	11
# Newton steps	233	176	128	90	55
# Linear solves	2958	2323	2271	2399	1805
# Time step cuts	6	17	0	0	0
# Wasted lin. solves	860	3934	0	0	0
Total time (sec):	24053	37388	16558	14727	10275
Linear solves	22570	35457	11697	11301	7899
Single-cell solves	0	0	4194	2996	2132

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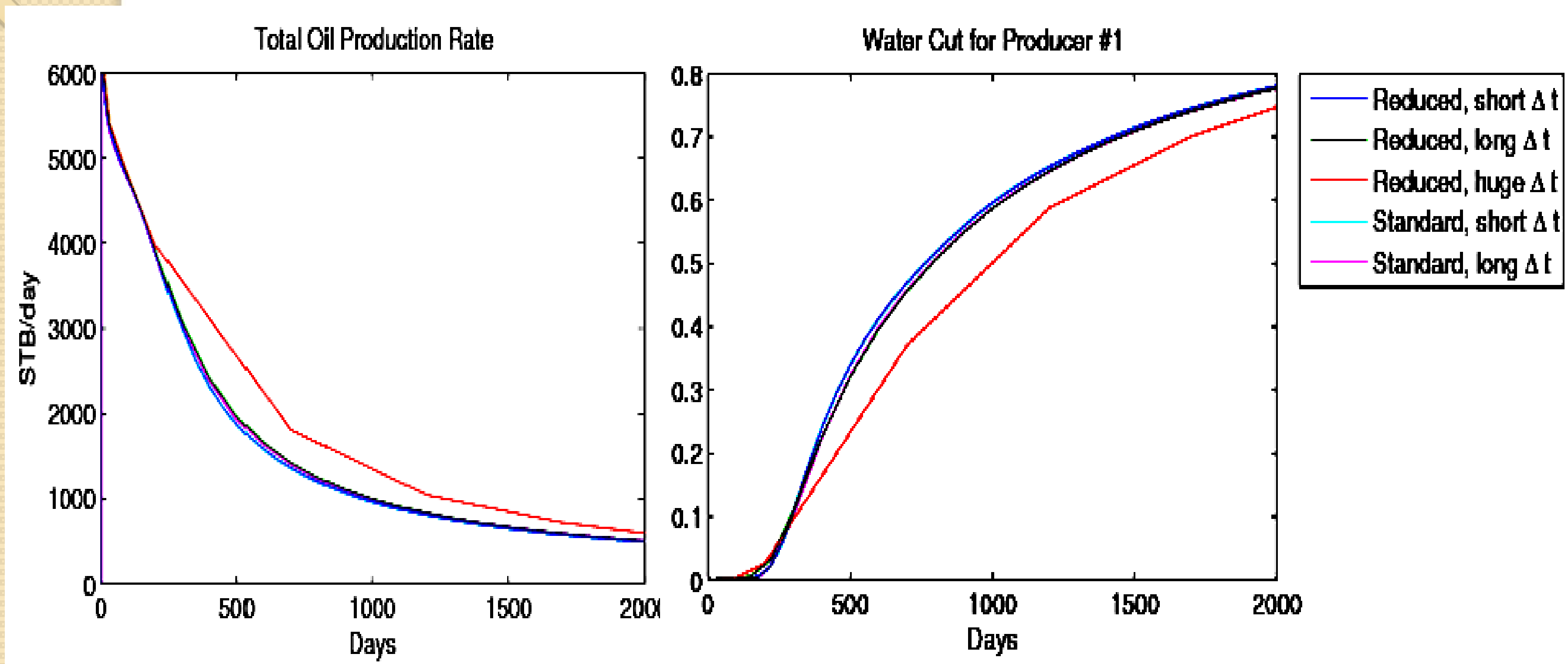


More Examples

CPR WITH POTENTIAL ORDERING

Linear Solvers

- Direct solvers are slow for large problems due to fill-in entries
- Off-the-shelf preconditioners do not work well with multiphase flow problems:
  - ILU: Slow for elliptic subproblem
  - AMG: Does not work well for transport problem
  - AINV: Does not work for matrices whose inverses have many dense entries

Multistage Preconditioners

- Two-stage preconditioner:

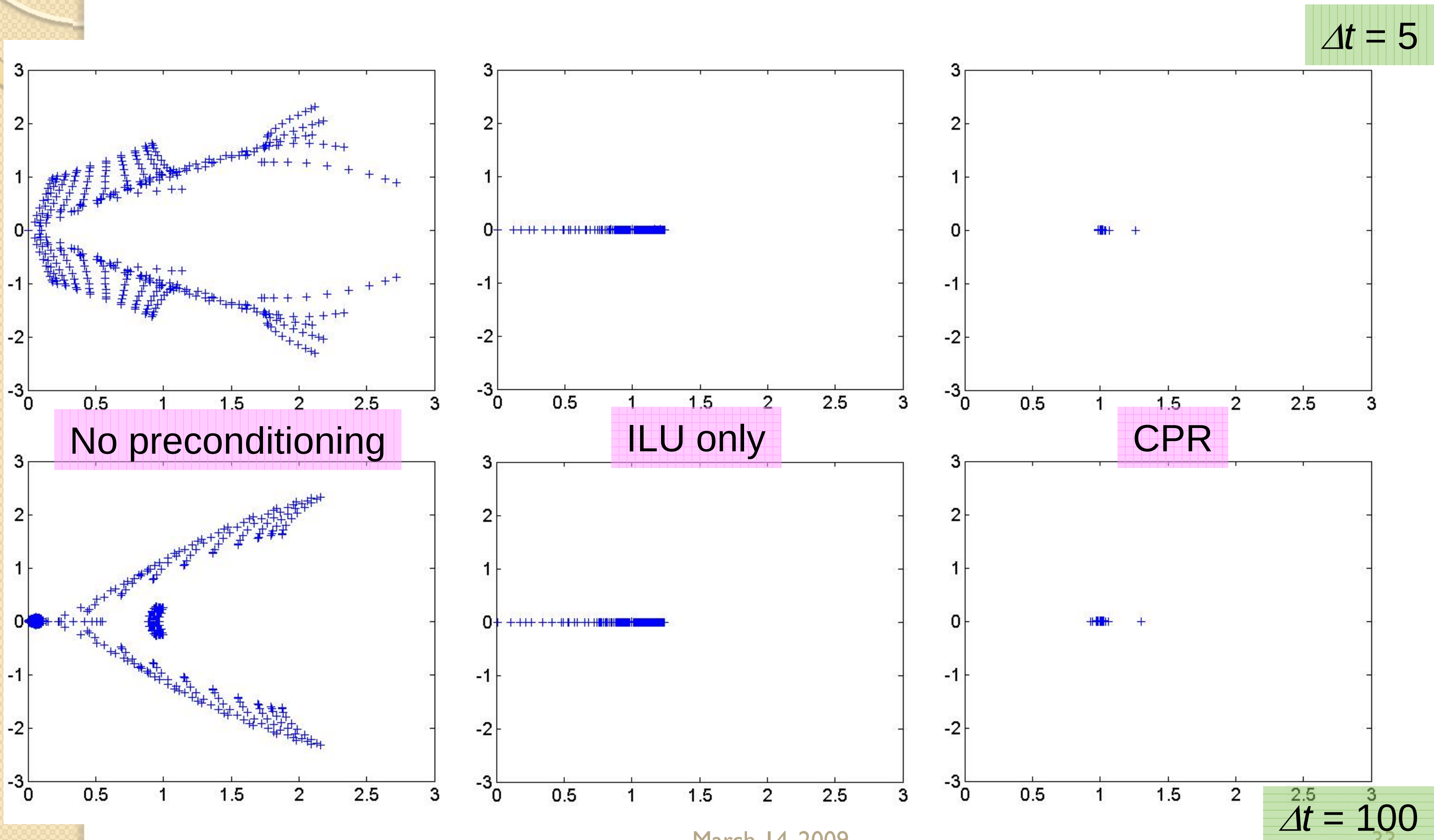
$$M_{1,2}^{-1} = M_2^{-1}(I - AM_1^{-1}) + M_1^{-1}$$

- Works well when M_1 and M_2 approximate different parts of the spectrum of A, since

$$I - AM_{1,2}^{-1} = (I - AM_2^{-1})(I - AM_1^{-1})$$

- Especially well-suited for problems of mixed character, e.g. multiphase flow problem:
  - M_1 : Flow problem (AMG on pressure matrix)
  - M_2 : Transport problem (ILU on full matrix)
- Constrained pressure residual method (Wallis 1983)

CPR vs. ILU



CPR with Potential Ordering

- Quality of second-stage ILU varies with ordering of equations/variables

- Exploit flow direction information:
  - M_1^{-1}: same as CPR (get approximate p)
  - M_2^{-1}: BILU(0) on *reordered* matrix, from upstream to downstream

CPR with Potential Ordering

1. Factorization is exact on the saturation part:

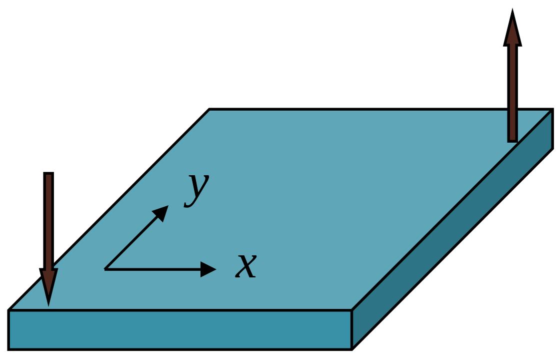
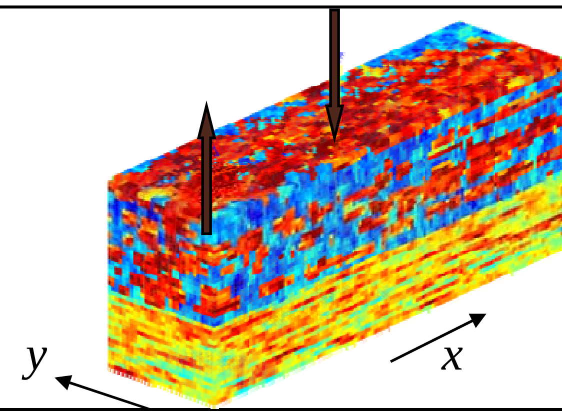
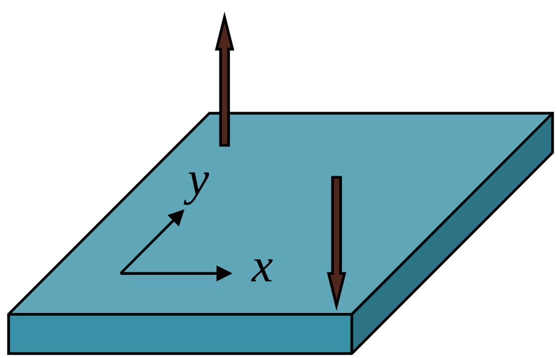
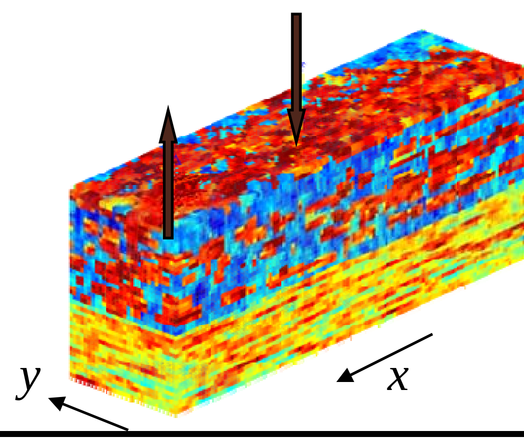
$$LU - A = \begin{bmatrix} 0 & E_{ps} \\ 0 & E_{pp} \end{bmatrix}$$

- Optimal choice of M_2 (since M_1 is exact on pressure)

2. Factorization is invariant over different potential orderings

- Reduced sensitivity over flow configurations

Test Cases

	Quarter 5-spot	Upscaled SPE10
(a)		
(b)		

Results – Quarter 5-spot

	Config. (a)		Config. (b)	
	Natural Ordering	Potential Ordering	Natural Ordering	Potential Ordering
# Time steps	21	21	21	21
# Newton steps	80	80	80	80
# GMRES iterations	254	254	346	246
# AMG V-cycles	286	286	368	274
Total running time (s)	1.37	1.45	1.49	1.43

Results – Upscaled SPE 10

	Config. (a)		Config. (b)	
	Natural Ordering	Potential Ordering	Natural Ordering	Potential Ordering
# Time steps	37	37	37	37
# Newton steps	106	106	106	106
# GMRES iterations	389	349	447	351
# AMG V-cycles	524	502	570	504
Total running time (s)	595.99	614.37	630.55	616.21

Conclusions

- Flow-based ordering allows:
  - Partial decoupling of saturation equations (both linear and nonlinear)
  - Rigorous convergence analysis based on convexity arguments
  - Improvements in quality of CPR preconditioner
- Reduced Newton method:
  - Retains local quadratic convergence
  - Converges for much larger Δt than standard Newton and avoids time step cuts
  - Works for complicated, large-scale problems

Ongoing work

- Derive effective hybridization criteria to handle strong countercurrent flow
- Extend framework to handle IMPSAT-based AIM formulations
- Spectral analysis of CPR preconditioner with ordering, esp. its sensitivity with respect to Δt

References

1. Kwok, F. and Tchelepi, H. Potential-based Reduced Newton algorithm for nonlinear multiphase flow in porous media. *J. Comput. Phys.*, 227:706–727, 2007.
2. Kwok, F. *Scalable Linear and Nonlinear Algorithms for Multiphase Flow in Porous Media.* PhD thesis, Stanford University, Stanford, CA, December 2007.