Part I: Degenerations and Integrable Systems

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Part I: Degenerations and Integrable Systems

In this part we give some general background. Everything is unoriginal.

1. What are integrable systems?
2. Degenerations
3. From degenerations to integrable systems
4. Horospherical degeneration/contraction

We conclude by describing horospherical degeneration, which introduces extra torus symmetry but is not, in general, a toric degeneration.

Part II: Symplectic contraction

We give symplectic geometric description of horospherical degeneration as a natural quotient in the symplectic category, analogous in many ways to symplectic reduction/implosion.
Integrable systems

Definition
An integrable system on a symplectic manifold \( (M^{2n}, \omega) \) is a set of functions \( f_1, \ldots, f_m \in C^\infty(M) \) such that

1. \( \{ f_i, f_j \} = \omega(X_{f_i}, X_{f_j}) = 0 \) for all \( i, j \).
2. Generically, \( \dim \langle X_{f_1}, \ldots, X_{f_m} \rangle = n \).

(1) + (2) \Rightarrow fibers of \( F = (f_1, \ldots, f_m) \) are generically Lagrangian submanifolds.

Common goals in study of integrable systems:

1. Construct new integrable systems.
2. Understand the singular Lagrangian foliations they define.
3. Apply to dynamics, quantization, symplectic topology, PDE, etc.

Results in part II concern item 2 for Gelfand-Zeitlin systems.
Example: toric manifolds

**Definition**

A *symplectic toric manifold* \((M^{2n}, \omega, \mu)\) is a connected symplectic manifold \(M^{2n}\) equipped with an effective Hamiltonian action of a compact torus \(T\) of dimension \(n\) generated by a moment map \(\mu: M \to t^*\).

- **Atiyah-Guillemin-Sternberg**: If \(M\) is compact, \(\mu(M)\) is a convex polytope.
- **Delzant**: Compact toric manifolds are classified by \(\mu(M)\), construction procedure (also cf. Karshon-Lerman).
- **Marle-Guillemin-Sternberg**: singular fibers of \(\mu\) are isotropic tori, local normal form.
Example: Gelfand-Zeitlin systems

Consider

\[
\begin{array}{ccccccc}
\uparrow p \downarrow s & \uparrow p \downarrow s & \cdots & \uparrow p \downarrow s \\
\Delta_n & \Delta_{n-1} & \cdots & \Delta_1 = t_1^* \\
\subseteq t_n^* & \subseteq t_{n-1}^* & \cdots & \\
\end{array}
\]

where \( s: u(k)^* \to \Delta_k, \xi \mapsto (U(k) \cdot \xi) \cap \Delta \). Explicitly, take \( u(k)^* \equiv \mathcal{H}_k \),

\[
p: \mathcal{H}_k \to \mathcal{H}_{k-1}, X \mapsto \text{upper left principal submatrix } X^{(k-1)}.
\]

\[
\Delta_k = \{ \text{diag}(\lambda_1, \ldots, \lambda_k): \lambda_1 \geq \cdots \geq \lambda_k \} \subseteq \mathbb{R}^k
\]

\[
s(p^{n-k}(X)) = \text{ordered eigenvalues } \lambda_1^{(k)} \geq \cdots \geq \lambda_k^{(k)} \text{ of } X^{(k)}
\]

\[
F(X) = (\lambda_1^{(n)}, \ldots, \lambda_1^{(1)}) \in \mathbb{R}^{n(n+1)/2}.
\]
Example: Gelfand-Zeitlin systems

**Guillemin-Sternberg:** Let $O_\lambda \subseteq u(n)^*$ with KKS form. Then,

- Image $F(O_\lambda)$ is a convex polytope (not Delzant).
- $F$ not smooth on all of $O_\lambda$! $O_\lambda$ not a toric manifold.
- Restriction of $F$ to open dense set $U \subseteq O_\lambda$ is a symplectic toric manifold for an action of $T_n \times T_{n-1} \times \cdots \times T_1$ ($T_n$ acts trivially).
- All fibers are connected smooth submanifolds.

**Questions about the singular Lagrangian foliation:**

- Fibers of non-smooth map $F$, how to study?
- What does the singular Lagrangian foliation look like in $O_\lambda \setminus U$? (cf. Cho-Kim-Oh)
- G-S observe “independence of polarization” if one counts integral fibers in $O_\lambda \setminus U$. Can we explain this geometrically? (cf. Hamilton-Harada-Kaveh)
Degenerations

**Definition**

A *degeneration* of $X$ to $X_0$ (affine, projective, quasiprojective varieties) is a flat morphism $\pi : X \to \mathbb{C}$ such that

1. $\pi^{-1}(0) \cong X_0$
2. $\pi^{-1}(z) \cong X$ for all $z \neq 0$.

**Examples:**

- $\pi : \mathbb{C}^2 \to \mathbb{C}, (z_1, z_2) \mapsto z_1 z_2$. $X \cong T^*S^1, X_0 = \{z_1 z_2 = 0\}$.
- $\pi : M_2(\mathbb{C}) \to \mathbb{C}, X \mapsto \det(X)$. $X = SL(2, \mathbb{C}) \cong T^*S^3, X_0 = \text{singular matrices}$.
- Given $A = \mathbb{C}[X]$ with filtration $\mathcal{F}$, Rees algebra construction $\to$ degeneration of $X$ to $\text{spec}(\text{gr}(A))$.

**Morally (for the examples we are interested in):**

- Geometry of $X \sim$ geometry of $X_0$ (study one via the other)
- $X_0$ is more singular than $X$, but also more symmetric.
Assume \((\mathcal{X}, \tilde{\omega})\) Kähler, \(\pi: \mathcal{X} \to \mathbb{C}\) holomorphic.

**Definition (going back to Ruan)**

The *gradient-Hamiltonian vector field* of the map (degeneration) \(\pi\) is

\[
V_\pi = \frac{-\nabla \text{Re}\pi}{||\nabla \text{Re}\pi||^2}
\]

where defined \((d\pi \neq 0)\).

Since \(\pi\) is holomorphic, \(\tilde{\omega}\) is Kähler, \(\nabla \text{Re}\pi = J\nabla \text{Im}\pi = -X_{\text{Im}\pi}\). Thus,

\[
V_\pi = \frac{X_{\text{Im}\pi}}{||X_{\text{Im}\pi}||^2}
\]

Thus the name “gradient-Hamiltonian.”
**Fact:** $\pi_*(V_\pi) = -\frac{\partial}{\partial x}$. The flow of $V_\pi$ maps fibers to fibers.

**Proof:**

$$d(\text{Re}_\pi)(V_\pi) = \frac{-1}{||\nabla \text{Re}_\pi||^2} g(\nabla \text{Re}_\pi, \nabla \text{Re}_\pi) = -1$$

$$d(\text{Im}_\pi)(V_\pi) = \frac{1}{||\nabla \text{Re}_\pi||^2} \tilde{\omega}(X_{\text{Im}_\pi}, X_{\text{Im}_\pi}) = 0$$
Endow $X_t = \pi^{-1}(t)$ with symplectic form $\omega_t = \tilde{\omega}|_{X_t}$. Let $\phi_t$ be the flow of $V_\pi$.

**Fact:** Where defined, $\phi_t : X_s \to X_{s-t}$ satisfies $\phi_t^* \omega_{s-t} = \omega_s$.

**Proof:** It’s sufficient to prove that for all vector fields $Y, Z \in \mathfrak{X}(\mathcal{X})$ such that $\pi_* Y = \pi_* Z = 0$, $(\mathcal{L}_{V_\pi} (Y, Z)) = 0$. Indeed,

\[
(\mathcal{L}_{V_\pi} (Y, Z)) = (d\iota_{V_\pi} (Y, Z))
= \mathcal{L}_Y (\omega (V_\pi, Z)) - \mathcal{L}_Z (\omega (V_\pi, Y)) - \omega (V_\pi, [Y, Z])
= 0
\]

where the last equality uses Hamiltons equation, the definition of $V_\pi$, and $\pi_* Y = \pi_* Z = 0$. 
Examples

- $\pi : \mathbb{C}^2 \to \mathbb{C}, (z_1, z_2) \mapsto z_1 z_2$. $\pi$ has critical pt at $0 \in X_0$.

  $\phi_1 : X_1 \setminus Z \to X_0$ is defined on the complement of zero section $Z \subset T^* S^1 \cong S^1$.

  Continuous extension $\tilde{\phi}_1 : X_1 \to X_0$ sends Lagrangian $Z$ to $(0, 0)$. “vanishing cycle.”

- $\pi : M_2(\mathbb{C}) \to \mathbb{C}, X \mapsto \det(X)$. $\pi$ has critical pt at $0 \in X_0$.

  Similar: continuous extension $\tilde{\phi}_1 : X_1 \to X_0$ sends Lagrangian $Z \cong S^3$ to $0$.

Morally: Symplectic geometry of $X_1$ and $X_0$ is the same on the complement of a “small” set. In general, more complicated than Lagrangian “vanishing cycle” unless critical pts. of $\pi$ are nondegenerate (cf. Seidel).
From Hilgert-Manon-Martens. Picture (real slice) of gradient-Hamiltonian flow for first example.
Time-1 gradient-Hamiltonian flow

In general, under various assumptions, (including some sort of compactness or properness that ensures the flows are defined) can prove something like the following:

**Theorem (very vague version of Harada-Kaveh)**

Let \((X, \tilde{\omega})\) Kähler, \(\pi : X \to \mathbb{C}\) holomorphic, such that

- \(\text{crit}(\pi) \subseteq X_0\)
- \((X_t, \omega_t)\) all isomorphic for \(t \neq 0\)
- ...

Then, \(\phi_1 : \mathcal{U} \subseteq X_1 \to X_0\) extends continuously to a map \(\tilde{\phi}_1 : X_1 \to X_0\) such that

- Restriction of \(\tilde{\phi}_1\) to \(\mathcal{U}\) is a symplectomorphism onto its image in \(X_0\).

E.g. given \(F : X_0 \to t^*\) toric, \(\tilde{\phi}_1^*F : X_1 \to t^*\) is toric on \(\mathcal{U}\), not smooth on complement. Very little understood about singular Lagrangian fibration of \(X_1\).
Horospherical degeneration of a $G$-variety (Popov)

Let $G = K^C$ connected, complex, linearly reductive. $\Lambda$ the lattice of weights.

Let $X$ an affine (quasiaffine) $G$ variety, $A = \mathbb{C}[X]$.

$U \subseteq G$ a maximal unipotent, $R^+$ corresponding positive roots, $U^-$ opposite max unipotent.

**A $G$-filtration of $A$**

Define

$$h: \Lambda \to \mathbb{Z}, \ h(\chi) = \sum_{\alpha \in R^+} \check{\alpha}(\chi) = \sum_{\alpha \in R^+} \frac{2(\alpha, \chi)}{(\alpha, \alpha)}.$$  

Observation: $\chi \leq_\Lambda \chi' \Rightarrow h(\chi) \leq h(\chi').$

$$A_n := \bigoplus_{\chi \in \Lambda, h(\chi) \leq n} A^{(\chi)}.$$
The associated graded algebra is the vector space

$$\text{gr} A = \bigoplus_n A_n/A_{n-1}$$

equipped with the multiplication

$$(x + A_{n-1}) \cdot (y + A_{m-1}) = xy + A_{m+n-1}.$$ 

Since filtration is $G$-invariant, $\text{spec}(\text{gr} A)$ is a $G$-variety.

**Rees algebra construction:** There is a degeneration of $A$ to $\text{gr}(A)$. The natural inclusion $\iota : \mathbb{C}[t] \to D = \bigoplus_n A_n t^n$ is a flat morphism. Moreover,

- The fiber of $\iota$ over $t = 0$ is $\cong \text{gr}(A)$
- The fiber of $\iota$ over $t \neq 0$ is $\cong A$

(see e.g. Grosshans for details).
**Properties of grA**

$G$ is a (quasi)affine $G \times G$ variety. $G/U^-$ is a $G \times T$ variety (since $T$ normalizes $U^-$).

**Theorem**

$$gr(A) \cong_G (A^U \otimes_{\mathbb{C}} \mathbb{C}[G]^{U^-})^T$$

where $T$ acts diagonally. Equivariance is w.r.t. $G$ action on $\mathbb{C}[G/U^-]$.

Thus, spec(gr($A$)) is a $G \times T$ variety. Extra torus symmetry!!

By definition of multiplication in gr($A$),

**Theorem**

$$gr(A)_\lambda \cdot gr(A)_\mu \subseteq gr(A)_{\lambda+\mu}$$
**Definition**

Horospherical $G$ varieties A subgroup $H \leq G$ is *horospherical* if it contains a maximal unipotent subgroup of $G$. A $G$ variety $X$ is *horospherical* if every stabilizer subgroup is horospherical.

**Theorem**

*A quasiaffine $G$ variety $\text{spec}(A)$ is horospherical iff*

$$A(\lambda) \cdot A(\mu) \subseteq A(\lambda + \mu).$$

Thus we see that the graded associative $\text{gr}(A)$ is a horospherical $G$-variety.

**Part II:** We study this degeneration from a symplectic perspective.