Random curves, scaling limits and Loewner evolutions

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2D lattice models: percolation, Ising model, random cluster models
Scaling limit of an interface

$\mathbb{P}$ is the law of a random simple curve in a simply connected domain $U$ and connecting $a, b \in \partial U$

Scaling limit: mesh $\to 0$. 
Goal: present a setting for proving the convergence.

Often: a sequence of \((U, a, b, P)\) so that \((U, a, b)\) approximates \((\hat{U}, \hat{a}, \hat{b}), \hat{P}\) from a chosen lattice model.

Useful to choose a reference domain, e.g. the unit-disc \(D\), and map conformally \(\phi : U \to D, \phi(a) = -1, \phi(b) = 1\).

We will consider a set \(\Sigma\), whose elements are triplets \((U, \phi, P)\).
Schramm-Loewner evolution

\[ W_t = g_t(\gamma(t)) \]

- \( g_t \) conformal mapping with a specific normalization:
  \[ g_t(z) = z + \frac{C(t)}{z} + \ldots \] capacity

- Loewner equation

\[
\frac{\partial g_t}{\partial t}(z) = \frac{C'(t)}{g_t(z) - W_t}
\]

where \( t \mapsto W_t \) continuous. Capacity param. \( C(t) = 2t \).

- SLE_\kappa, \( \kappa > 0 \), a random curve s.t. \( W_t = \sqrt{\kappa} B_t \) (\( B_t \)) standard, one-dimensional Brownian motion.
A *crossing* of an annulus

\[ A(z_0, r, R) = \{ z : r < |z - z_0| < R \} \]

is a subcurve such that its end points are in the different components of

\[ \mathbb{C} \setminus A(z_0, r, R). \]
Crossings of an annulus

There are three types of crossings of an annulus $A = A(z_0, r, R)$.

**Unforced**

**Forced**

**Ambiguous**
Condition assumed to hold for the random curve

Choose a parametrization for all curves, $\gamma : [0, 1] \rightarrow \mathbb{C}$.

<table>
<thead>
<tr>
<th>Condition</th>
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<tbody>
<tr>
<td>$\exists C &gt; 1$ s.t. for any $(U, \phi, \mathbb{P}) \in \Sigma$, for any stopping time $\tau$ and for any annulus $A = A(z_0, r, R)$, $0 &lt; Cr \leq R$,</td>
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<tr>
<td>$\mathbb{P}(\gamma[\tau, 1] \text{ crosses any } V \in U(U_\tau, A) \mid \gamma[0, \tau]) &lt; \frac{1}{2}$</td>
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<tr>
<td>for almost every $\gamma[0, \tau]$.</td>
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where $U_t = U \setminus \gamma(0, t]$ and

$$U(U_t, A) = \begin{cases} 
\emptyset & \text{if } A \text{ is not on } \partial U_t \\
\text{those components of } U_t \cap A \text{ that don’t separate } \gamma(t) \text{ and } b & \text{otherwise}
\end{cases}$$
Condition assumed to hold for the random curve

Choose a parametrization for all curves, \( \gamma : [0, 1] \rightarrow \mathbb{C} \).

**Condition**

\[ \exists K > 0, \Delta > 0 \text{ s.t. for any } (U, \phi, \mathbb{P}) \in \Sigma, \text{ for any stopping time } \tau \text{ and for any annulus } A = A(z_0, r, R), 0 < r < R, \]

\[ \mathbb{P}(\gamma[\tau, 1] \text{ crosses any } V \in U(U_\tau, A) \mid \gamma[0, \tau]) < K \left( \frac{r}{R} \right)^\Delta \]

for almost every \( \gamma[0, \tau] \).

where \( U_t = U \setminus \gamma(0, t] \) and

\[ U(U_t, A) = \begin{cases} \emptyset & \text{if } A \text{ is not on } \partial U_t \\ \text{those components of } U_t \cap A \text{ that don't separate } \gamma(t) \text{ and } b & \text{otherwise} \end{cases} \]
Main theorem, shortly

- Condition assumed is carried nicely under conformal mappings. Hence it holds in chosen reference domain(s).

- Under Condition the following holds for $\Sigma_D = \{ \phi\mathbb{P} : (U, \phi, \mathbb{P}) \in \Sigma \}$: $\exists F_n$ so that $\mathbb{P}(F_n) \to 1$ as $n \to \infty$ uniformly in $\mathbb{P} \in \Sigma_D$ and so that each $F_n$ is
  - precompact in the path topology: The space of curves $X$, the equivalence classes of $C([0,1], \mathbb{C})$ under increasing reparametrizations. A metric in $X$:
    \[ d_X(\gamma, \hat{\gamma}) = \inf \left\{ \| f - \hat{f} \|_\infty : f, \hat{f} \text{ parametr. of } \gamma, \hat{\gamma} \right\}. \]
  - precompact in the driving process topology: The capacity parametrization and the norm $\| \cdot \|_\infty$
  - On $\overline{F_n}$ the above descriptions are the same: the tip is uniformly visible from the target point.
There exists $\alpha > 0$ s.t. each curve can be parametrized $\alpha$-Hölder continuously with the Hölder norm being a tight random variable on $\Sigma_D$.

- Tight r.v. $Y$ on $\Sigma_0$: for each $\varepsilon > 0$, $\exists M > 0$ s.t. $\mathbb{P}(Y \in [-M, M]) > 1 - \varepsilon$, for any $\mathbb{P} \in \Sigma_0$.

- Aizenman&Burchard [1999]: the above holds for a collection of probability measures $\Sigma_0$ on $X$ if

$$
\mathbb{P}(\exists \ n \ \text{crossings of } A(z_0, r, R)) \leq K_n \left(\frac{r}{R}\right)^{\Delta_n}
$$

for each $\mathbb{P} \in \Sigma_0$, where $\Delta_n$ large when $n$ large.

- Half of the crossings are unforced. $\Delta_n \geq \left((n - 2)/12\right) \cdot \Delta$
Precompactness in the driving process convergence

**Theorem**

*For any $0 < \beta < 1/2$, the driving processes of the curves are $\beta$-Hölder continuous with a tight Hölder norm on $\Sigma_D$.*

- The increments of the driving process have exponential tails:
  \[
  \mathbb{P}[|W_t - W_s| \geq L \mid W[0, s]] \leq Ke^{-c \frac{L}{\sqrt{t-s}}}
  \]
  (note: capacity parametrization)
Excluding a six-arms event

For fixed $\rho$, define $E(r, R)$ as the event that $\exists (s, t) \in [0, 1]^2$, $s < t$, s.t.

1. $\text{diam}(\gamma[s, t]) \geq R$ and
2. $\exists$ a crosscut $C$, $\text{diam}(C) < r$, that separates $\gamma(s, t)$ from $B(1, \rho)$ in $\mathbb{D} \setminus \gamma(0, s)$.

**Theorem**

Uniformly for $\mathbb{P} \in \Sigma_{\mathbb{D}}$, $\mathbb{P}\left( E(r, R) \right) = o(1)$ as $r \to 0$. 
The condition assumed is uniform over the scales. Scale invariance is not assumed.

Conformal maps were used (mostly for resolving a problem with non-smooth boundary near $a$ and $b$), but conformal invariance is not assumed.

Condition assumed is a natural property to check. Gives a conceptual way to prove the existence of subsequential limits and that the limits are well-described by Loewner equation.

Should be applicable for any random curve converging to $\text{SLE}_\kappa$, $\kappa < 8$.

Generalizes to several points, $\text{SLE}_\kappa(\rho_1, \rho_2, \ldots)$ processes.

Thank you for your attention!