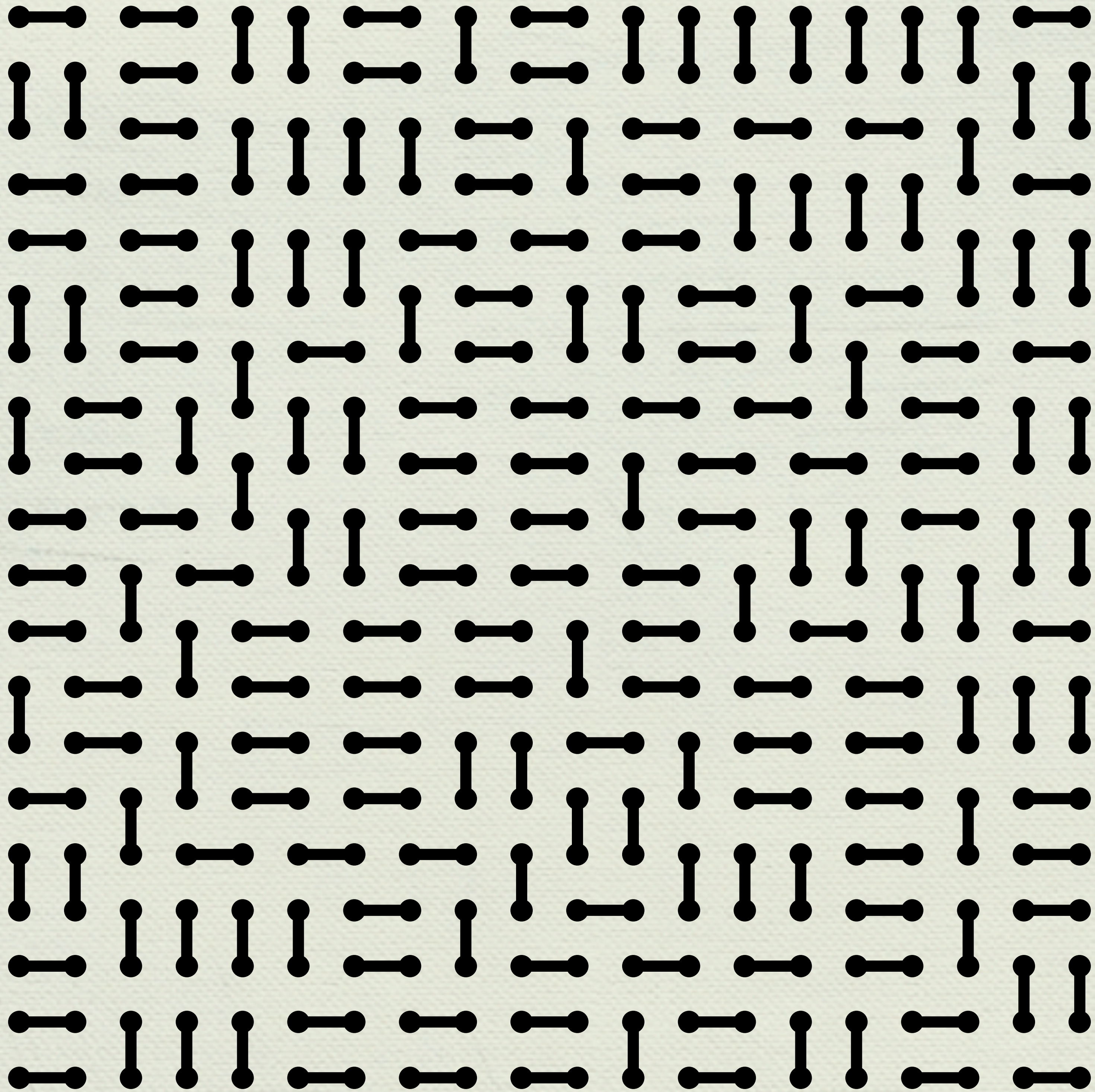
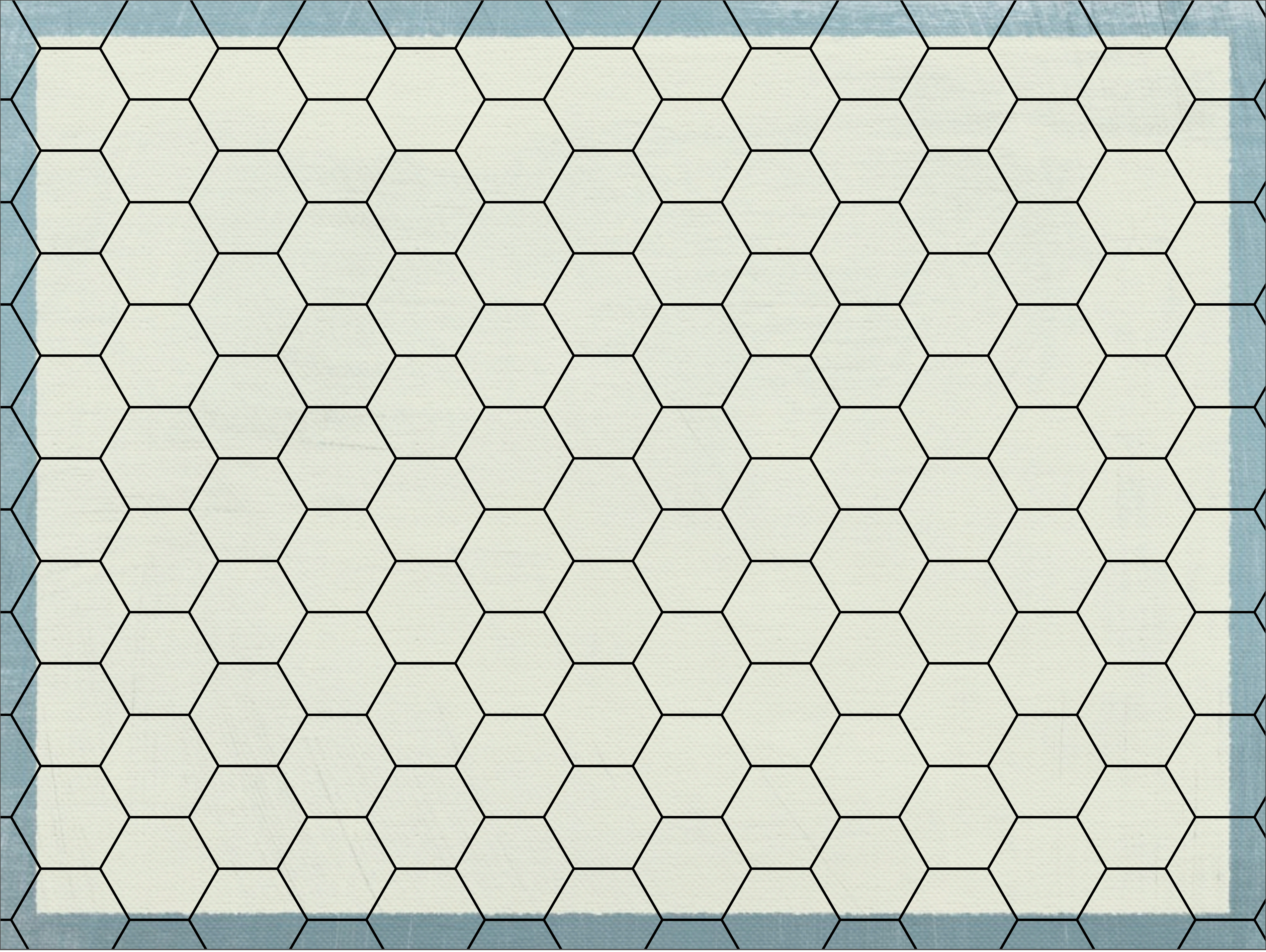
Dimers and integrability

R. Kenyon



Dimer model on $\mathbb{Z}^2$

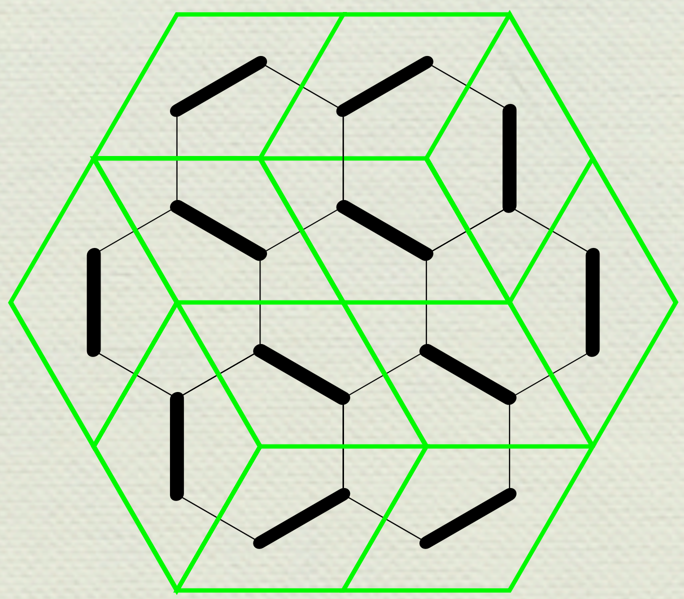
random dimer covering = random perfect matching



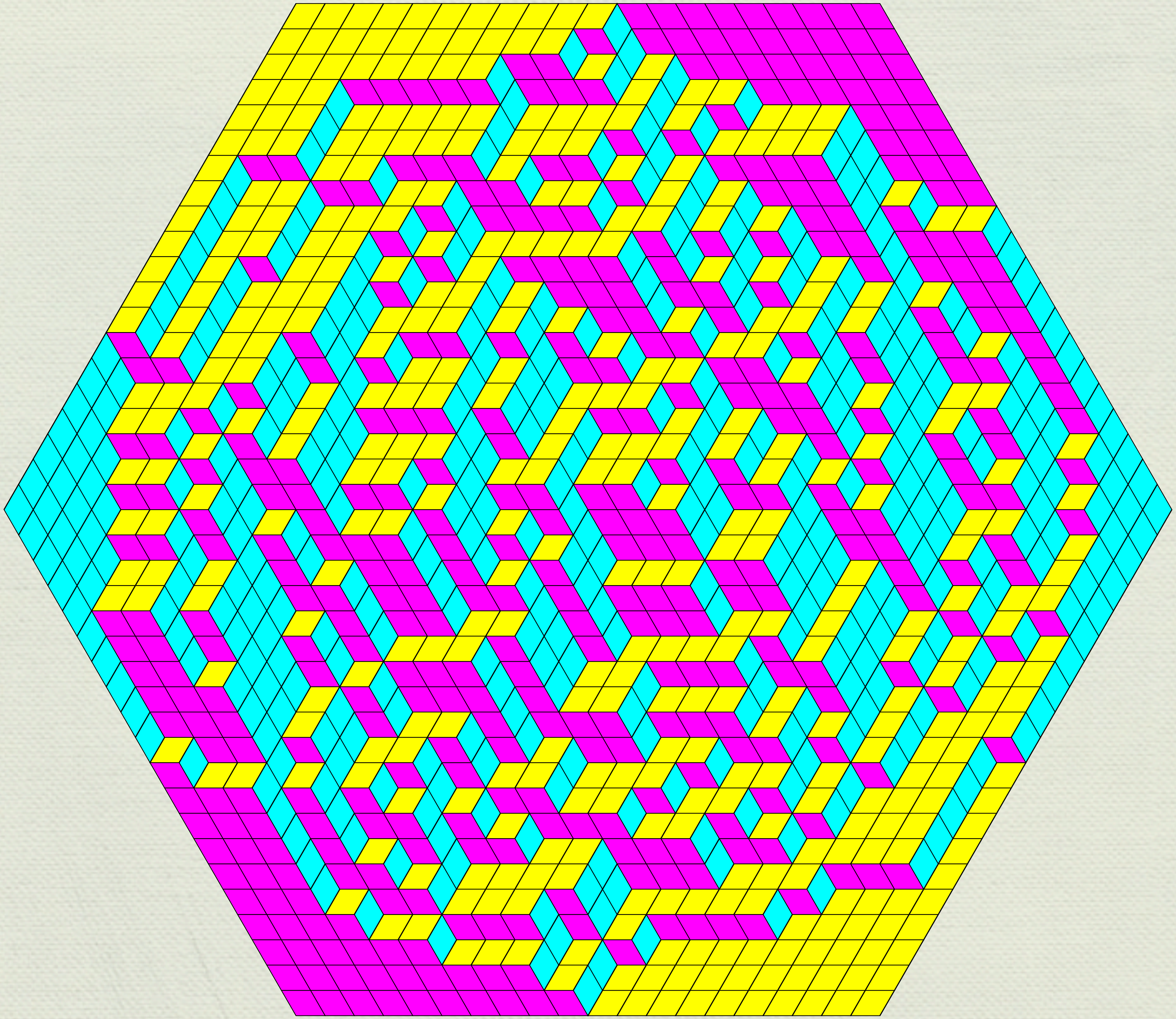
Thm (Kasteleyn 1965) For $G \subseteq$ honeycomb, let $K =$ adjacency matrix,

$$k_{ij} = \begin{cases} 1 & i \sim j \\ 0 & \text{else.} \end{cases}$$

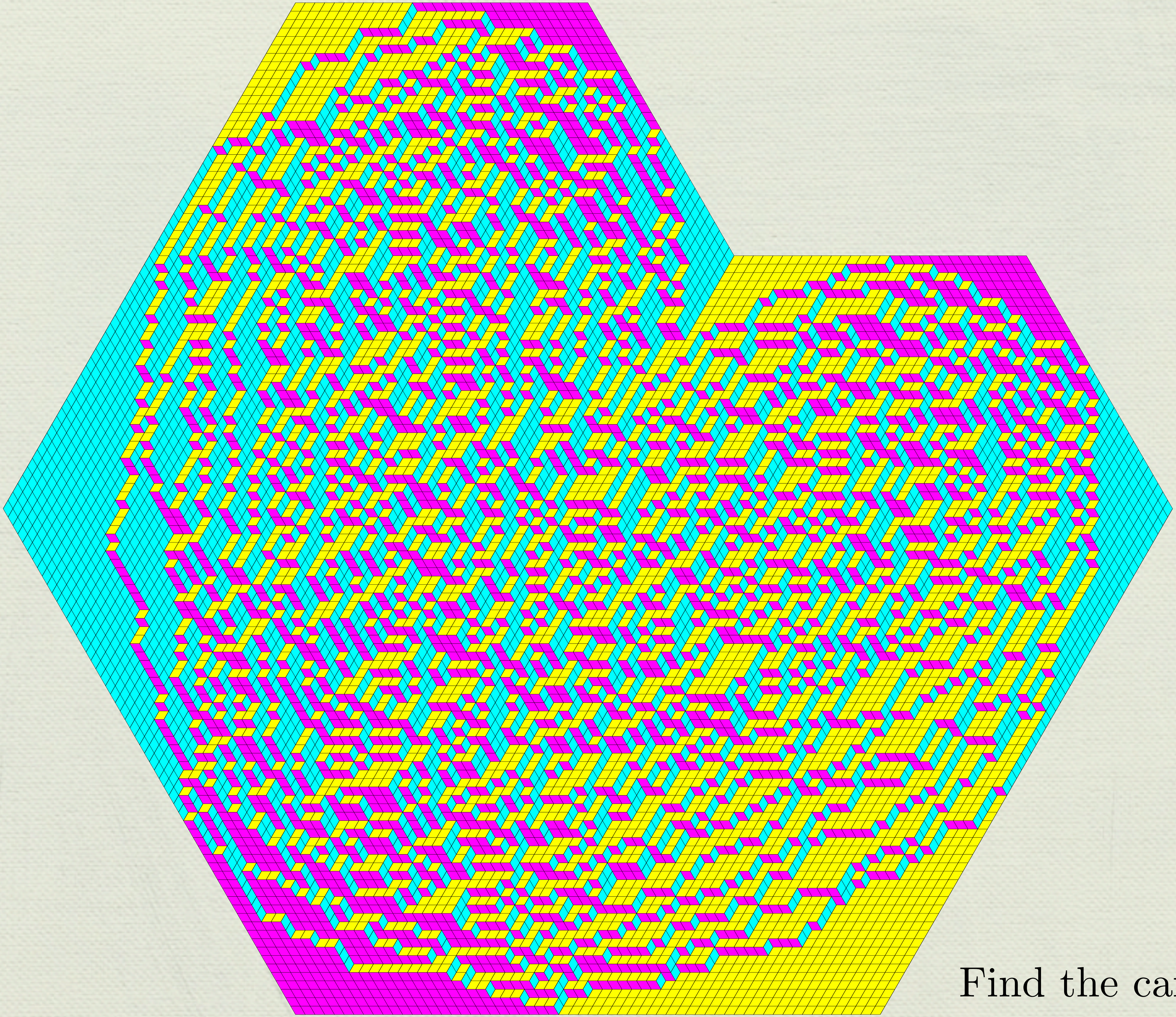
Then the number of dimer coverings is $\sqrt{\det K}$.



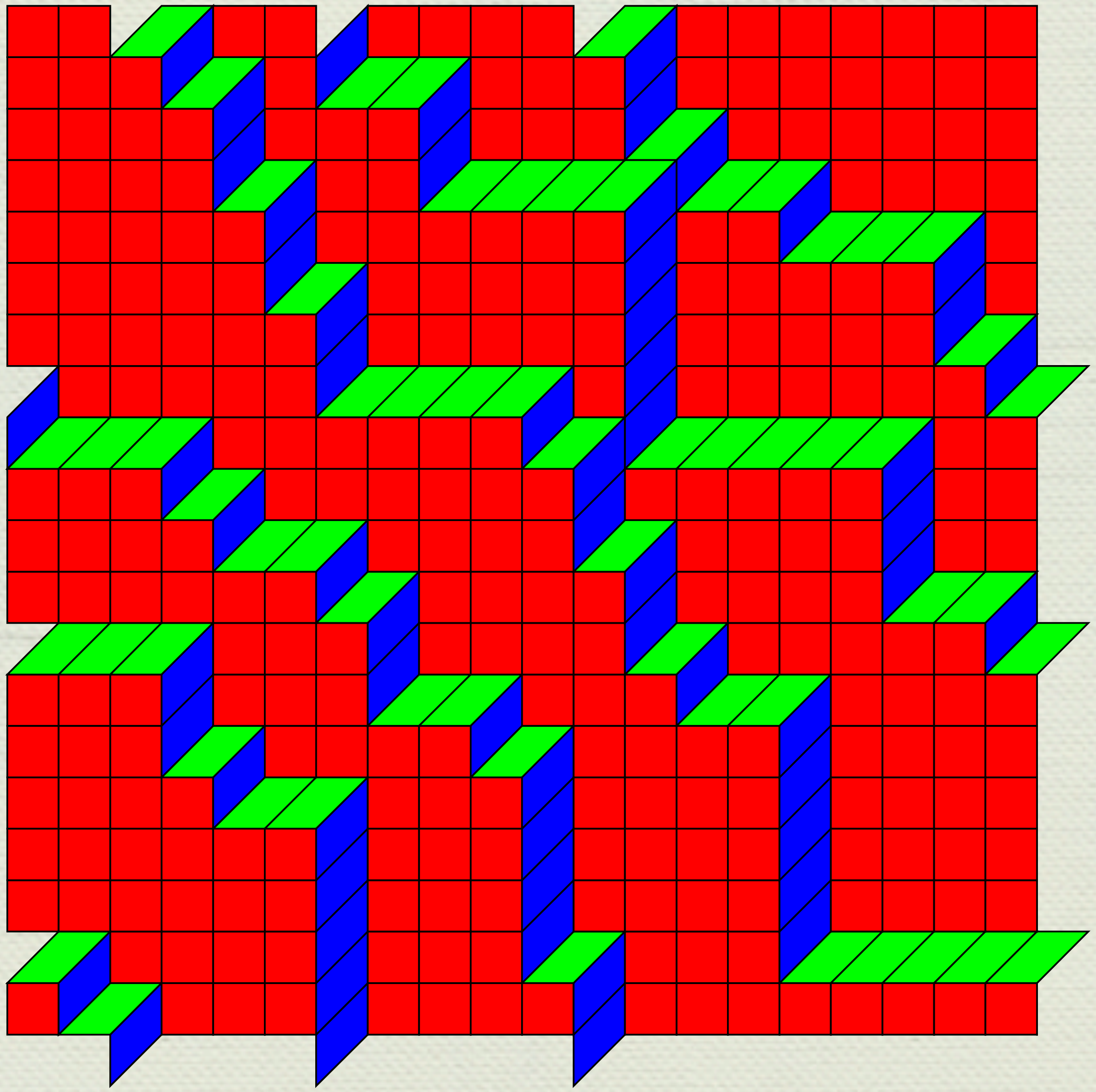
Example: K is 24×24 and $\det K = 400$.



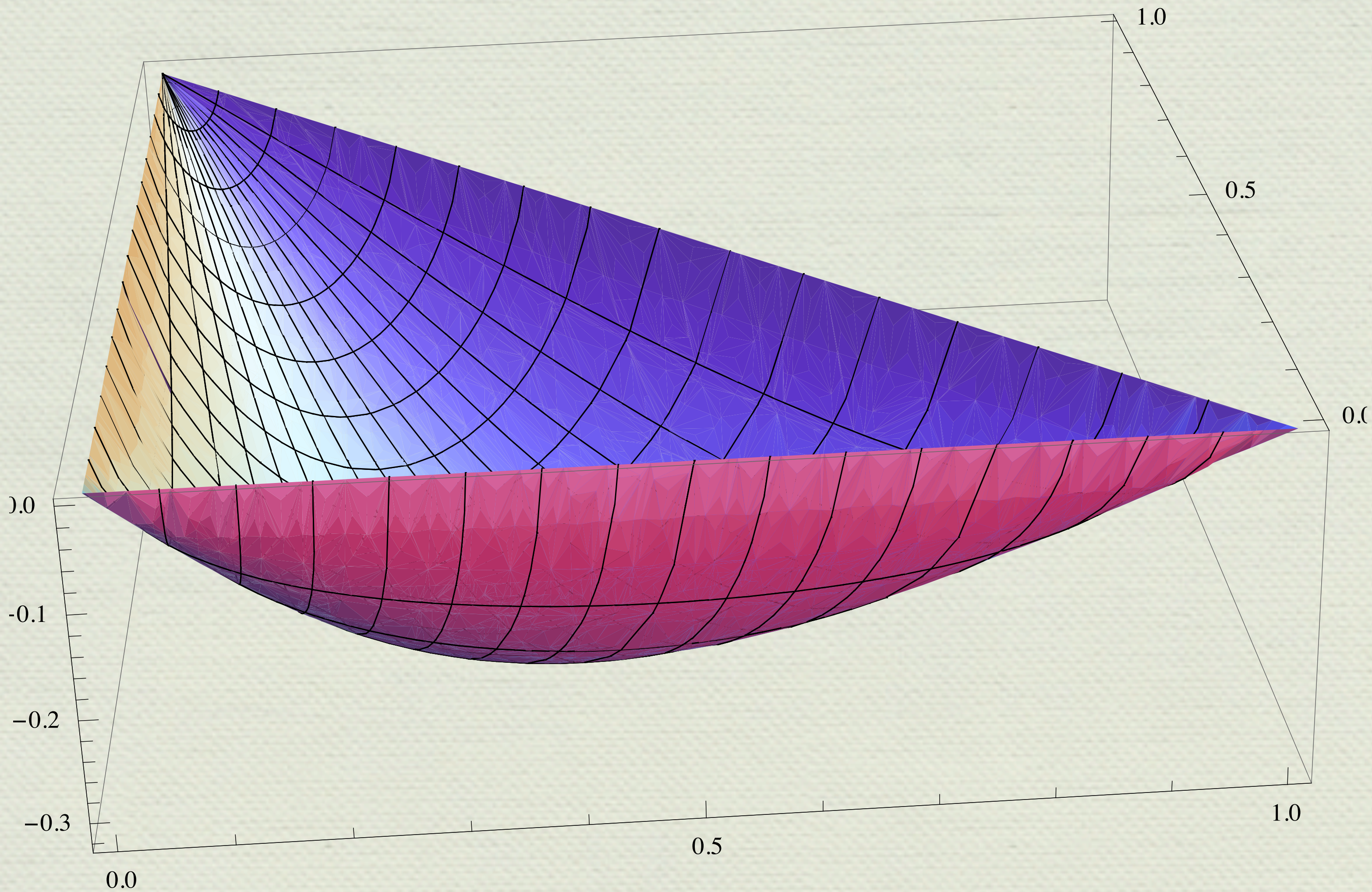
Boxed plane partition



Find the cardiod

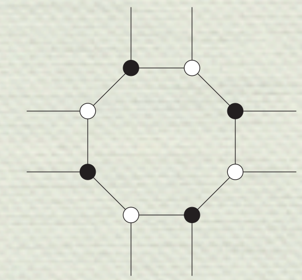


different gradients have different growth rates per unit area.

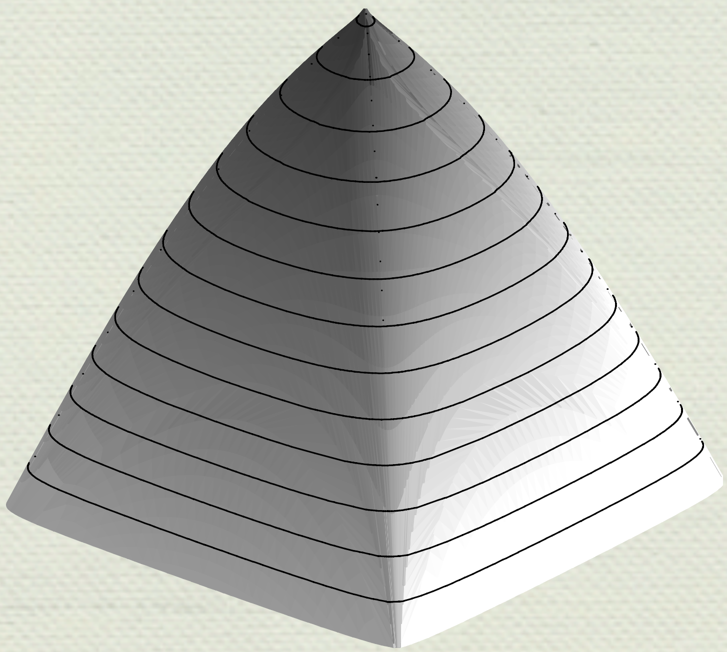


Honeycomb dimer surface tension
(a function of the gradient of the height function).

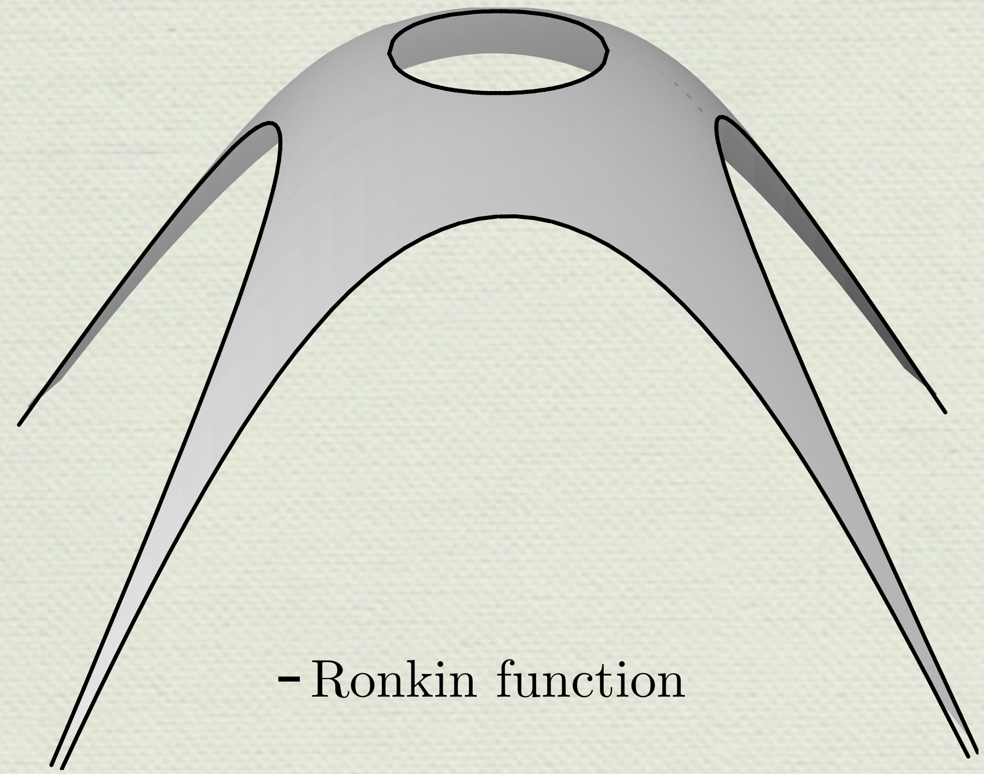
Square-octagon lattice



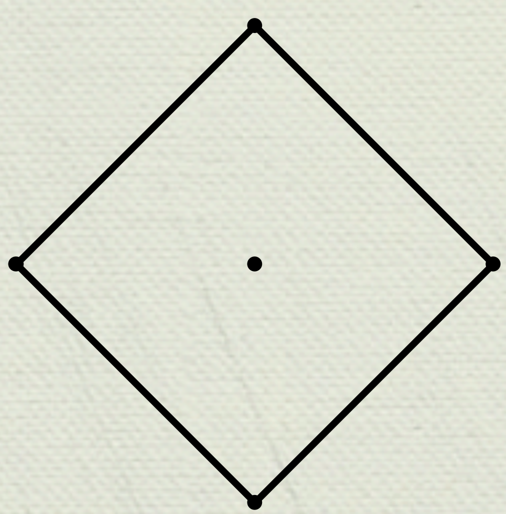
$$P(z,w) = 5 + z + 1/z + w + 1/w$$



minus the surface tension

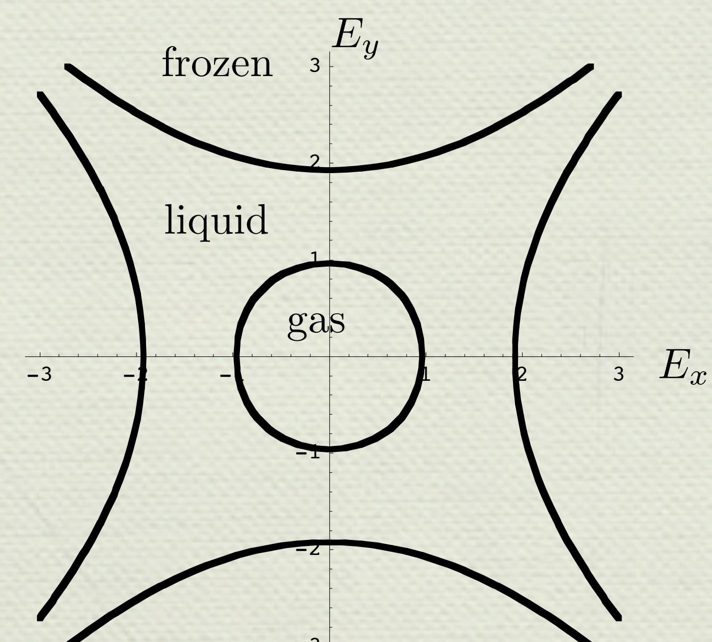


- Ronkin function



Newton polygon = allowed slopes

Phase space

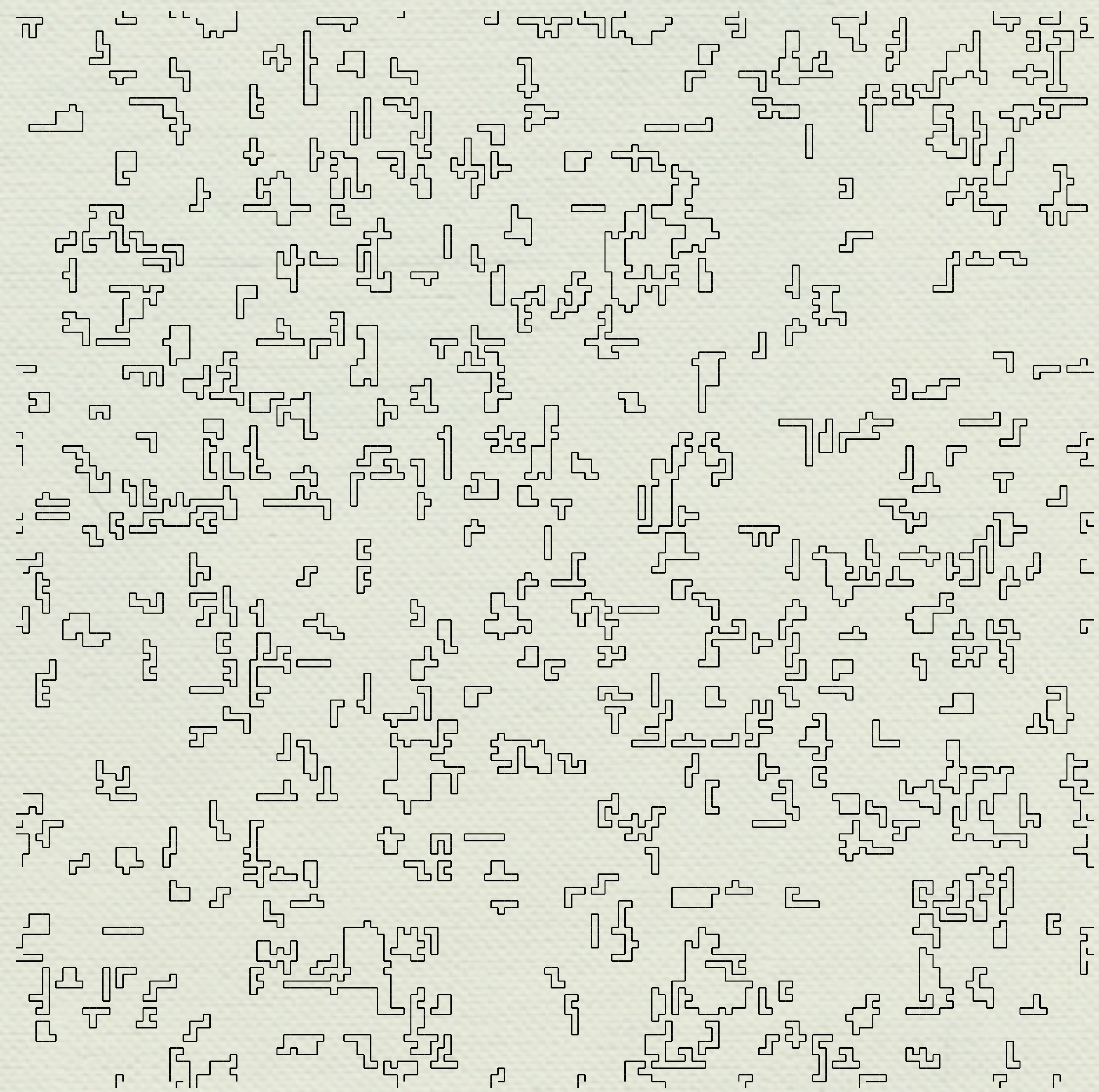


Amoeba of P

Three phases of measures

liquid phase contours

Conformally invariant ([K, 2010])



gaseous phase contours

height contours

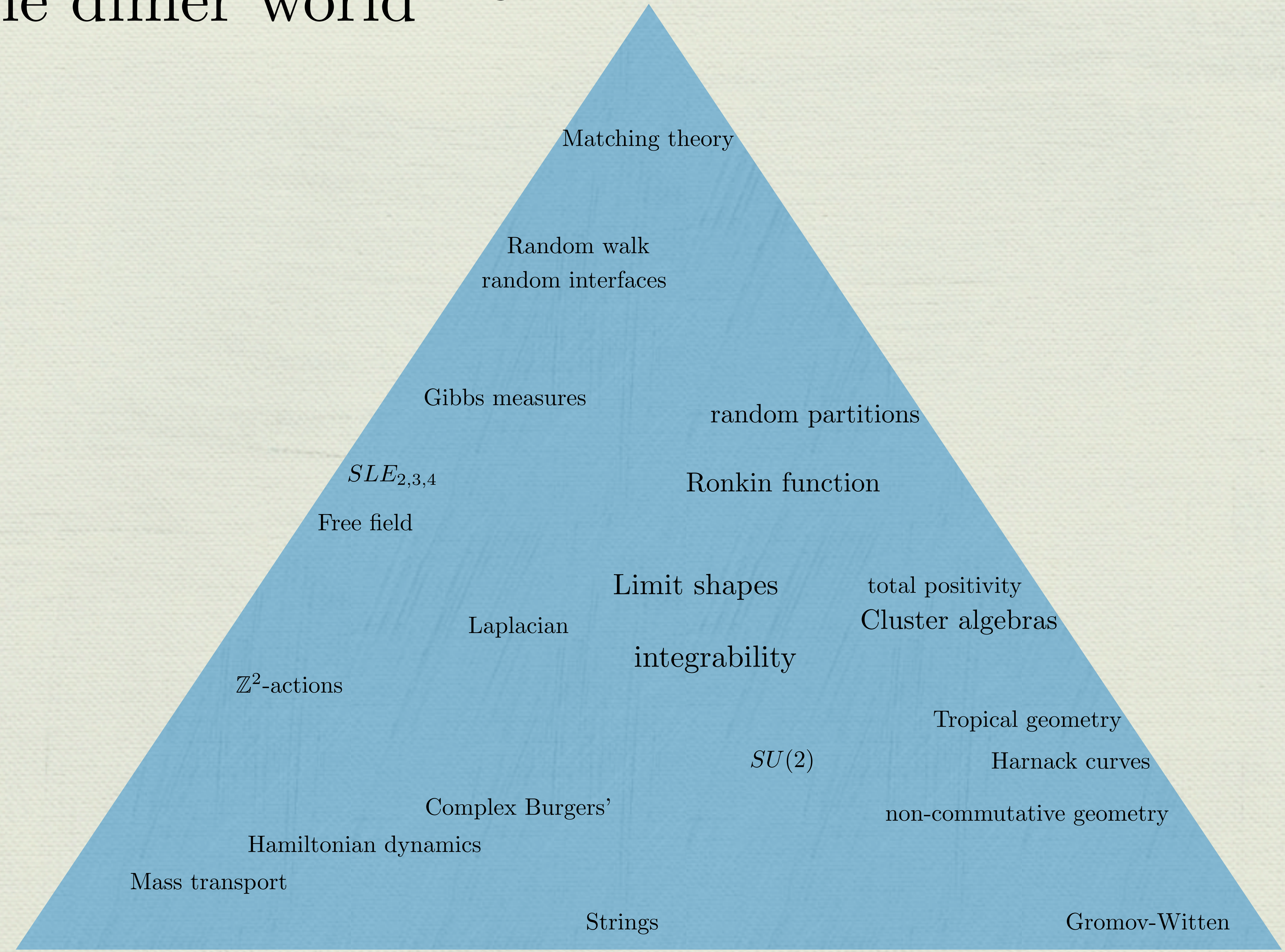
Three phases of measures

frozen phase contours

height contours

The dimer world

Combinatorics



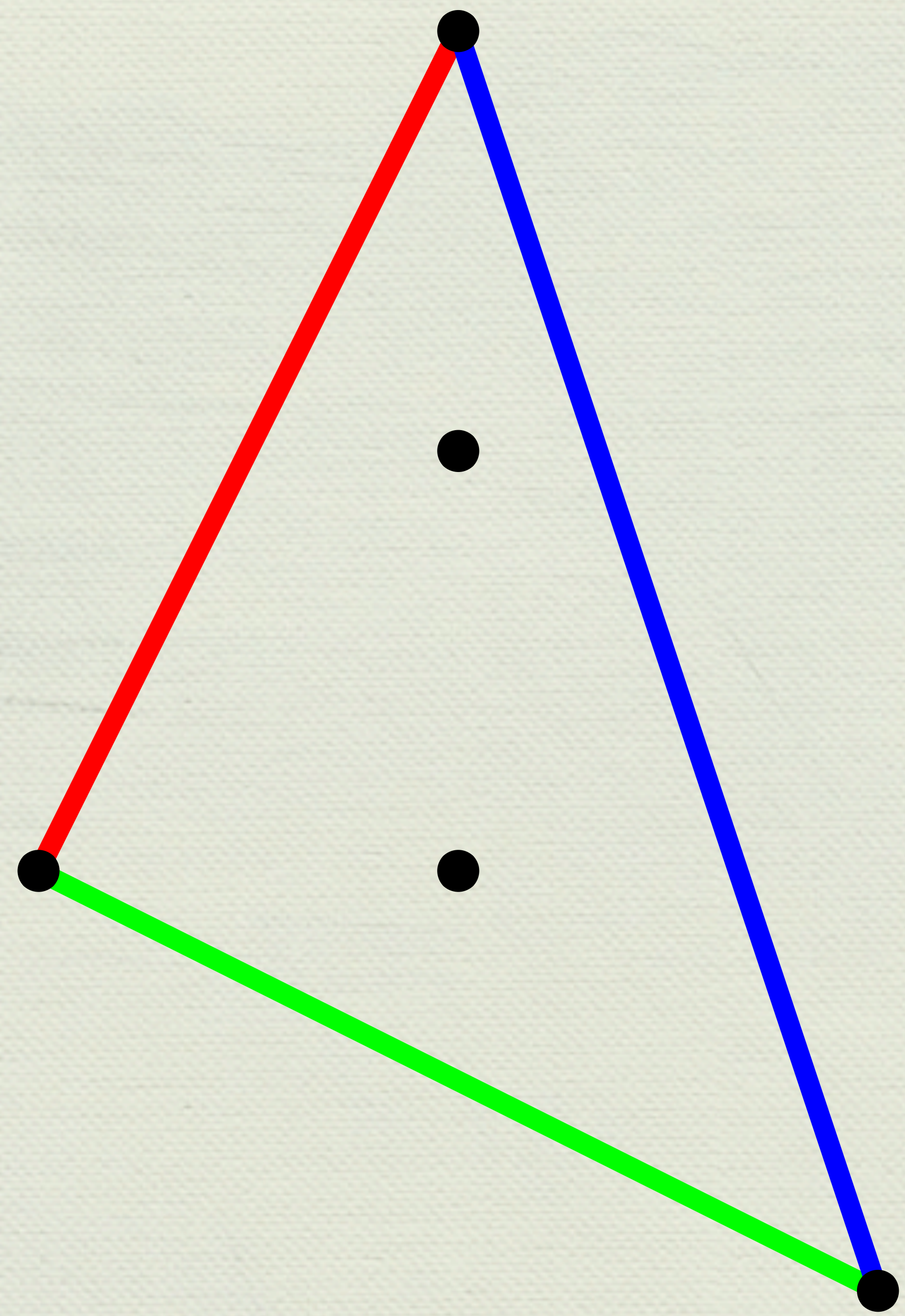
Analysis

Algebraic Geometry

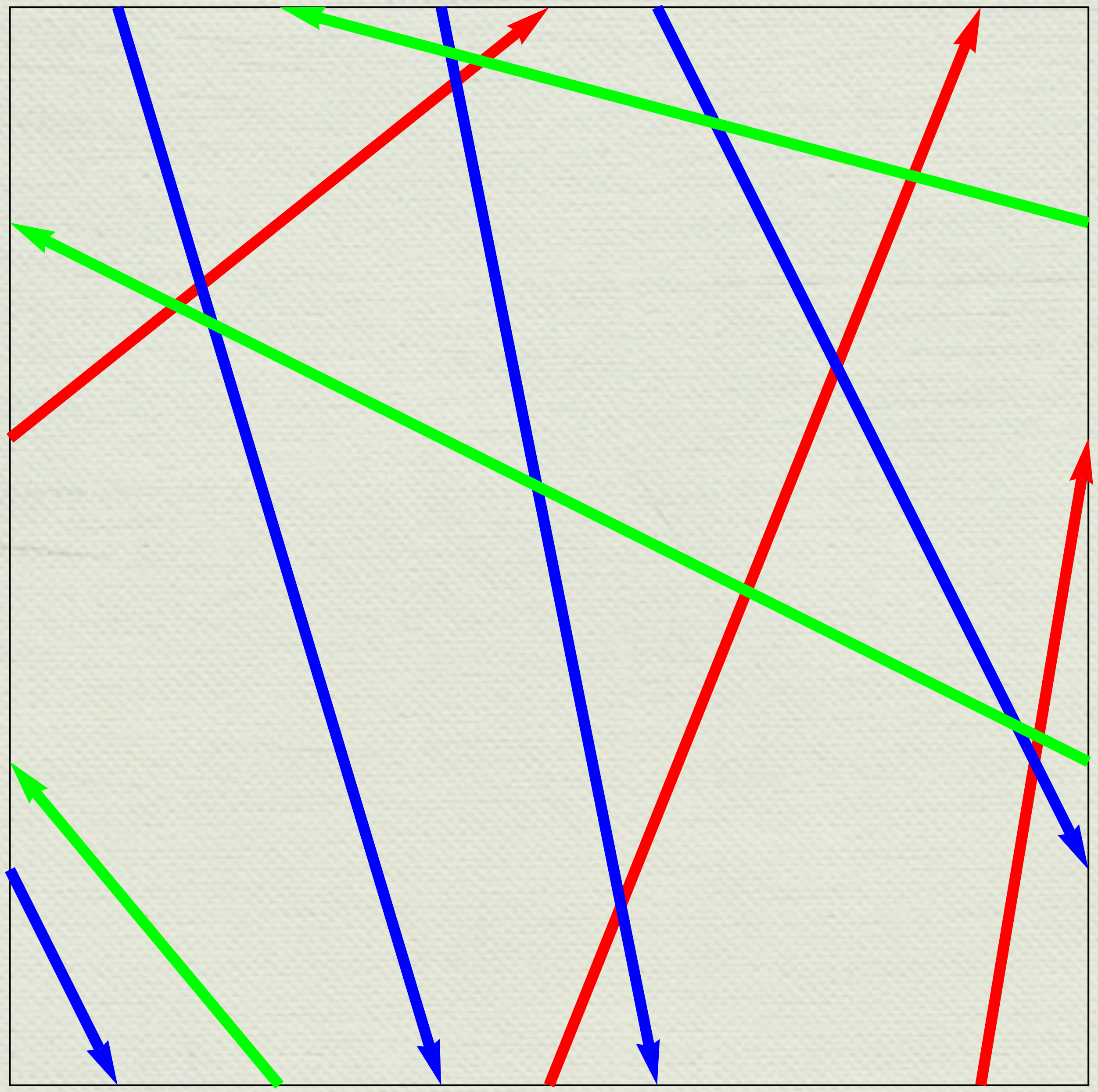
Study the structure of dimer models on periodic planar graphs.

(with A. Goncharov)

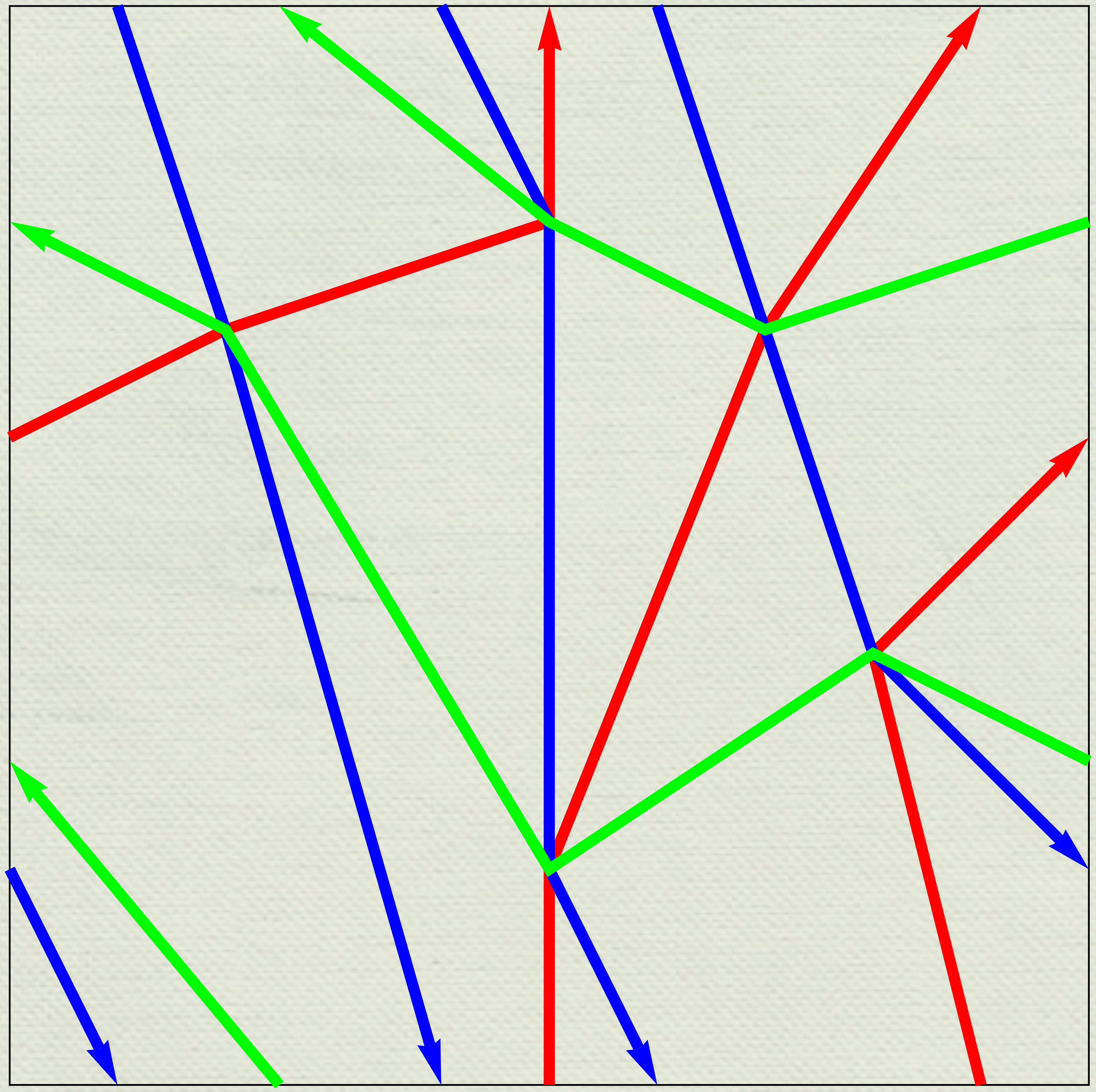
Dimers	Riemann Surfaces
A convex $\mathbb{Z}^2$ polygon	(g, n)
bipartite graph on torus	ideal triangulation
mutation/ urban renewal	flip move
face weights	cross ratios
Harnack curve + divisor	Conformal structure
Dimer Teichmüller space	Teichmüller space $\cong \mathbb{R}^{6g-6+2n}$
tropical Harnack curve	measured lamination
cluster algebra	cluster algebra
commuting Hamiltonians	

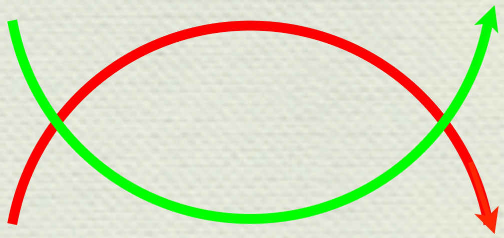


Start: a convex polygon with vertices in $\mathbb{Z}^2$.

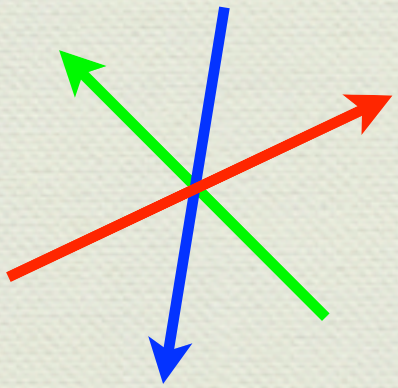


Geodesics on the torus, one for each primitive edge of N.



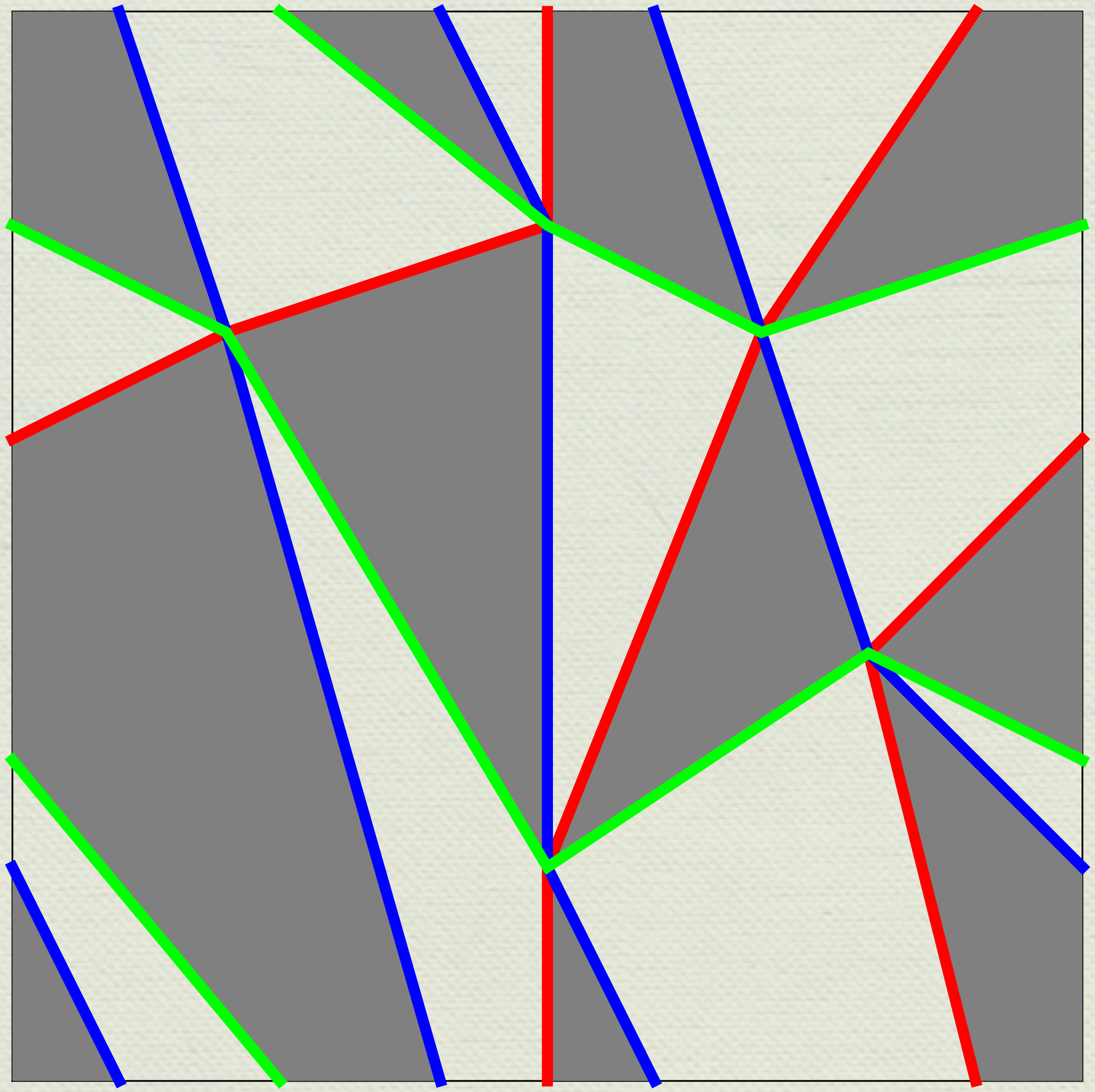


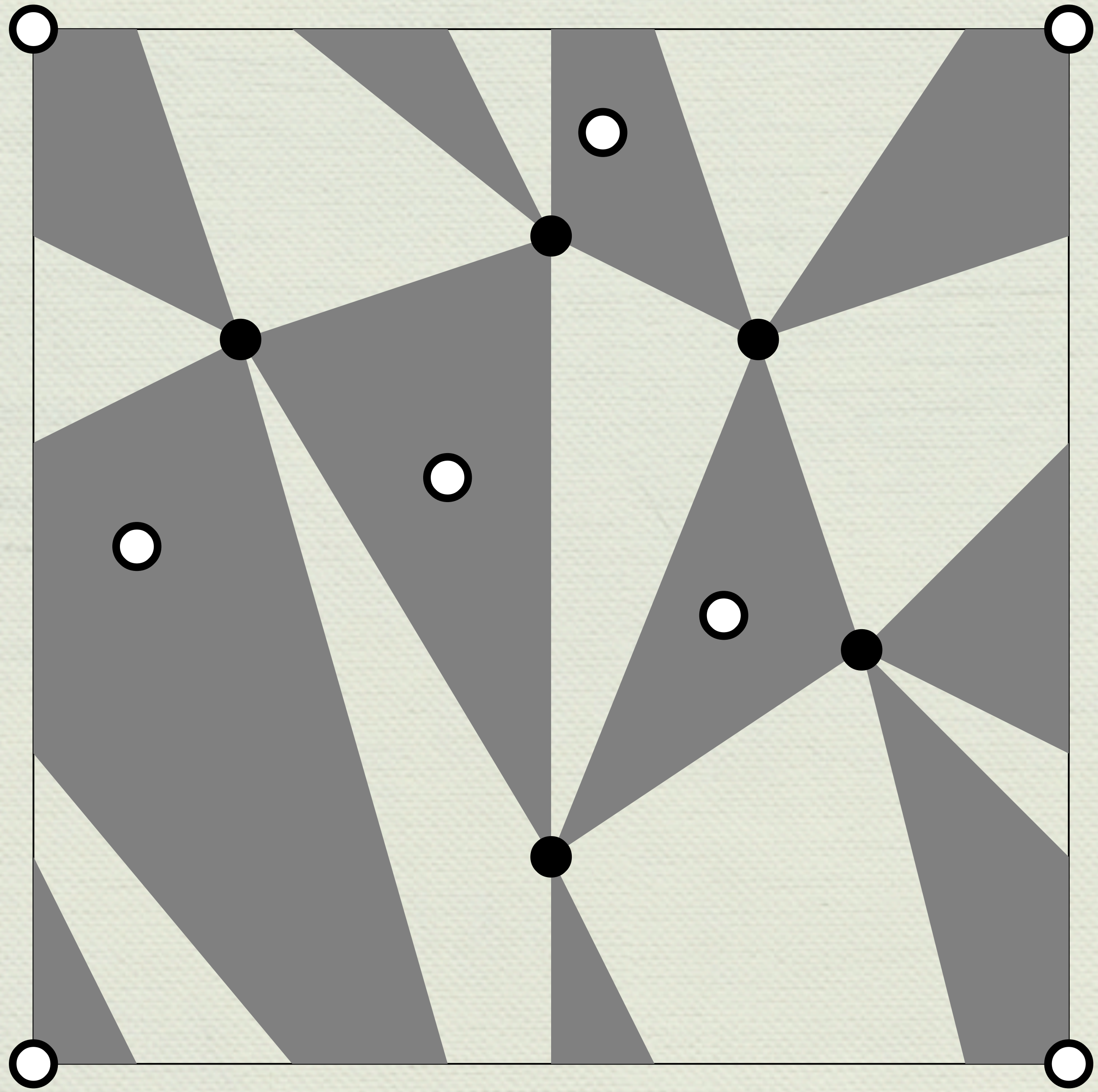
no parallel double crossings

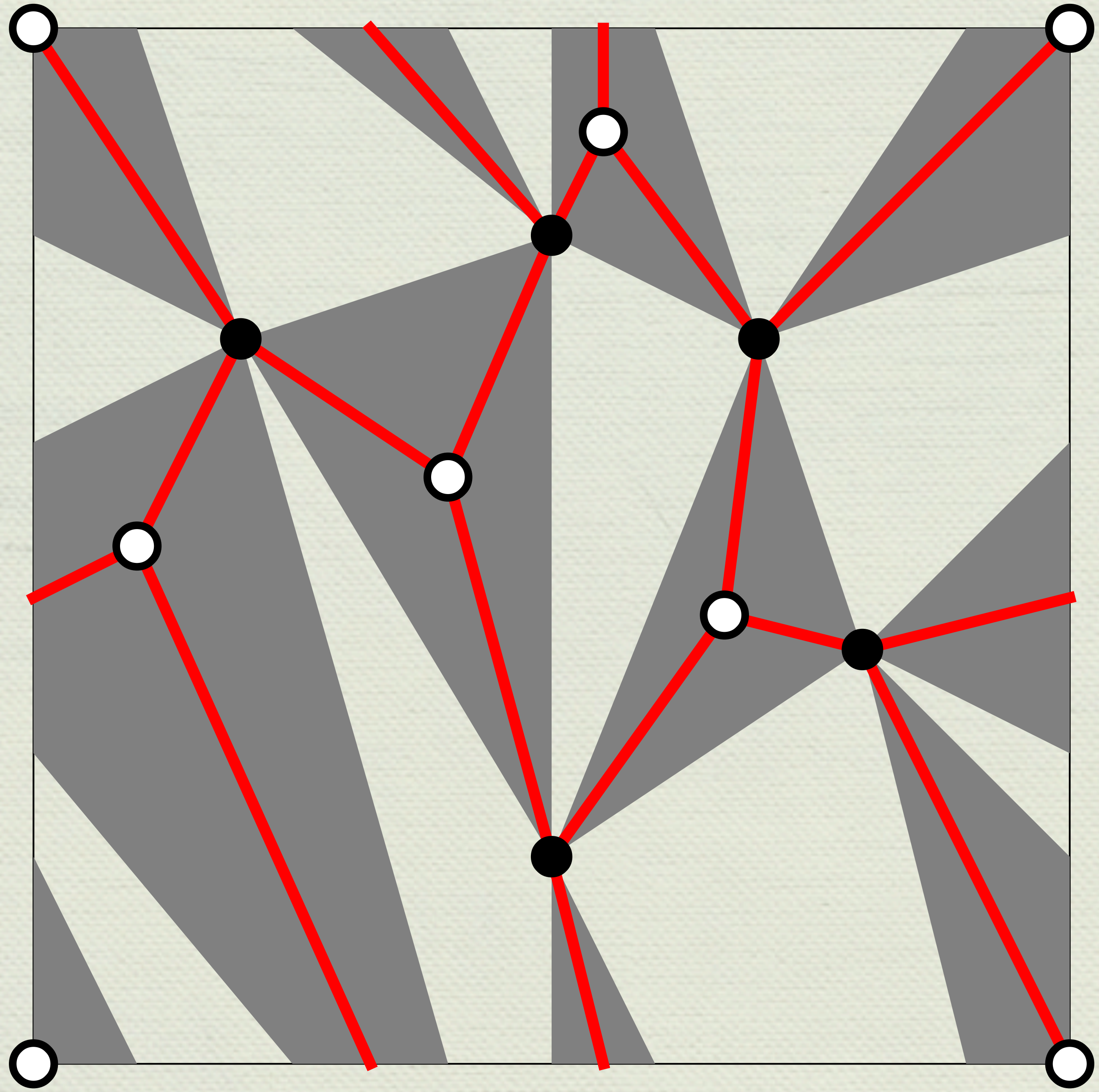


respect circular order

Isotope to a "triple-crossing diagram" [D. Thurston]

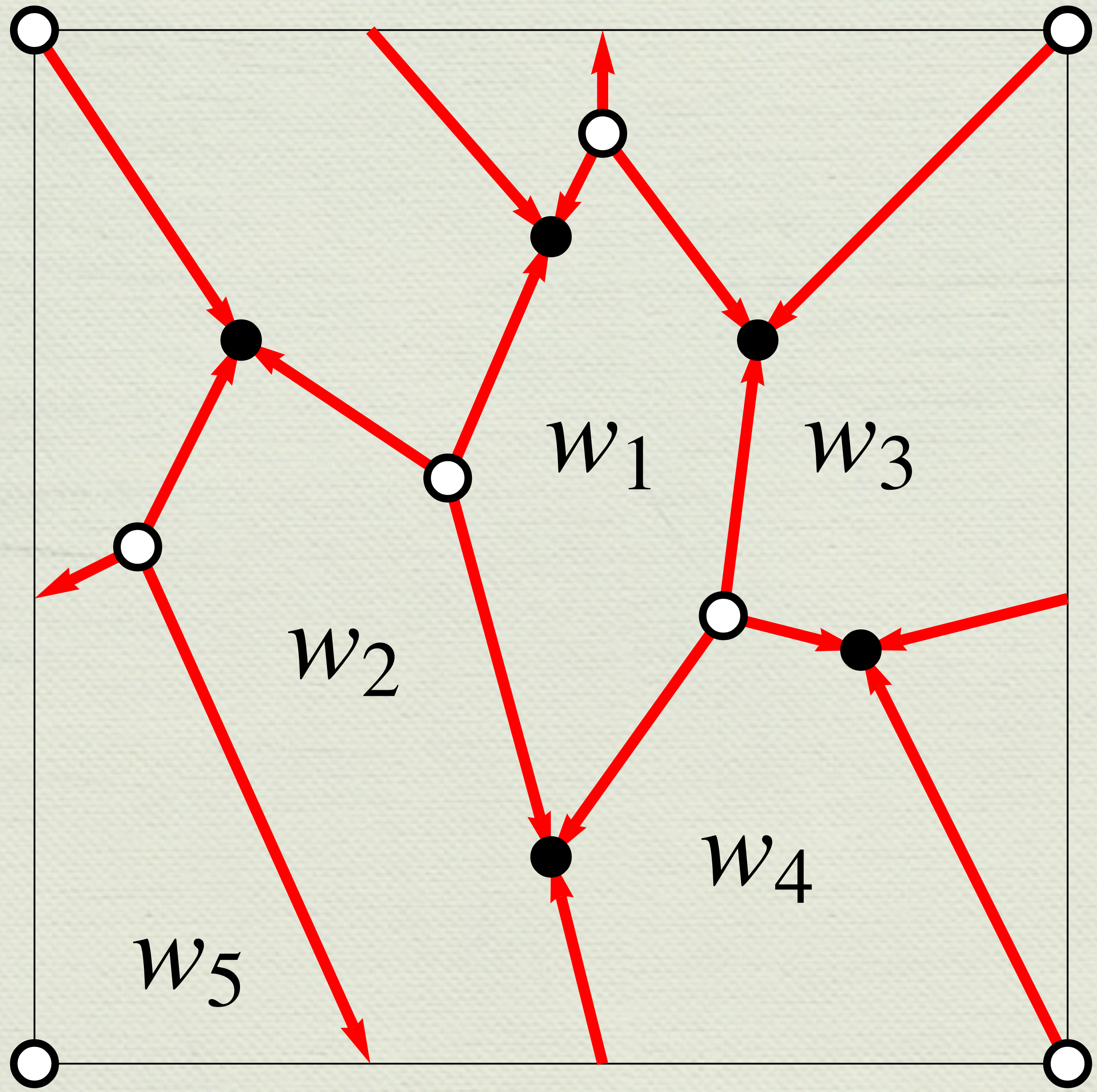






Obtain a bipartite graph

Lemma: $|\text{white vertices}| = |\text{black vertices}| = |\text{faces}| = 2\text{Area}(N)$.



Label faces.

Matchings and homology

Let $\Omega(G) \subset [0,1]^E$ be the matching polytope.

$$f \in \Omega \implies \partial f(v) = \begin{cases} 1 & v \text{ is white} \\ -1 & v \text{ is black.} \end{cases}$$

Lemma: Vertices of Ω are dimer covers of G.

If $f_0, f_1 \in \Omega$ then $\partial(f_1 - f_0) = 0$

so defines a homology class $[f_1 - f_0]$ in $\mathbb{R}^2 = H_1(\mathbb{T}^2, \mathbb{R})$.

Lemma: The image of Ω under $f \mapsto [f - f_0]$ is (a translate of) N.

Let $M(G)$ be the set of dimer covers of G.

Fix $m_0 \in M(G)$. For any $m \in M(G)$, $[m - m_0] \in H_1(G, \mathbb{R})$.

But $H_1(G)$ is generated by the $[w_i]$ and $[z_1], [z_2]$.

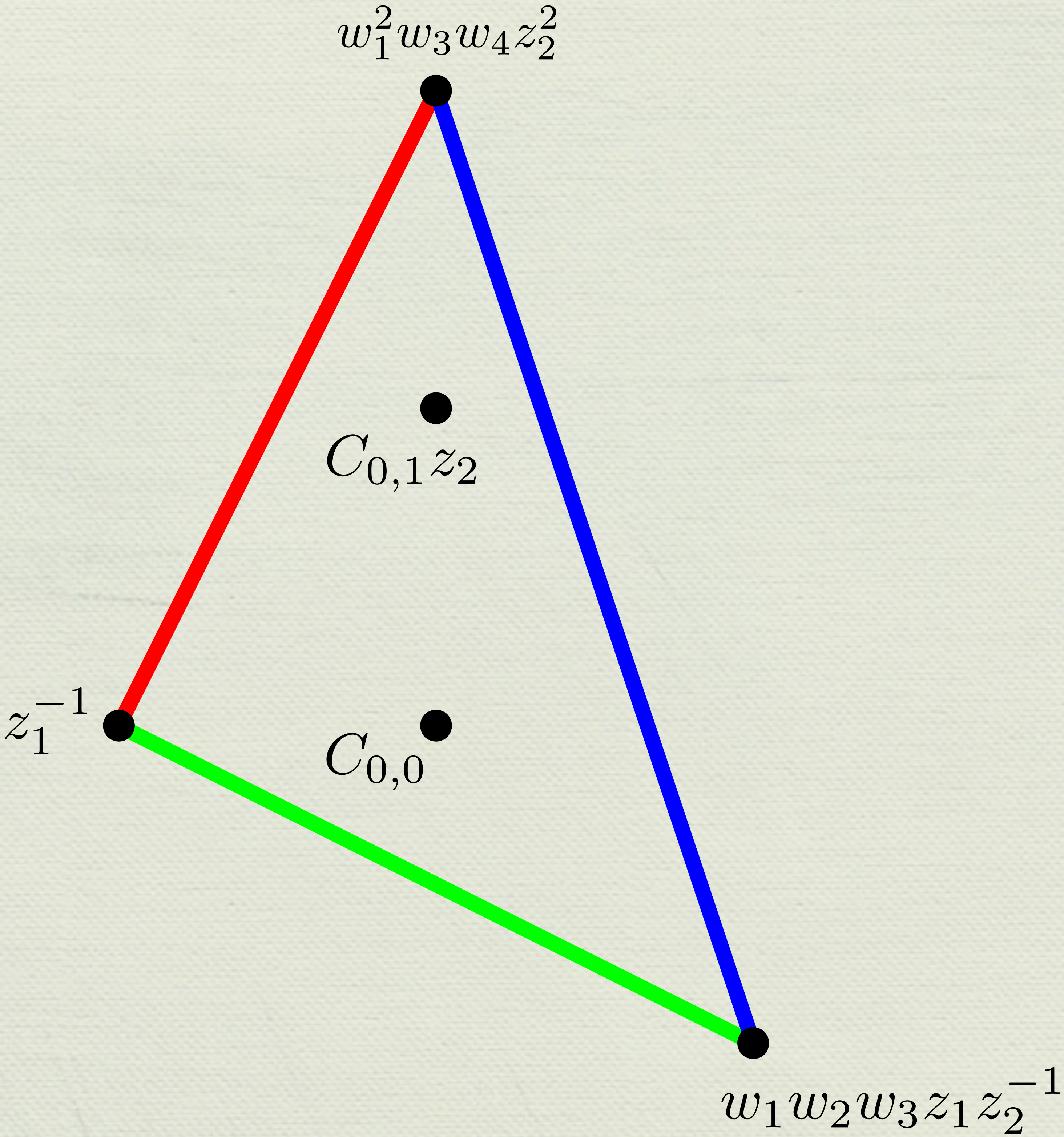
So $[m - m_0] = \sum \alpha_i [w_i] + h_x [z_1] + h_y [z_2]$.

Define the **weight** of m to be

$$\nu(m) = \prod w_i^{\alpha_i} z_1^{h_x} z_2^{h_y}.$$

and the "partition function"

$$P(z_1, z_2; w) = \sum_{m \in M(G)} \nu(m) (-1)^{h_x h_y}.$$



$$C_{0,1} = w_1(1 + w_3 + w_3 w_4 + w_1 w_3 w_4 + w_1 w_2 w_3 w_4)$$

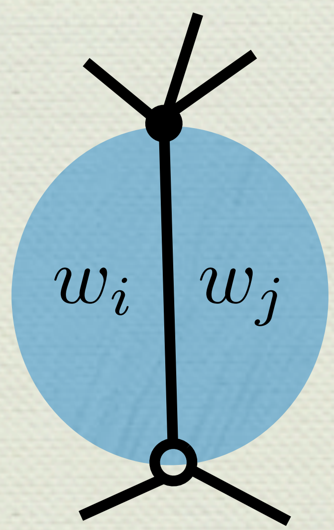
$$C_{0,0} = 1 + w_1 + w_1 w_3 + w_1 w_2 w_3 + w_1 w_2 w_3 w_4$$

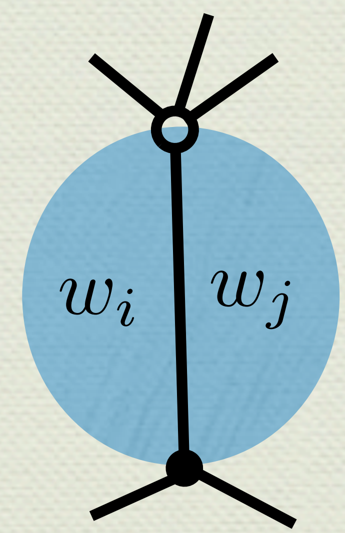
Define a Poisson structure on $(\mathbb{C}^*)^{n+2}$

(a Poisson bracket on $\mathbb{C}[w_1^{\pm1}, \ldots, w_n^{\pm1}, z_1^{\pm1} z_2^{\pm1}])$

$$\{w_i, w_j\} = \varepsilon_{ij} w_i w_j$$

where ε is a skew-symmetric form

$\varepsilon_{ij} = 1$ if 

$\varepsilon_{ij} = -1$ if 

$$\varepsilon_{ij} = 0 \text{ else.}$$

A similar rule for $\{w_i, z_j\}$ and $\{z_i, z_j\}$.

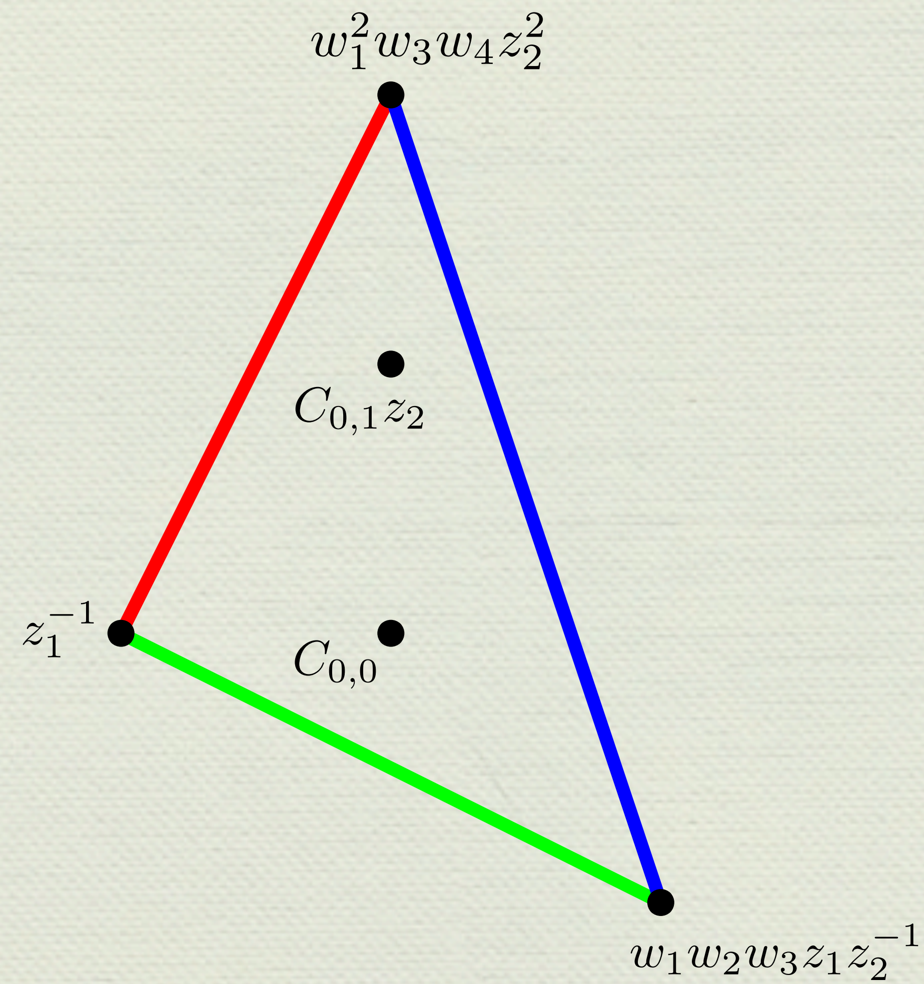
Theorem [Goncharov-K]

This Poisson bracket defines a completely integrable system of dimension $2 + 2\text{Area}(N)$, with symplectic leaves of dimension $2\text{int}(N)$, (twice the number of interior vertices). A basis for the Casimir elements is given by (ratios of) boundary coefficients of P.

The commuting Hamiltonians are the 'interior' coefficients of P.

A quantum integrable system can be defined using q-commuting variables:

$$w_i w_j = q^{2\varepsilon_{ij}} w_j w_i.$$

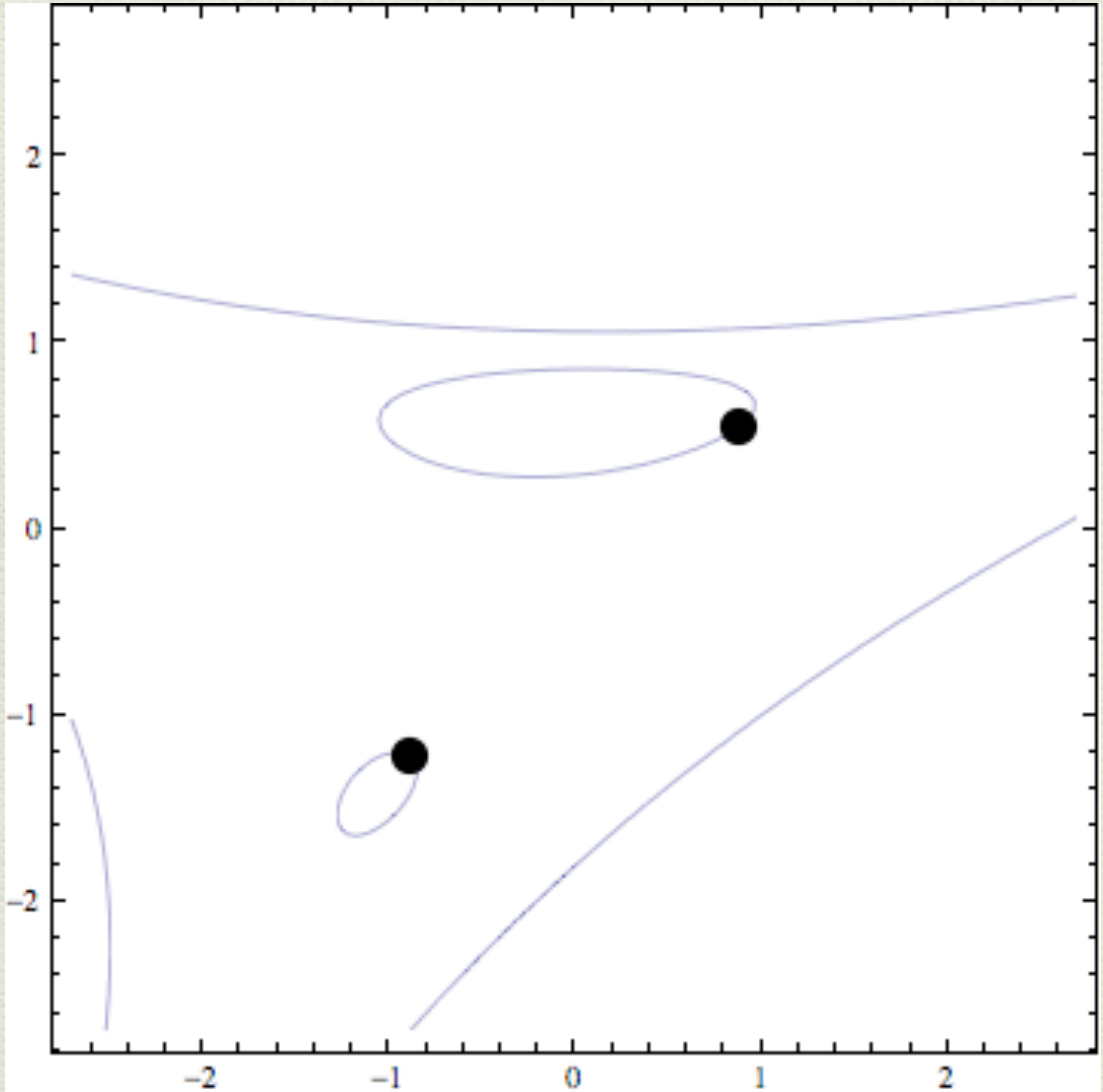


$$C_{0,1} = w_1(1 + w_3 + w_3 w_4 + w_1 w_3 w_4 + w_1 w_2 w_3 w_4)$$
$$C_{0,0} = 1 + w_1 + w_1 w_3 + w_1 w_2 w_3 + w_1 w_2 w_3 w_4$$

Casimirs: $w_1 w_3 w_4 z_2^2 z_1$, $w_1 w_2 w_3 z_1^2 z_2^{-2}$ (commute with everything)

Hamiltonians: $H_0 = C_{0,0} z_1$, $H_1 = C_{0,1} z_1$ (commute with each other)

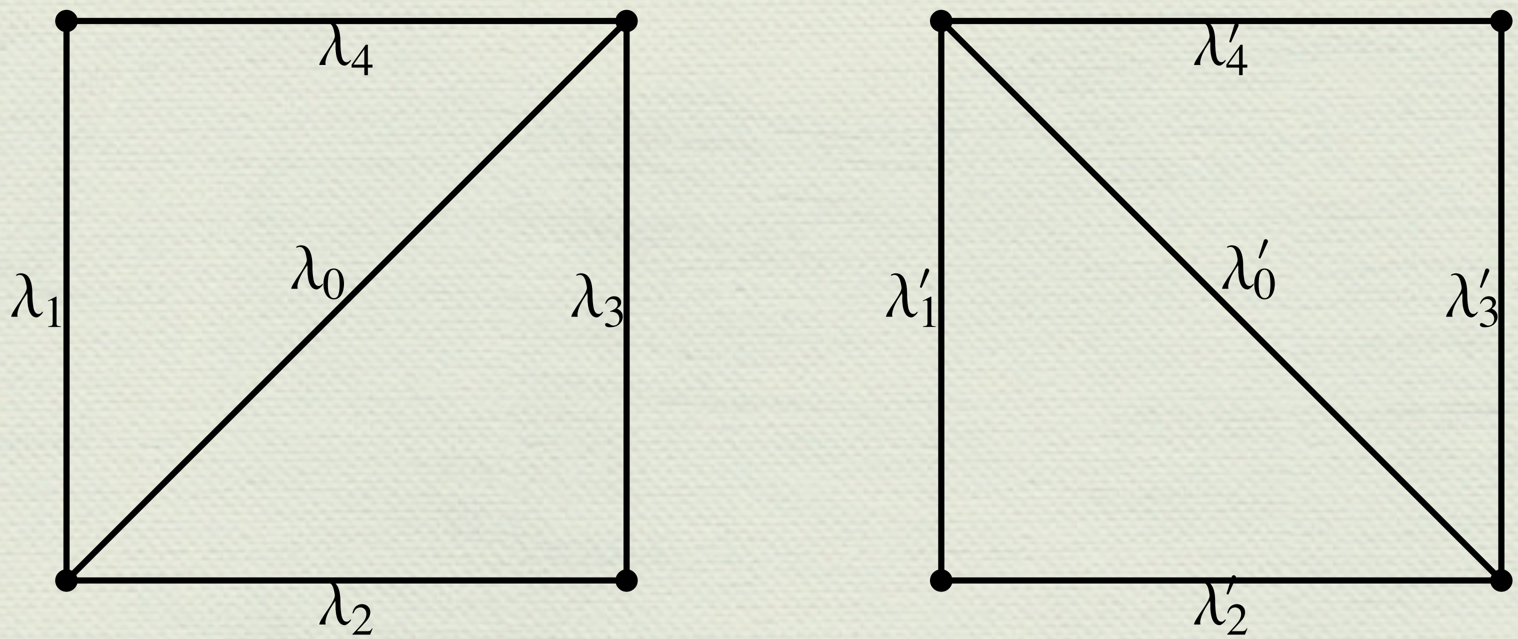
$6 = 2 + 2 * 2$ Complete integrability



$$\frac{\partial z_i}{\partial t} = \{z_i, H\}$$

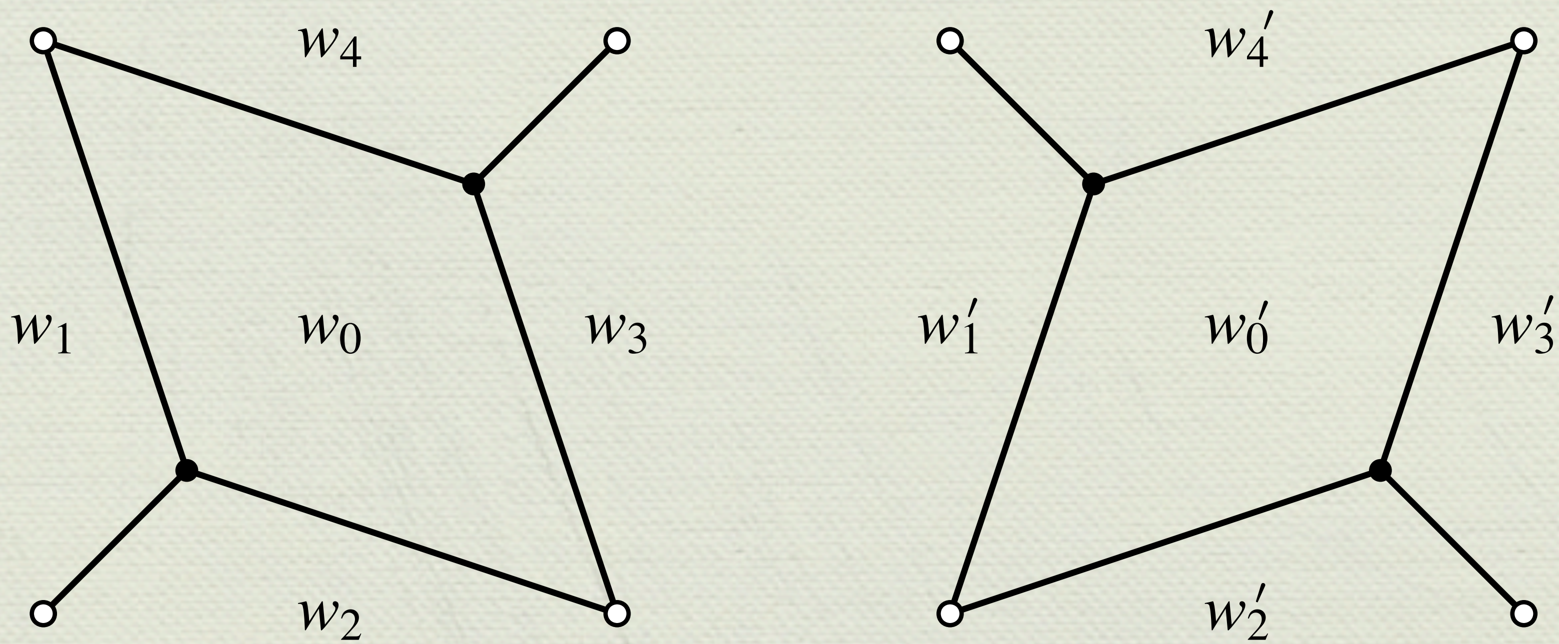
$$\frac{\partial w_i}{\partial t} = \{w_i, H\}$$

Triangle flip



$$\begin{aligned}
\lambda_0' &= \lambda_0^{-1} \\
\lambda_1' &= \lambda_1(1+\lambda_0) \\
\lambda_2' &= \lambda_2(1+\lambda_0^{-1})^{-1} \\
\lambda_3' &= \lambda_3(1+\lambda_0) \\
\lambda_4' &= \lambda_4(1+\lambda_0^{-1})^{-1}
\end{aligned}$$

Urban renewal



$$\begin{aligned}
w_0' &= w_0^{-1} \\
w_1' &= w_1(1+w_0) \\
w_2' &= w_2(1+w_0^{-1})^{-1} \\
w_3' &= w_3(1+w_0) \\
w_4' &= w_4(1+w_0^{-1})^{-1}
\end{aligned}$$