UNIVERSITÉ
DE GENÈVE

FACULTÉ DES SCIENCES

Discrete complex analysis and probability

Stanislav Smirnov

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Complex analysis studies **holomorphic** and **harmonic** functions on the subdomains of the complex plane $\mathbb{C}$ and Riemann surfaces

Discrete complex analysis studies their discretizations, often called **preholomorphic** and **preharmonic** functions
on **planar graphs embedded into** $\mathbb{C}$
(or on discrete Riemann surfaces)

Sometimes terms *discrete analytic* and *discrete harmonic* are used.

We will talk about applications of preholomorphic functions to **probability** and **mathematical physics** using examples from our recent work with
Dmitry Chelkak, **Clément Hongler** and **Hugo Duminil-Copin**.

Preholomorphic or **discrete holomorphic functions** appeared implicitly already in the work of **Kirchhoff** in **1847**.

- A **graph** models an **electric network**.
- Assume all edges have **unit resistance**.
- Let $F(\vec{uv}) = -F(\vec{vu})$ be the **current** flowing from u to v

Then the **first** and the **second Kirchhoff laws** state that the sum of currents flowing from a vertex is zero:

$$\sum_{u:\ \text{neighbor of}\ v} F(\vec{uv}) = 0 , \tag{1}$$

the sum of the currents around any oriented closed contour γ is zero:

$$\sum_{\vec{uv}\in\gamma} F(\vec{uv}) = 0 . \tag{2}$$

Rem For planar graphs contours around faces are sufficient

The **second** and the **first Kirchhoff laws** are equivalent to

$\boldsymbol{F}$ being given by the **gradient** of a **potential function** $\boldsymbol{H}$:

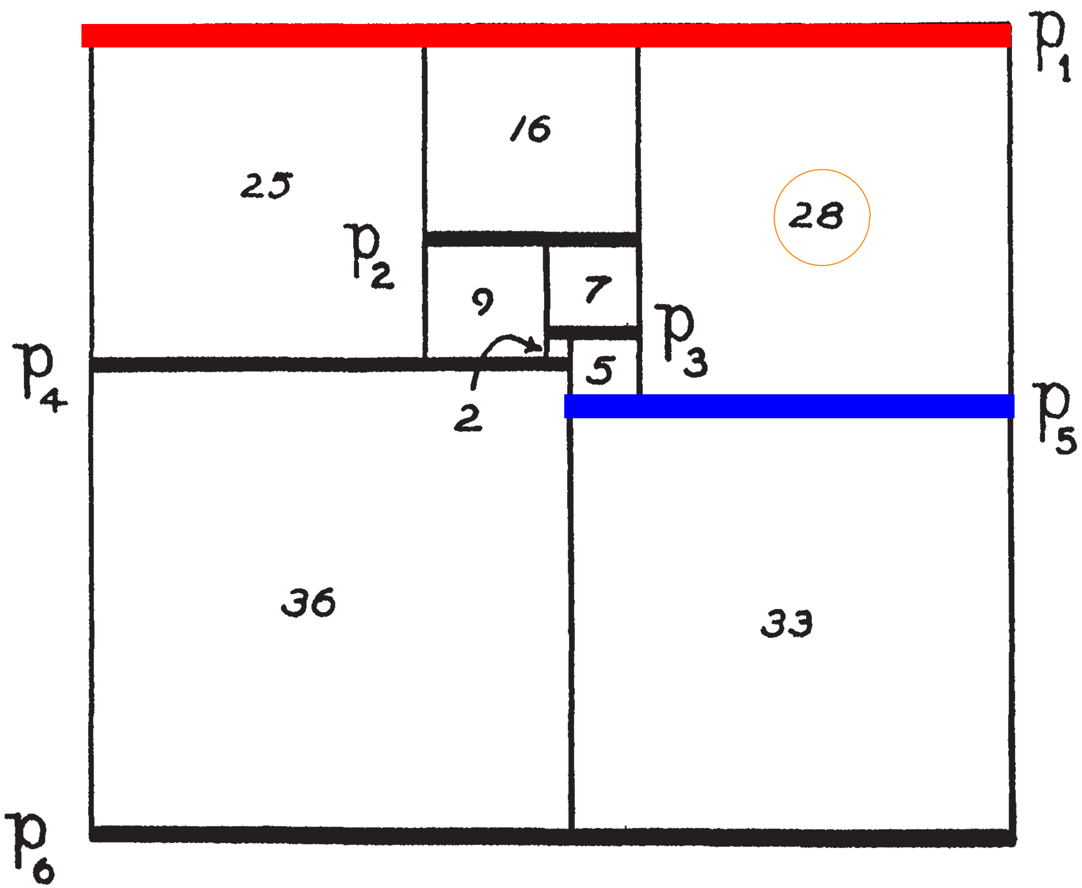
$$F(\vec{uv}) \;=\; H(v) - H(u)\;, \tag{2'}$$

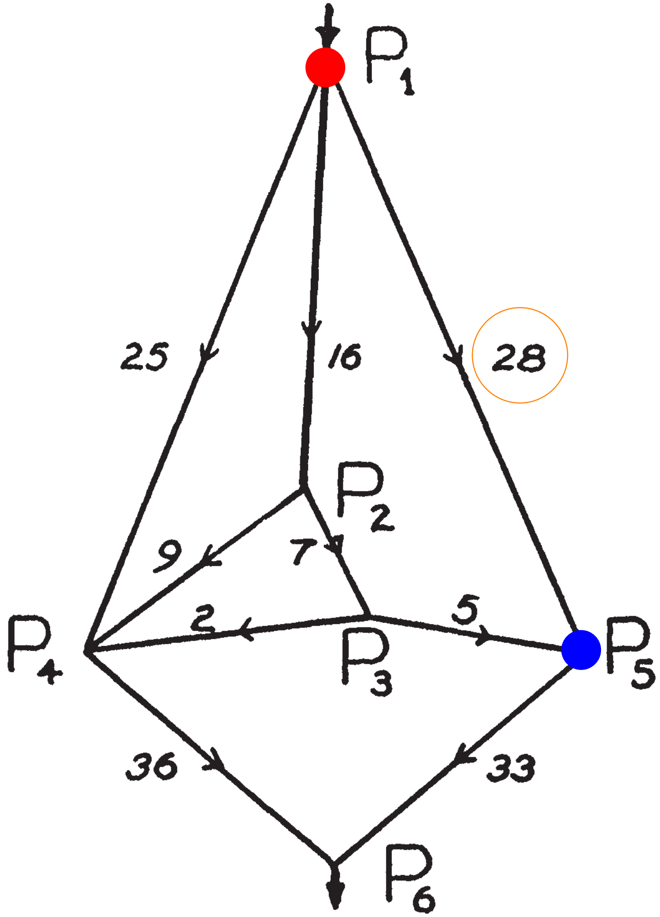
and the latter being **preharmonic**:

$$0 \;=\; \Delta H(u) \;:=\; \sum_{v:\ \text{neighbor of}\ u} \left(H(v) - H(u)\right)\,. \tag{1'}$$

- Different resistances amount to putting weights into **(1')**.
- Preharmonic functions can be defined on any graph, and have been very well studied.
- On planar graphs preharmonic gradients are **preholomorphic**, similarly to harmonic gradients being holomorphic.

Besides the original work of Kirchhoff, the first notable application was perhaps the famous article **[Brooks, Smith, Stone & Tutte, 1940]** *"The dissection of rectangles into squares"* which used preholomorphic functions to construct tilings of rectangles by squares.



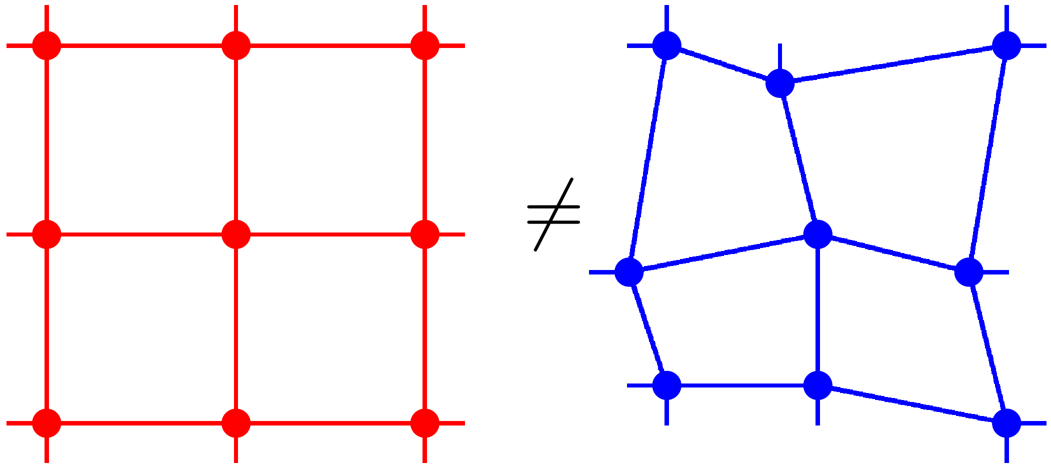


tilings by squares $\leftrightarrow$ **preholomorphic functions on planar graphs**

There are several other ways to introduce discrete structures on graphs in parallel to the usual complex analysis.

We want such discretizations that
restrictions of **holomorphic** (or **harmonic**) functions become *approximately* **preholomorphic** (or **preharmonic**).

Thus we speak about
- a **planar graph**,
- its **embedding** into $\mathbb{C}$
  or a Riemann surface,
- a **preholomorphic** definition.



The applications we are after require passages to the **scaling limit** (as mesh of the lattice tends to zero), so we want to deal with discrete structures that converge to the usual complex analysis as we take finer and finer graphs.

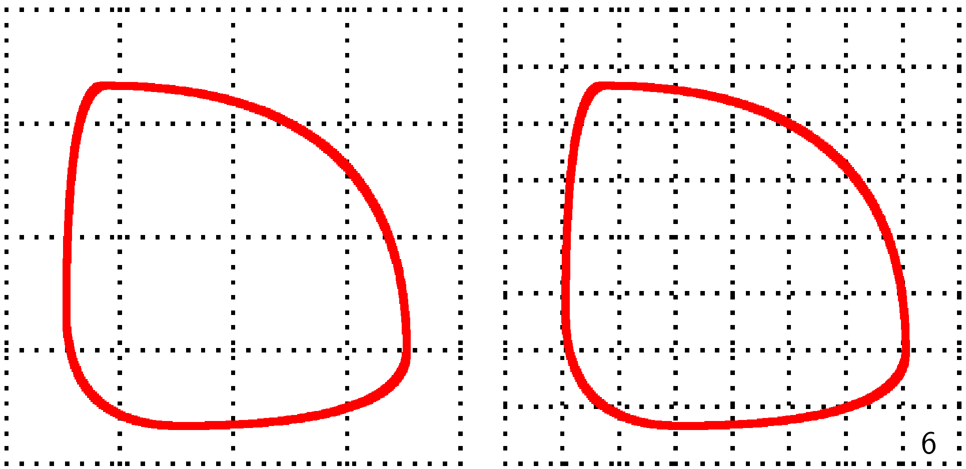
Preharmonic functions on the **square lattice** with decreasing mesh fit well into this context.

They were studied in a number of papers in early twentieth century: **[Phillips & Wiener 1923, Bouligand 1926, Lusternik 1926 . . .]**, culminating in the seminal **[Courant, Friedrichs & Lewy 1928]** studying the **Dirichlet Boundary Value Problem**:

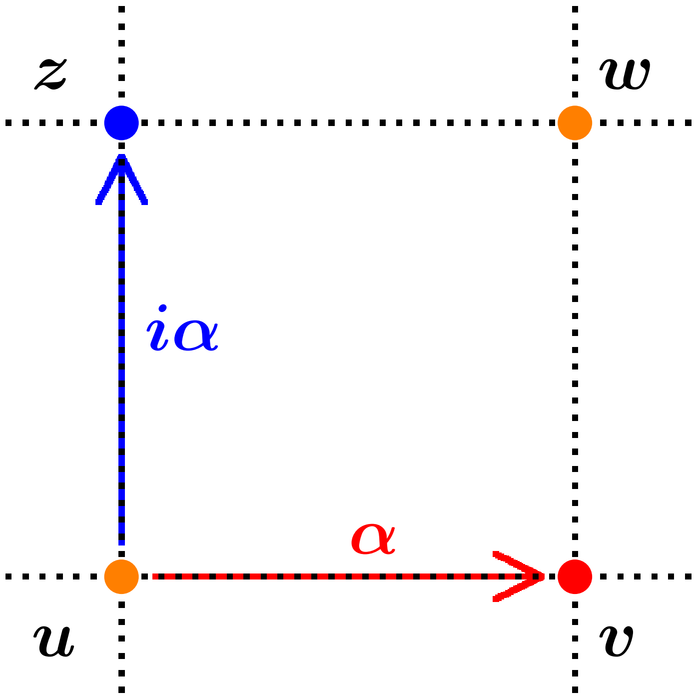
Theorem [CFL] Consider a smooth domain and boundary values. Then, as the square lattice mesh tends to zero, (discrete) preharmonic solution of the Dirichlet BVP converges to (continuous) harmonic solution of the same BVP along with all its partial derivatives.

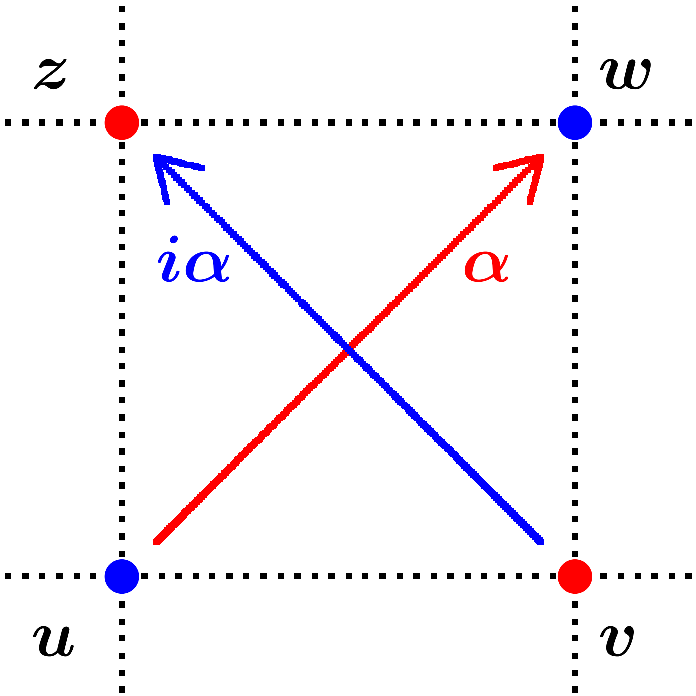
Rem Proved for discretizations of a general elliptic operator

Rem Relation with the **Random Walk** explicitly stated



Preholomorphic functions were explicitly studied in **[Isaacs, 1941]** under the name "monodiffric". Issacs proposed two ways to discretize the **Cauchy-Riemann equations** $\partial_{i\alpha}F = i\partial_{\alpha}F$ on the square lattice:





$$\frac{F(z)-F(u)}{z-u} = \frac{F(v)-F(u)}{v-u} \quad (1^{st})$$

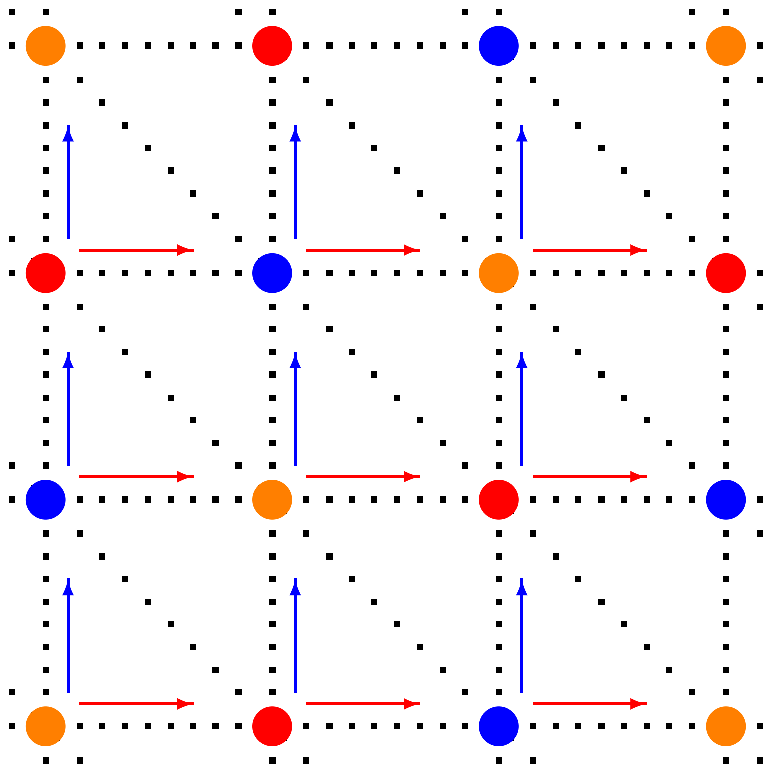
$$F(z) - F(u) = i\,(F(v) - F(u))$$

$$\frac{F(z)-F(v)}{z-v} = \frac{F(w)-F(u)}{w-u} \quad (2^{nd})$$

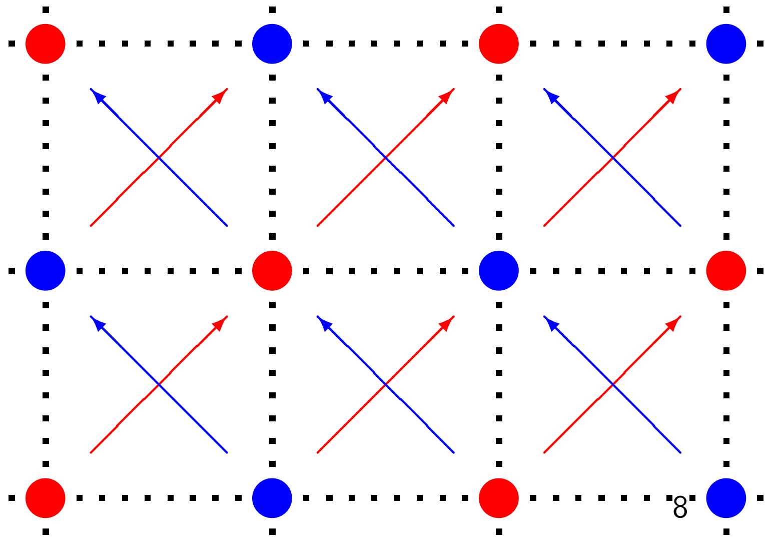
$$F(z) - F(v) = i\,(F(w) - F(u))$$

Rem There are more possible definitions

Isaacs' first definition is **asymmetric** on the **square lattice**.
If we add the diagonals in one direction, it provides one difference relation for every other triangle and becomes **symmetric** on the **triangular lattice**.
The first definition was studied by **Isaacs** and others, and recently it was reintroduced by **Dynnikov** and **Novikov**.

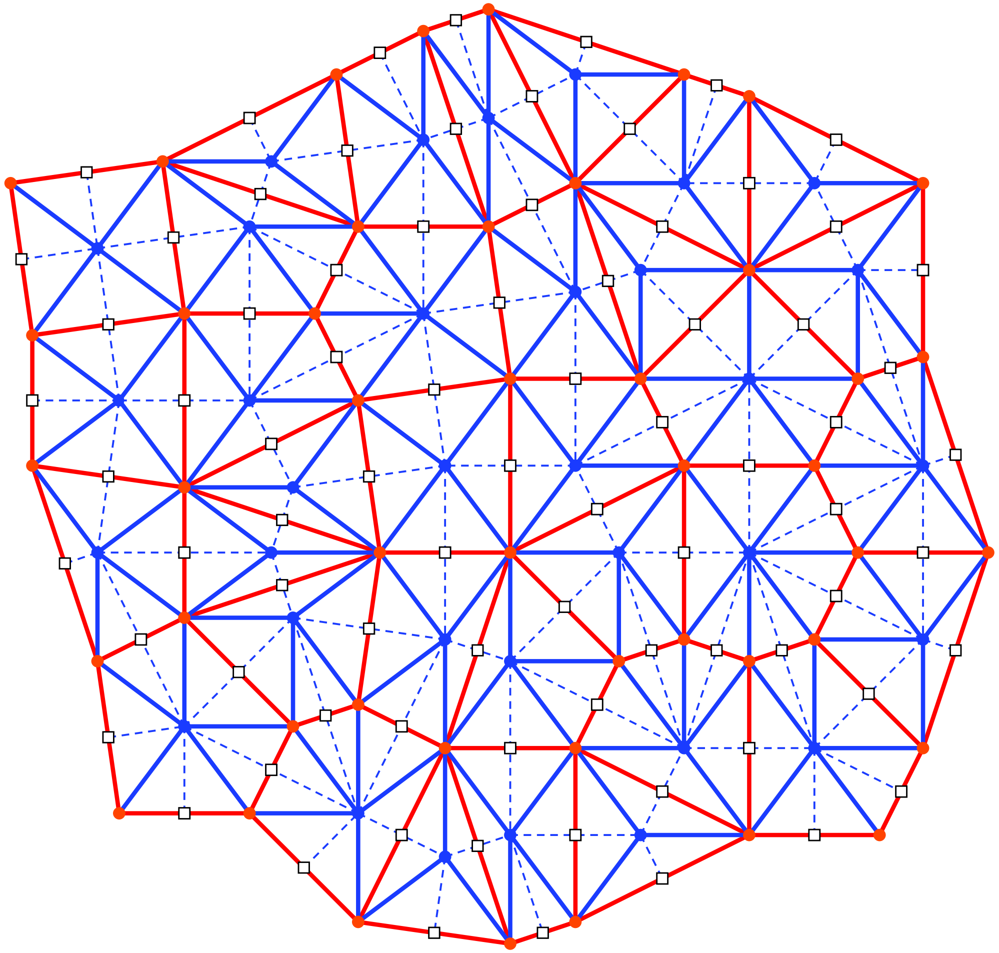


Isaacs' second definition is **symmetric** on the **square lattice**.
Note that the Cauchy-Riemann equation relates the real part on the red vertices to the imaginary part on the blue vertices, and vice versa.



The second definition was reintroduced by **Lelong-Ferrand** in 1944.

She studied the **scaling limit**, giving new proofs of the Riemann uniformization and the Courant-Friedrichs-Lewy theorems.



This was followed by extensive studies of **Duffin** and others.

Duffin extended the definition to **rhombic lattices** – graphs, with rhombi faces. Equivalently, blue or red vertices form **isoradial graphs**, whose faces can be inscribed into circles of the same radius.

Many results were generalized to this setting by **Duffin, Mercat, Kenyon, Chelkak & Smirnov**.

With most **linear** definitions of preholomorphicity,
discrete complex analysis starts like the usual one.

On the square lattice it is easy to prove that **if** $F, G \in \mathrm{Hol}$, then

- $F \pm G \in \mathrm{Hol}$
- **derivative** F' is well-defined and $\in \mathrm{Hol}$ (on the dual lattice)
- **primitive** $\int^z F$ is well-defined and $\in \mathrm{Hol}$ (on the dual lattice)
- $\oint F = 0$ for closed contours
- **maximum principle**
- $F = H + i\tilde{H} \Rightarrow H$ **preharmonic** (on even sublattice)
- H preharmonic $\Rightarrow \exists\ \tilde{H}$ such that $H + i\tilde{H} \in \mathrm{Hol}$

Problem: On the square lattice $F, G \in \mathrm{Hol} \not\Rightarrow F \cdot G \in \mathrm{Hol}$.
On rhombic lattices even $F \in \mathrm{Hol} \not\Rightarrow F' \in \mathrm{Hol}$.
Thus we cannot easily mimic continuous proofs.

Rem There are also **non-linear** definitions, e.g. in circle-packings.

There are several expositions about the applications of the **discrete complex analysis** to **geometry**, **combinatorics**, **analysis**:

- **L. Lovász**: *Discrete analytic functions: an exposition*, in Surveys in differential geometry. Vol. IX, Int. Press, 2004.

- **K. Stephenson**: *Introduction to circle packing. The theory of discrete analytic functions*, CUP, 2005

- **C. Mercat**: *Discrete Riemann surfaces*, in Handbook of Teichmüller theory. Vol. I, EMS, 2007

- **A. Bobenko and Y. Suris**: *Discrete differential geometry*, AMS, 2008

We will concentrate on its applications to **probability** and **statistical physics**.

New approach to 2D integrable models of statistical physics

We are interested in **scaling limits**, i.e. we consider some statistical physics model on a planar lattice with mesh ε tending to zero.

We need an **observable** F_ε (edge density, spin correlation, exit probability,. . .) which is **preholomorphic** and solves some **Boundary Value Problem**. Then we can argue that in the scaling limit F_ε converges to a holomorphic solution F of the same BVP.

Thus F_ε has a **conformally invariant scaling limit**, also $F_\varepsilon \approx F$ and we can deduce other things about the model at hand.

Several models were approached in this way:

- **Random Walk** – **[Courant, Friedrich & Lewy, 1928]**
- **Dimer model, UST** – **[Kenyon, 2001]**
- **Critical percolation** – **[Smirnov, 2001]**
- **Uniform Spanning Tree** – **[Lawler, Schramm & Werner, 2003]**
- **Random cluster model with $q = 2$** – **[Smirnov, 2006]**

An example: critical percolation

to color every hexagon we toss a coin:

tails $\Rightarrow$ blue, **heads $\Rightarrow$ yellow**

Blue hexagons are "holes" in a yellow rock.

Can the water sip through? Hard to see!

The reason: clusters (connected blue sets) are complicated fractals of dimension $91/48$

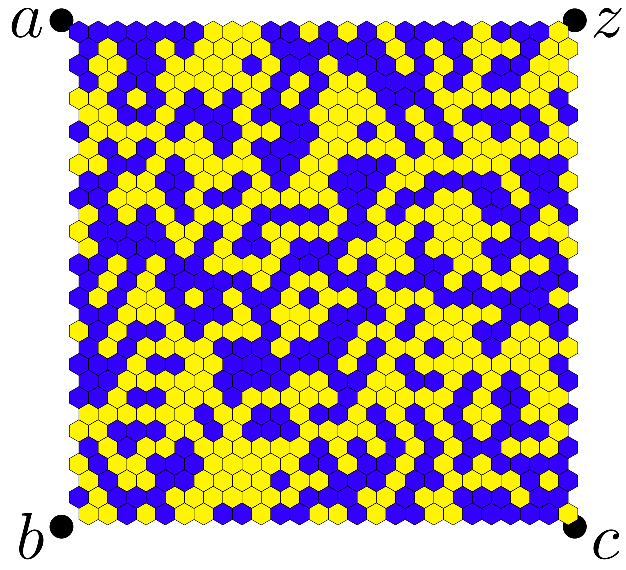
(a cluster of diam D on average has $\approx D^{91/48}$ hexagons), Numerical study and conjectures by Langlands, Pouilot & Saint-Aubin; Aizenman

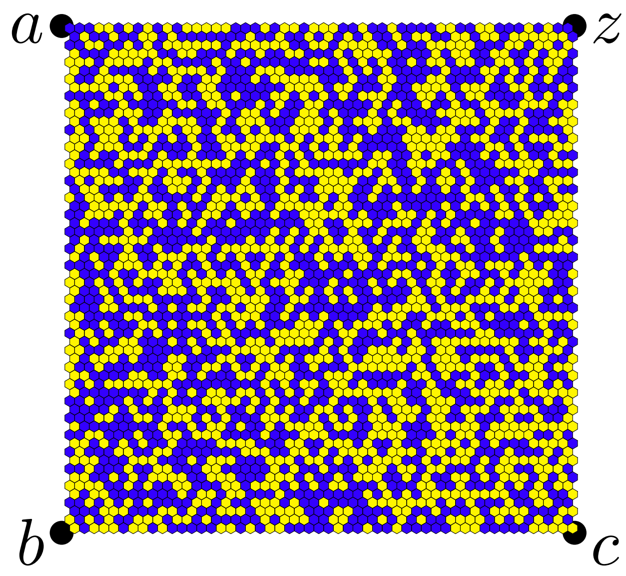
Cardy's prediction: in the scaling limit for a rectangle of conformal modulus m

$$\mathbb{P}\,(\text{crossing}) = \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{4}{3}\right)}\, m^{1/3}\, {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{4}{3}; m\right)$$

Thm [Smirnov 2001] holds on hex lattice

Proof by allowing z to move inside the rectangle and showing that complexified $\mathbb{P}$ is approximately preholomorphic solution of a DBVP





We will discuss newer applications, with examples from our work with **Dmitry Chelkak**, **Clément Hongler**, **Hugo Duminil-Copin**:

Stanislav Smirnov: *Towards conformal invariance of 2D lattice models*, in Proceedings of the ICM 2006 (Madrid)

Stanislav Smirnov: *Conformal invariance in random cluster models. I. Holomorphic fermions in the Ising model*, Ann. Math. 172 (2010)

Dmitry Chelkak & S. S.: *"Discrete complex analysis on isoradial graphs"*, Adv. in Math., to appear

Dmitry Chelkak & S. S.: *"Universality in the 2D Ising model and conformal invariance of fermionic observables"*, Inv. Math., to appear

Clément Hongler & S. S.: *"The energy density in the planar Ising model"*, arXiv:1008.2645

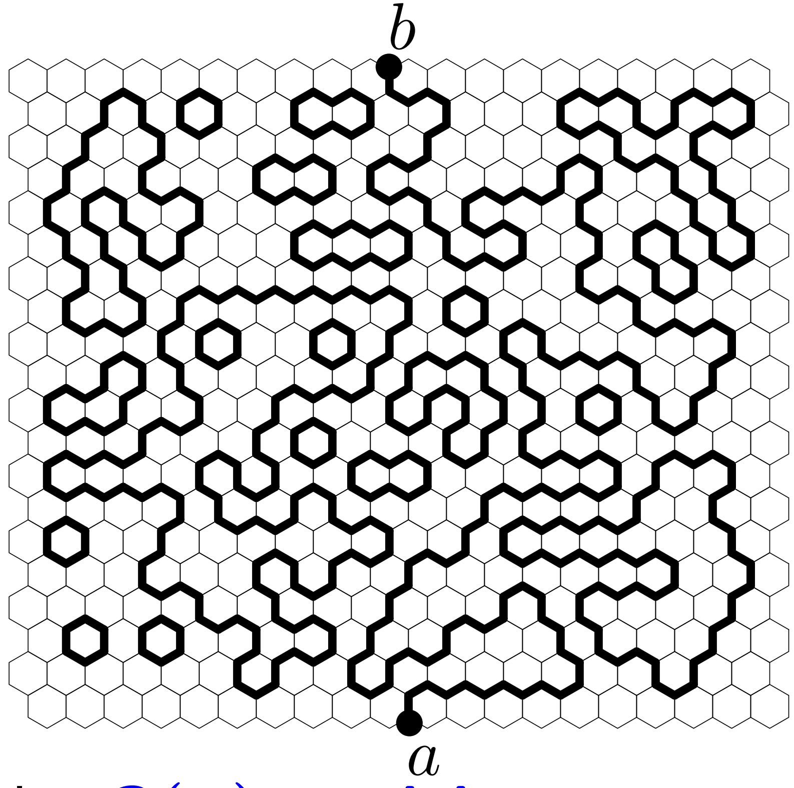
Hugo Duminil-Copin & S. S.: *"The connective constant of the honeycomb lattice equals* $\sqrt{2+\sqrt{2}}$, arXiv:1007.0575

A model: Loop gas on hexagonal lattice
Configurations of disjoint simple loops
Loop-weight $n \in [0,2]$, edge-weight $x > 0$
Partition function given by

$$\boldsymbol{Z = \sum_{\text{configs}} n^{\#\ \text{loops}}\, x^{\#\ \text{edges}}}$$

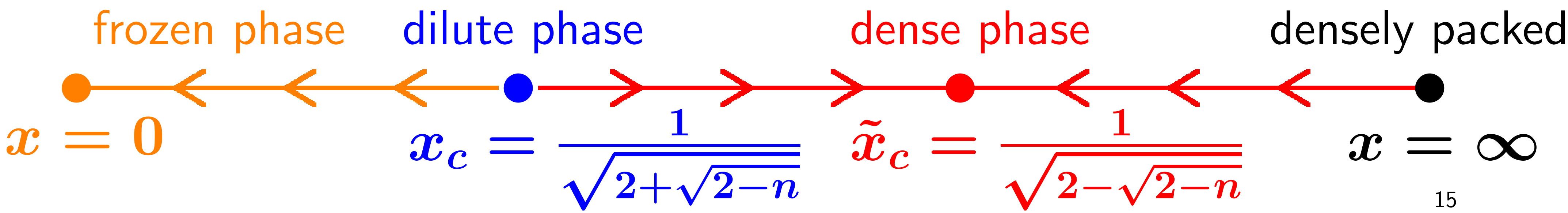
Probability of a configuration is

$$\mathbb{P}(\text{config}) \ = \ \boldsymbol{n^{\#\ \text{loops}}\, x^{\#\ \text{edges}}\ /\ Z}$$

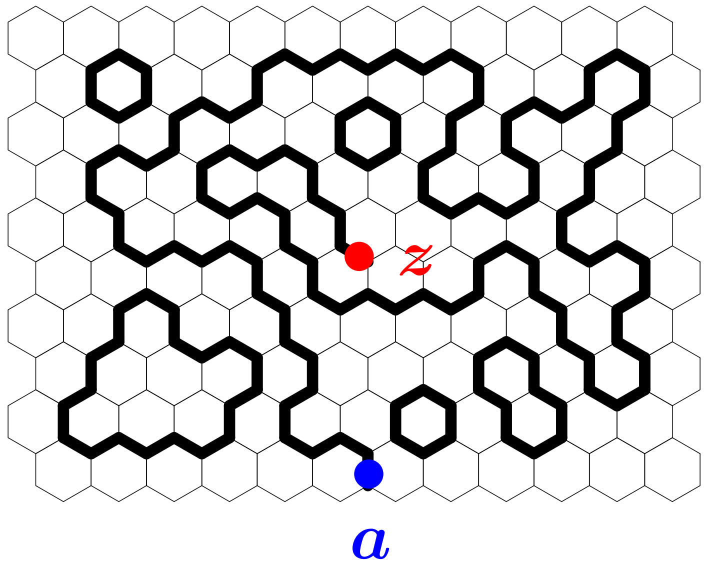


- This is **high-temperature expansion** of the **$O(n)$ model**.
- **Dobrushin boundary conditions**: loops + an interface $\gamma : a \leftrightarrow b$.

Nienhuis proposed the following **renormalization** picture:
for fixed $\boldsymbol{n}$, rescaling amounts to changing $\boldsymbol{x}$:



Loop gas preholomorphic observable



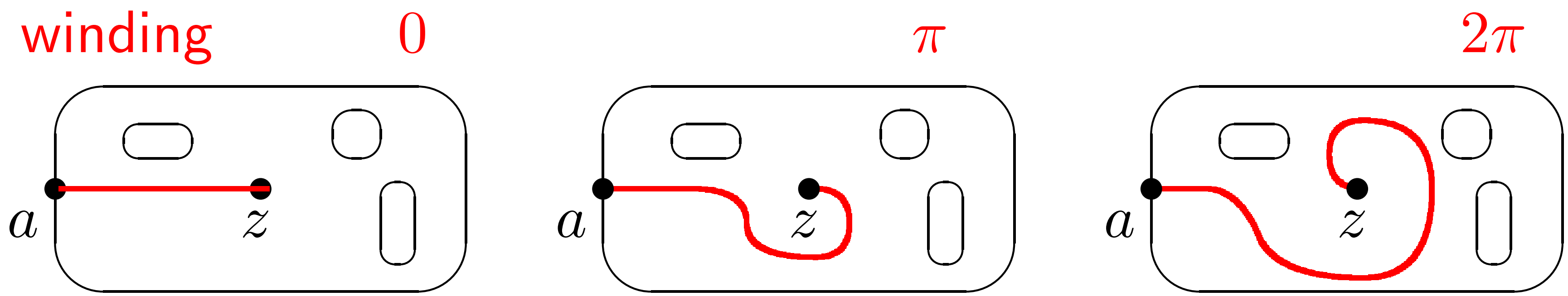
- consider configurations ω which have loops plus an interface from a to z.
- introduce **parafermionic** complex weight with **spin** parameter $\sigma \in \mathbb{R}$:

$$\mathcal{W} := \exp(-i\,\sigma\,\mathrm{winding}(\gamma, a \to z))$$
$$= \lambda^{\#\text{ signed turns of }\gamma}, \quad \lambda := e^{-i\sigma\pi/3}$$

- Define the observable by $F(z) := \sum_\omega n^{\#\text{loops}} x^{\#\text{edges}} \mathcal{W}(\omega)$

Rem Actually a **spinor** or a **parafermion** $F(z)\,(dz)^\sigma$

Rem Removing complex weight $\mathcal{W}$ one obtains correlation of spins at a and z in the $O(n)$ model

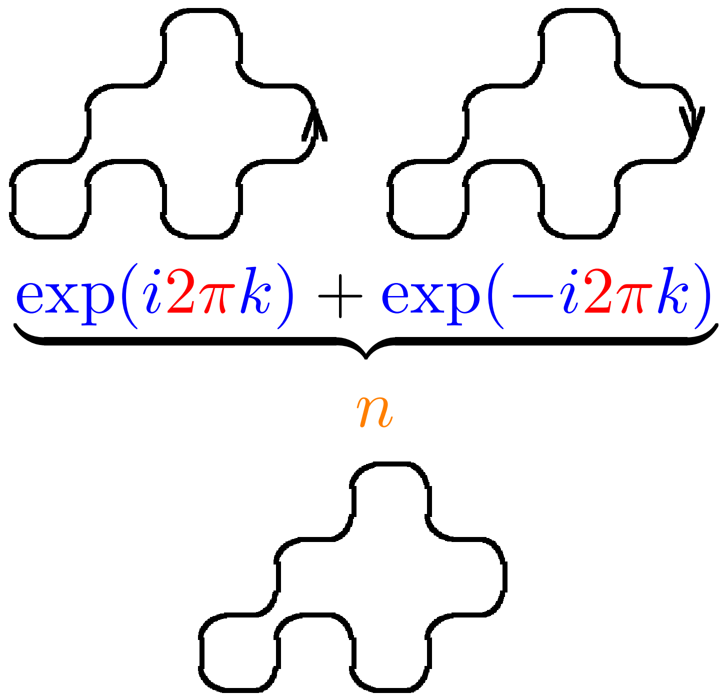


Why complex weights? [cf. **Baxter**]

Set $\mathbf{2\cos(2\pi k) = n}$. Orient loops
$\Leftrightarrow$ height function changing by $\pm\mathbf{1}$
whenever crossing a loop *(think of a geographic map with contour lines)*
New $\mathbb{C}$ partition function (local!):

$$Z^{\mathbb{C}} = \sum \prod_{\text{sites}} x^{\#\text{edges}} e^{(i \ \text{winding} \cdot k)}$$

Forgetting orientation projects onto
the original model: $\mathrm{Proj}\left(Z^{\mathbb{C}}\right) = Z$



Oriented interface $a \longrightarrow z \Leftrightarrow +1$ monodromy at z
Can rewrite our observable as $F(z) = Z^{\mathbb{C}}_{+1 \text{ monodromy at } z}$
Note: being attached to $\partial\Omega$, γ is weighted differently from loops

$+$ **more reasons coming from physics, analysis, combinatorics**

Preholomorphic observable

It is convenient to use the Kirchhoff approach.

- Define F with interface γ joining the centers of edges.
- Rewrite F as a **complex** flow f on edges by setting
  $f(\vec{uv}) := F(z)\,(u-v)$, with z the center of the edge uv.

The first Kirchhoff law for f takes the form

$$(p-v)F(p)+(q-v)F(q)+(r-v)F(r) \;=\; 0\;,$$

for a vertex v with neighboring edge centers p, q, r.

Recall that $2\cos(2\pi k)=n$. Our main observation is

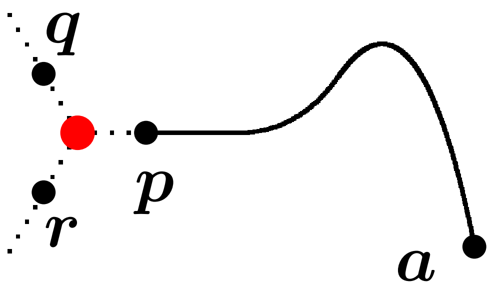
Key lemma. *Observable F satisfies the first Kirchhoff law if*

- $\sigma = 1/4+3k/2$ *and* $x = x_c(n) := 1/\sqrt{2+\sqrt{2-n}}$, *or*
- $\sigma = 1/4-3k/2$ *and* $x = \tilde{x}_c(n) := 1/\sqrt{2-\sqrt{2-n}}$.

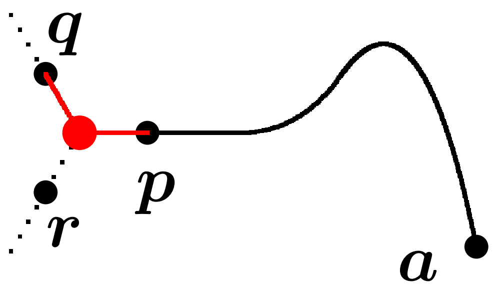
Rem For other parameter values "massive" relations.

Proof: local rearrangements Consider configurations with an interface ending near v and are their contributions to F:

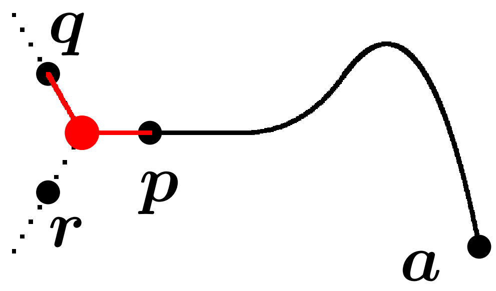
$\mathbf{1} C_1$ to $F(p)$



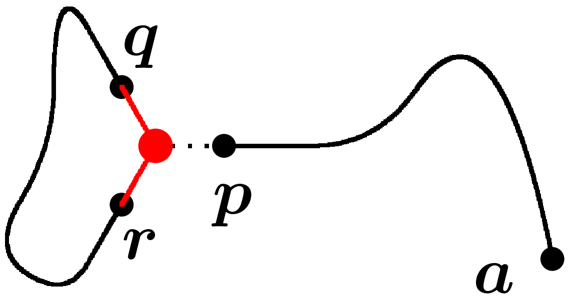
$x\bar{\lambda}\, C_1$ to $F(q)$



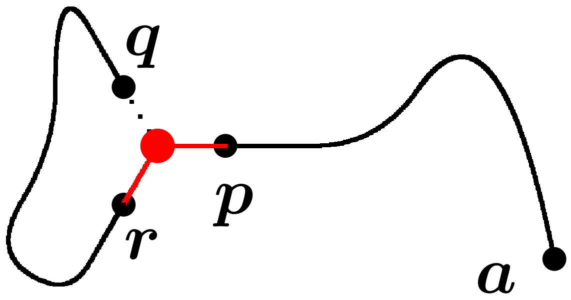
$x\lambda\, C_1$ to $F(r)$



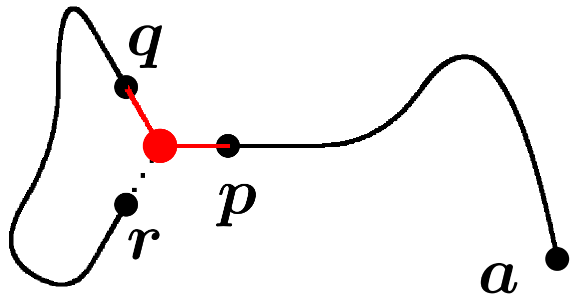
$n\, C_2$ to $F(p)$



$\bar{\lambda}^4\, C_2$ to $F(q)$



$\lambda^4\, C_2$ to $F(r)$



Plug into $$\underset{1}{(p-v)}F(p) + \underset{e^{i2\pi/3}}{(q-v)}F(q) + \underset{e^{-i2\pi/3}}{(r-v)}F(r) = 0$$

Proof: verifying the first Kirchhoff law we must check that

$$\begin{aligned} \mathbf{1}\cdot\mathbf{1}+e^{i2\pi/3}\,x\bar{\lambda}+e^{-i2\pi/3}\,x\lambda &= 0 \\ \mathbf{1}\cdot n+e^{i2\pi/3}\,\bar{\lambda}^4+e^{-i2\pi/3}\,\lambda^4 &= 0 \end{aligned}$$

Recalling that $\lambda=\exp\left(-i\sigma\pi/3\right)$ and $n=2\cos(2\pi k)$, rewrite

$$\begin{aligned} 1+x\left(e^{i2\pi/3+i\sigma\pi/3}+e^{-i2\pi/3-i\sigma\pi/3}\right) &= 0 \\ 2\cos(2\pi k)+\underbrace{\left(e^{i2\pi/3+i4\sigma\pi/3}+e^{-i2\pi/3-i4\sigma\pi/3}\right)}_{2\cos(2\pi/3+4\sigma\pi/3)} &= 0 \end{aligned}$$

or equivalently

$$\begin{aligned} x^{-1} &= -\,2\cos\left(2\pi/3+\sigma\pi/3\right) \\ 2\pi k &= \pm\left(2\pi/3+4\sigma\pi/3\right)\;+\;\pi\;+\;2\pi\mathbb{Z} \end{aligned}$$

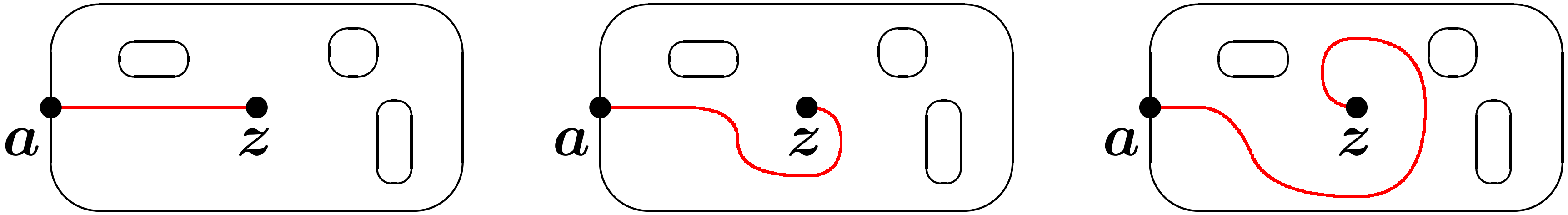
which produces the **promised values** of σ and x !!! $\square$

Question: is F preholomorphic? (= second Kirchhoff law)

- **Yes** for the the **2D Ising model at critical temperature**:
  $n = 1$, $x = x_c = 1/\sqrt{3}$, $\sigma = 1/2$.

The complex weight is **Fermionic**: if we know the direction from which the interface came from, we can determine $\mathcal{W}$ up to ± 1:

weight $\mathcal{W}$ $\quad 1 \quad -i \quad -1$



This allows to deduce the second law from the first one.

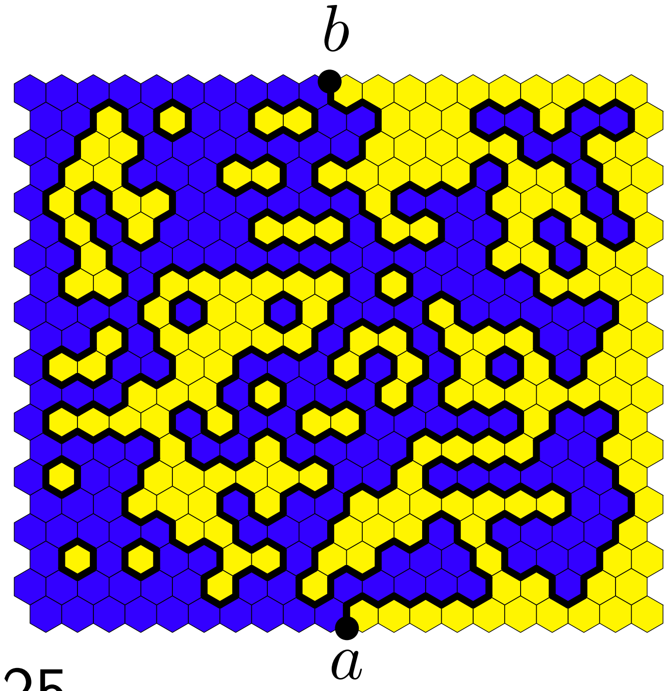
- **No** exact pre-holomorphicity for other models.

Questions: Approximate pre-holomorphicity?
Another definition? Different observable?

Loop gas at $n = 1$: the Ising model with $x = \exp(-2\beta)$

Ising spins $s(v) = \pm 1$ – hexagons of two colors which change whenever a loop is crossed.

$$
\begin{aligned}
Z &= \sum n^{\#\ \mathbf{loops}} x^{\#\ \mathbf{edges}} = \sum x^{\#\ \mathbf{edges}} \\
&= \sum \exp\left(-2\beta\ \#\left\{\text{neighbors } \mathbf{u}, \mathbf{v} \text{ with } \mathbf{s}(\mathbf{u}) \neq \mathbf{s}(\mathbf{v})\right\}\right) \\
&\asymp \sum \exp\left(-\beta \sum_{\text{neighbors } u,v} s(u)s(v)\right)
\end{aligned}
$$



- $\boldsymbol{n = 1}$, $\boldsymbol{x = 1/\sqrt{3}}$: **Ising model at T_c**

Note: critical value of x is known [Wannier]

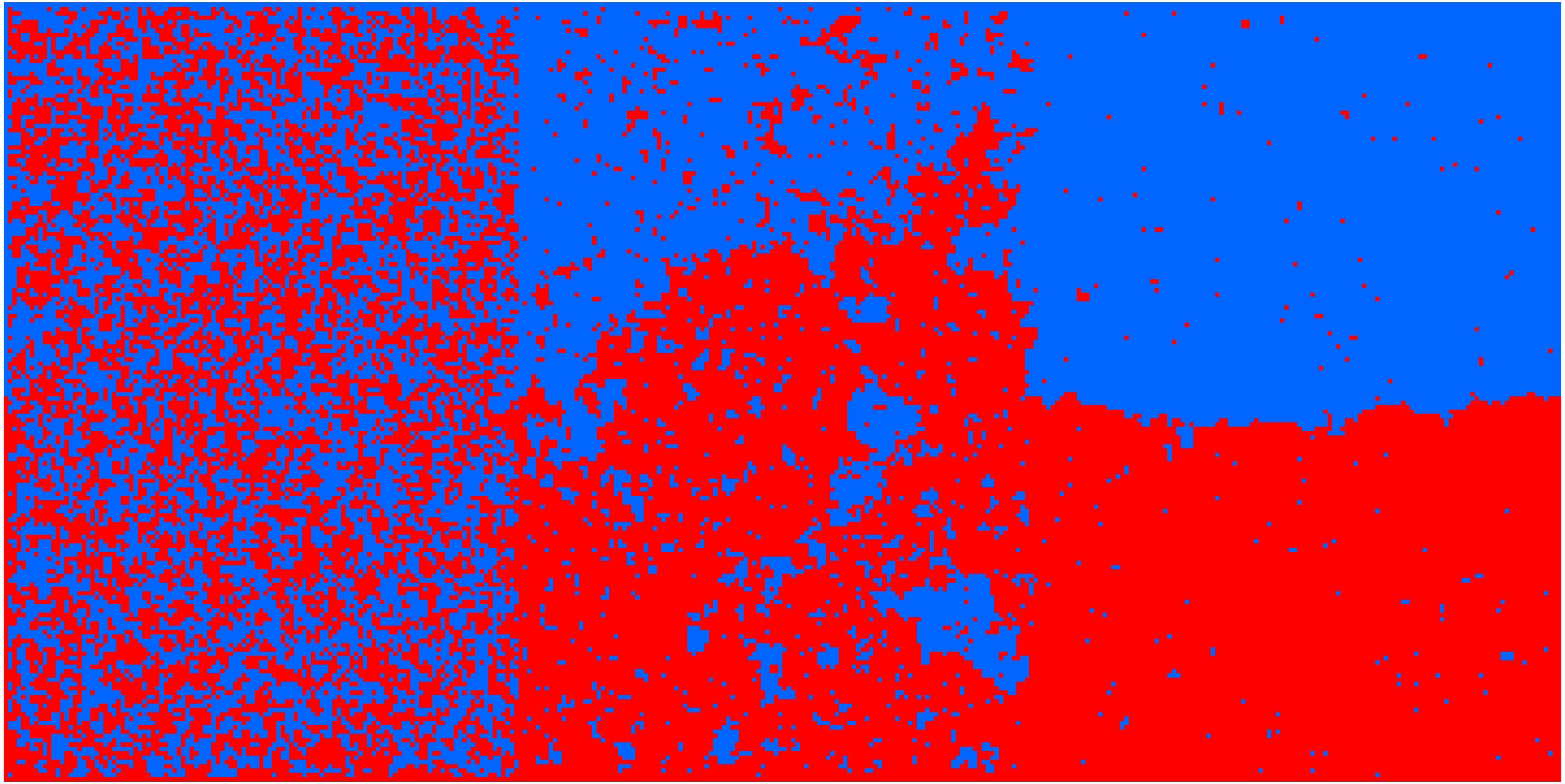
- $\boldsymbol{n = 1}$, $\boldsymbol{x = 1}$: **critical percolation**

All configs are equally likely ($p_c = 1/2$ [Kesten]).

The model was introduced by **Lenz**, and in 1925 his student **Ising** proved that there is no phase transition in 1D

Phase transition in the **2D Ising model:** $\mathbb{P}(\text{config}) \asymp x^{\text{loops length}}$

$x > x_c$, $T > T_c$ $\quad$ $x = x_c$, $T = T_c$ $\quad$ $x < x_c$, $T < T_c$



dense phase $\quad$ **dilute phase** $\quad$ **frozen phase**

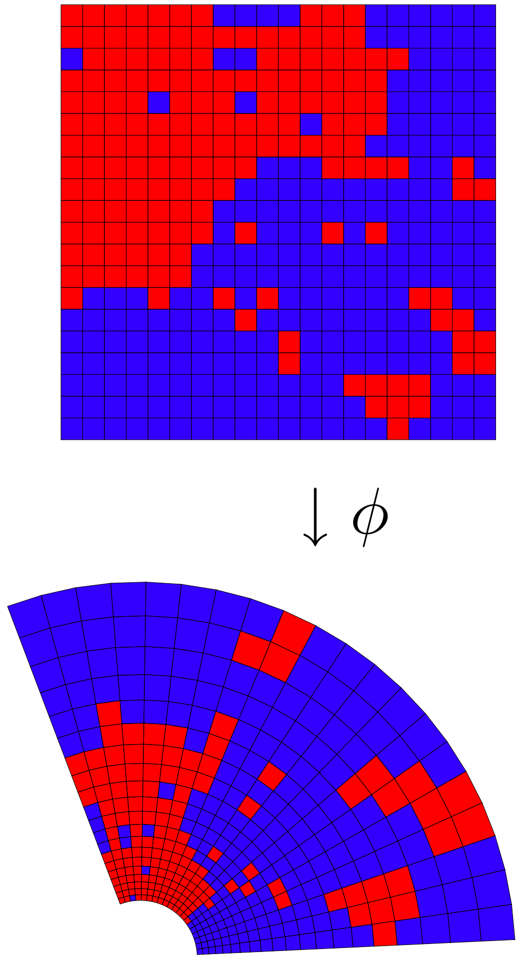
Universality: same behavior on all lattices, though different x

[Kramers, Wannier]: $x_c^{\text{square}} = 1/(1+\sqrt{2})$ **and** $x_c^{\text{hex}} = 1/\sqrt{3}$

• Physically "realistic model" of order–disorder **phase transitions**

• **"Exactly solvable"** – many parameters computed exactly, but usually non-rigorously **[Onsager, Kaufman, Yang, Kac, Ward, Potts, Montroll, Hurst, Green, Kasteleyn, Vdovichenko, Fisher, Baxter, . . .]**

• Connections to **Conformal Field Theory** – allow better description in a more general setting **[den Nijs, Nienhuis, Belavin, Polyakov, Zamolodchikov, Cardy, Duplantier, . . .]**



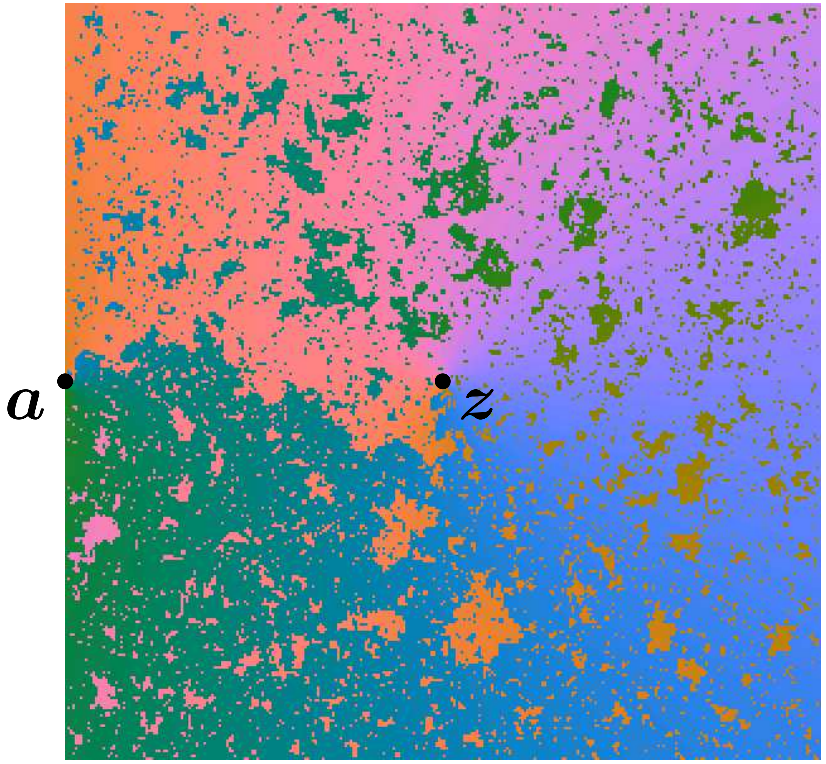
• At criticality one expects to see: existence of the **scaling limit** (as mesh $\to 0$), its **universality** (lattice-independence) and **conformal invariance** (for *all* conformal maps), though it was **never** fully and rigorously established

• We construct new objects of physical interest and prove that they have **a universal, conformally invariant scaling limit**

Theorem [Chelkak & Smirnov]. **For isoradial Ising model at T_c, F is a preholomorphic solution of a Riemann-Hilbert boundary value problem. Its scaling limit is universal and conformally invariant: when mesh $\varepsilon \to 0$,**

$$\boldsymbol{F(z)\,/\,\sqrt{\varepsilon} \rightrightarrows \sqrt{P'(z)}} \text{ inside } \boldsymbol{\Omega}.$$

Here P is the Schwarz ($=$ complexified Poisson) kernel at a: a conformal map $\Omega \to \mathbb{C}_+$ with $a \mapsto \infty$.



Rem F & P normalized in the same chart

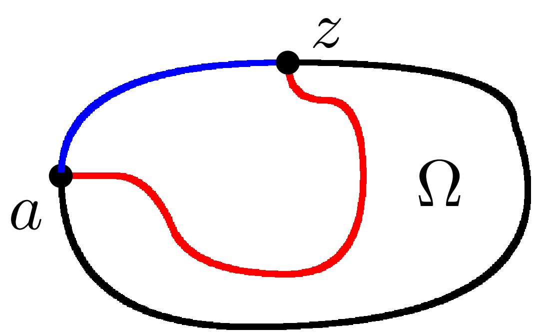
Rem $F(z)\sqrt{dz}$ is a **fermion** or a **spinor**

Rem For Ising one can define $\boldsymbol{F}$ by creating a disorder operator, i.e. a monodromy at z: when one goes around, spins $+1$ become -1 and vice versa.

Rem Off criticality **massive holomorphic:** discrete $\boldsymbol{\bar\partial\, F = im(x - x_c)\bar F}$, cf. **[Makarov, Smirnov]**

Proof: Hol **solution of Riemann-Hilbert boundary value problem**

When z is on the boundary, winding of the interface $a \to z$ is uniquely determined, and coincides with the winding of $\partial\Omega$, $a \to z$.
So we know $\mathrm{Arg}(F)$ on $\partial\Omega$.



F solves the discrete version of the covariant Riemann BVP

$$\mathbf{Im}\left(\boldsymbol{F(z)\cdot}(\textbf{tangent to }\boldsymbol{\partial\Omega})^{\boldsymbol{\sigma}}\right)\boldsymbol{=0} \quad \text{with } \boldsymbol{\sigma = 1/2}.$$

$$\Rightarrow \quad \boldsymbol{F\sqrt{dz}} \in \mathbb{R} \text{ along } \boldsymbol{\partial\Omega} \quad \Rightarrow \quad \boldsymbol{F^2\,dz} \in \mathbb{R}_+ \text{ along } \boldsymbol{\partial\Omega}$$

$$\Rightarrow \quad \boldsymbol{H(z)} \;=\; \mathbf{Im}\boldsymbol{\int_{z_0}^{z} F^2(u)du} \;=\; \textbf{const} \text{ along } \boldsymbol{\partial\Omega}, \quad \text{pole at } \boldsymbol{a}$$

Dirichlet problem, in continuum case solved by the Poisson kernel:

$$\boldsymbol{H(z) =} \,\mathrm{Im}\boldsymbol{P(z)} \;\Rightarrow\; \boldsymbol{F(z) = \sqrt{P'(z)}},$$

where $\boldsymbol{P}$ is a conformal map $\boldsymbol{\Omega} \longrightarrow \mathbb{C}_+$, $\boldsymbol{a} \mapsto \boldsymbol{\infty}$.

Big problem: in the discrete case F^2 is no longer analytic!!!

Proof of convergence: set $H := \frac{1}{2\varepsilon}\mathrm{Im}\int^z F(z)^2 dz$

- well-defined
- approximately discrete harmonic: $\Delta H = \pm\left|\partial F\right|^2$
- $H = 0$ on the boundary, blows up at a

$\Rightarrow \boldsymbol{H \rightrightarrows \mathrm{Im}\, P}$ where $\boldsymbol{P}$ is the complex Poisson kernel at $\boldsymbol{a}$

$\Rightarrow \boldsymbol{\nabla H \rightrightarrows P'} \;\; \Rightarrow \;\; \boldsymbol{\frac{1}{\varepsilon} F^2 \rightrightarrows P'} \;\; \Rightarrow \;\; \boldsymbol{\frac{1}{\sqrt{\varepsilon}} F \rightrightarrows \sqrt{P'}}$ $\square$

Rem: we approximate the integral by the Riemann sums, hence division by ε and after the square root by $\sqrt{\varepsilon}$.

Problems: we must do all sorts of estimates (Harnack inequality, normal familes, harmonic measure estimates, . . .) for *approximately* discrete harmonic or holomorphic functions in the absence of the usual tools. For general isoradial graphs even worse, moreover there are no known Ising estimates to use.

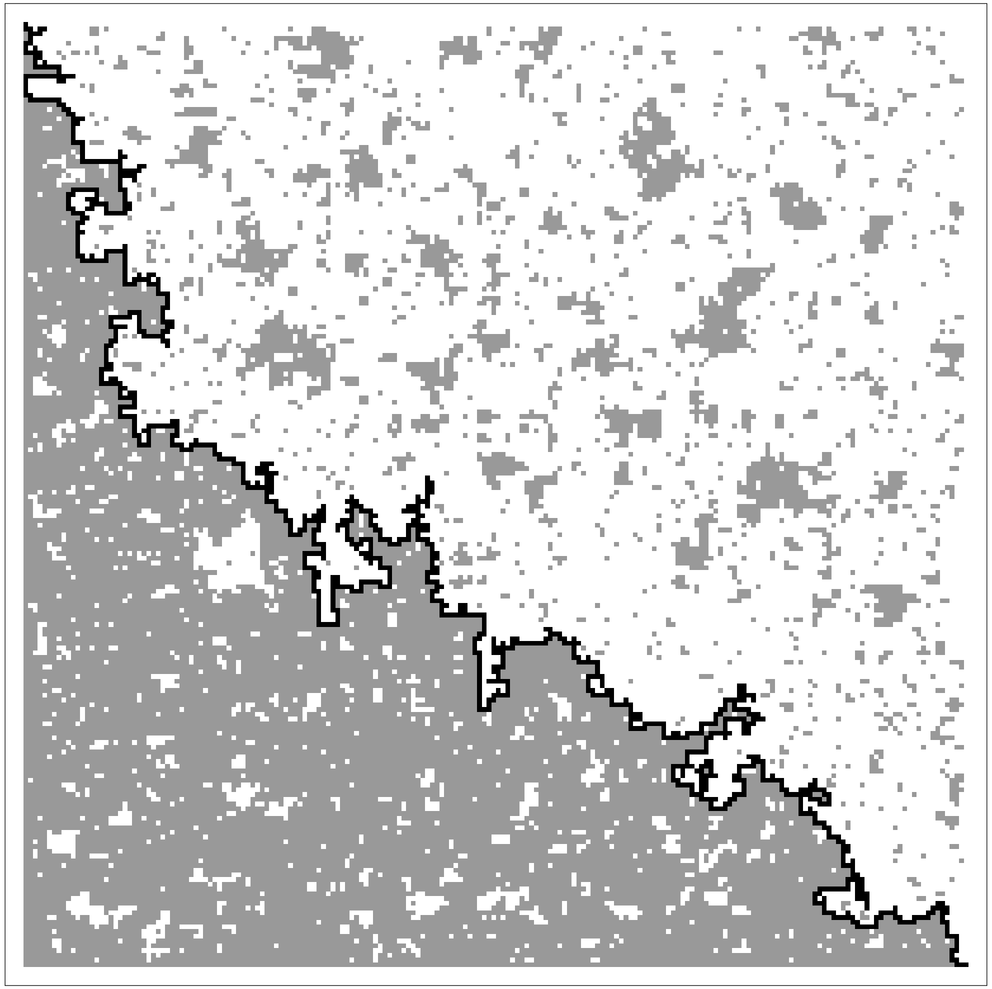
Question: what is the most general discrete setup when one can get the usual complex analysis estimates? (without using multiplication)

Theorem [Chelkak & Smirnov]. **Ising model on** ***isoradial graphs*** **at T_c has a conformally invariant scaling limit as mesh $\varepsilon \to 0$. Interfaces in spin and random cluster representations converge to Schramm's SLE(3) and SLE$(16/3)$**

- More can be deduced from convergence of interfaces
- **[Pfister-Velenik]** at $T < T_C$ interface converges to an interval
- **Conjecture** at $T > T_C$ interface converges to SLE(6), same as percolation.

Known only for hexagonal lattice and $T = \infty$ **[Smirnov 2001]**.

Idea of proof: trace interface while sampling the observable.

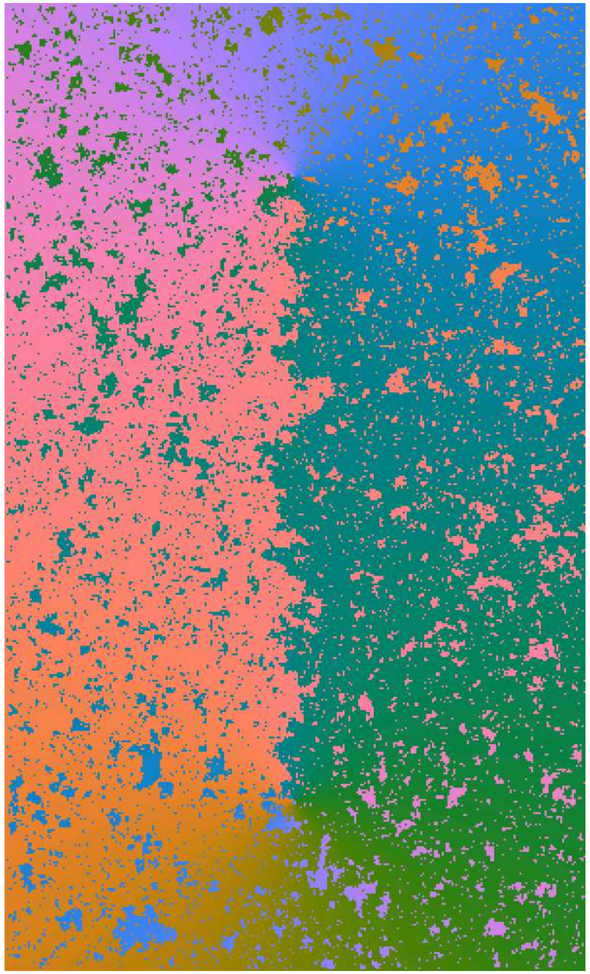


Ising interface $\to$ SLE(3), $\mathrm{Dim} = 11/8$

Can we deduce more from this observable?

Interfaces converge to Schramm's SLE curves. Then one can use the machinery of SLE and Itô calculus to calculate almost anything.

But even without SLE we can do things. Putting both points a and b inside, we obtain a discrete version of Green's function with Riemann-Hilbert BV.



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Theorem [Hongler - Smirnov]. At T_c the correlation of neighboring spins $s(u)$, $s(v)$ (spin-pair or energy field) satisfies

$$\mathbb{E}\, s(u)\, s(v) \;=\; \frac{1}{\sqrt{2}} \pm \frac{1}{\pi}\, \rho_\Omega(u)\, \varepsilon + O(\varepsilon^2),$$

where ρ is the element of the hyperbolic metric, and the sign $\pm$ depends on the boundary conditions ("$+$" or free).

[Hongler 2010]: formula for many spin-pairs (energy) correlation

The Self Avoiding Walk = a walk **without self-intersections** was proposed by chemist **Flory** as a model for **polymer chains**, and turned out to be an interesting mathematical object.

Let $C(k)$ be the number of length k SAW on a given lattice.
It is easy to see that $C(k+l) \leq C(k) \cdot C(l)$ and hence there is a (lattice-dependent) **connective constant** μ such that

$$C(k) \approx \mu^k, \quad k \to \infty .$$

Using Coulomb gas formalism, physicist **Nienhuis** argued that for the **hexagonal lattice** $\mu = \sqrt{2+\sqrt{2}}$, and moreover

$$C(k) \approx \left(\sqrt{2+\sqrt{2}}\right)^k k^{11/32}, \quad k \to \infty .$$

Note that while μ is lattice-dependent, the power law correction is supposed to be universal. We prove part of his prediction:

Theorem [Duminil-Copin & Smirnov]. $\mu = \sqrt{2+\sqrt{2}}$.

Proof: Self Avoiding Walk as the loop gas at $n = 0$

There are no loops, just one interface $a \leftrightarrow z$, weighted by x^{length}

The first Kirchhoff law holds for $\sigma = 5/8$ and $x_c = 1/\sqrt{2+\sqrt{2}}$:

$$(p - v)F(p) + (q - v)F(q) + (r - v)F(r) \;=\; 0\,.$$

Sum it over Ω, all interior contributions cancel out:

$\sum_{z\in\partial\Omega} F(z)n(z) \;=\; 0$, where $n(z)$ are the normal vectors.

- by definition $F(a) = 1$.
- for other $z \in \partial\Omega$ the complex weight is uniquely determined.

Considering the real part of F we get positive weights and

$$\sum_{z\in\partial\Omega\setminus\{a\}} \;\sum_{\omega(a\to z)} x_c^{\text{length of contours}} \;\asymp\; 1\,,$$

regardless of the size of the domain Ω.

A simple counting argument then shows that the series

$$\sum_k C(k)\, x^k \;=\; \sum_{\text{simple walks from } a \text{ inside } \mathbb{C}} x^{\text{length}}\,,$$

converges when $x < x_c$ and **diverges** when $x > x_c$.

This clearly implies that $\mu = 1/x_c = \sqrt{2+\sqrt{2}}$ $\square$

WHAT'S NEXT?

Problem Establish the full holomorphicity of F. This would allow to relate self-avoiding walk to the Schramm's **SLE** with $\kappa = 8/3$ and together with the work of **Lawler**, **Schramm** and **Werner** to establish the precise form of the **Nienhuis** prediction.

- **Other models and observables?**
- **Analysis on vector bundles? cf. [Kenyon arXiv:1001.4028]**
- **Connection to Yang-Baxter integrability?**
- **Random planar graphs? Related talks:**

  Today: Itai Benjamini, Thursday: Scott Sheffield

THANK YOU!

Itai Benjamini “Random Planar Metrics”

in 1.03 right after this talk