UNIVERSITÉ DE GENÈVE
FACULTÉ DES SCIENCES
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MΠΦ ICMP|09 PRAGUE

Massive Schramm-Loewner Evolutions

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based on joint work with
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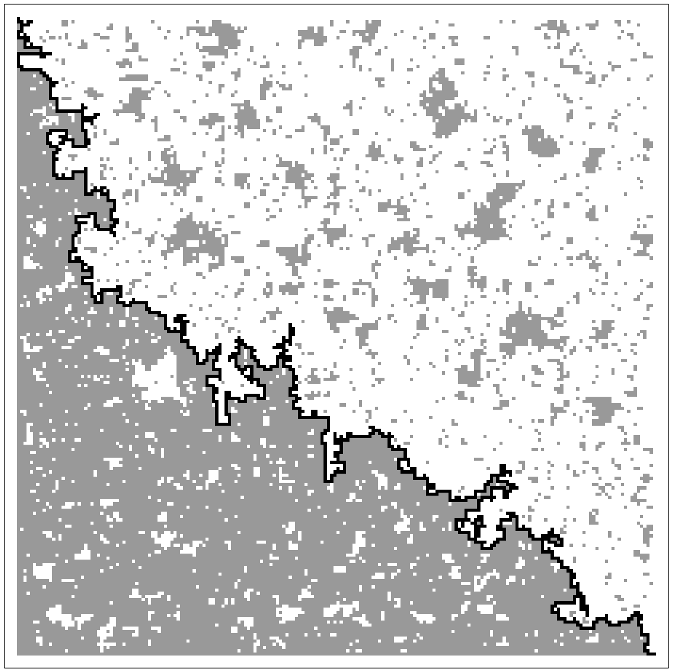
SLE = Schramm-Löwner evolution

a way to construct random curves
introduced by Oded Schramm in 1999
while looking for a proof of spectacular
CFT predictions for 2d lattice models at T_c

O. Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. Israel J. Math., 118, 221-288 (2000), arxiv math/9904022

- Rigorous proofs and/or better understanding of many predictions
- New theory, different from CFT and interesting in itself
- Heavily relies on CONFORMAL INVARIANCE



critical Ising interface → SLE(3)

How to describe such a curve?

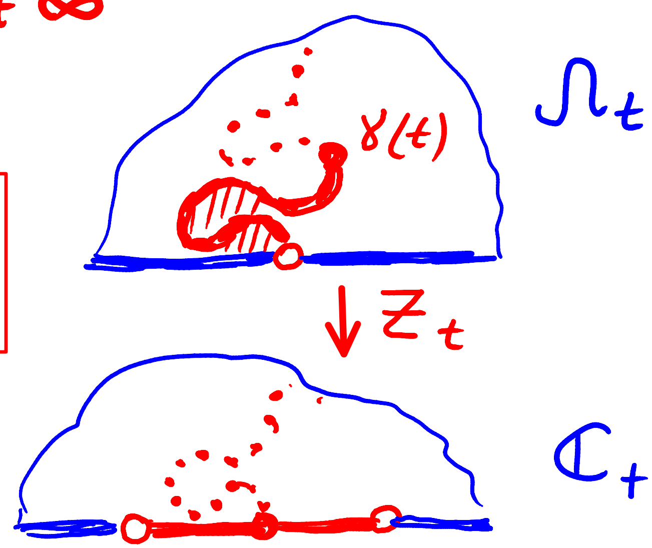
K. Löwner, Untersuchungen über schlichte konforme Abbildungen des Einheitskreises, I. ***Math. Ann.*** **89, 103-121 (1923).**

Löwner theorem when a "nice" slit γ is parametrized by capacity, the conformal map $G_t : \mathbb{C}_+ \setminus \gamma[0,t] \to \mathbb{C}_+$ satisfies

$$Z_t(z) = z - w(t) + \frac{2t}{z} + \dots \text{ at } \infty$$

$$\gamma(t) \mapsto 0$$

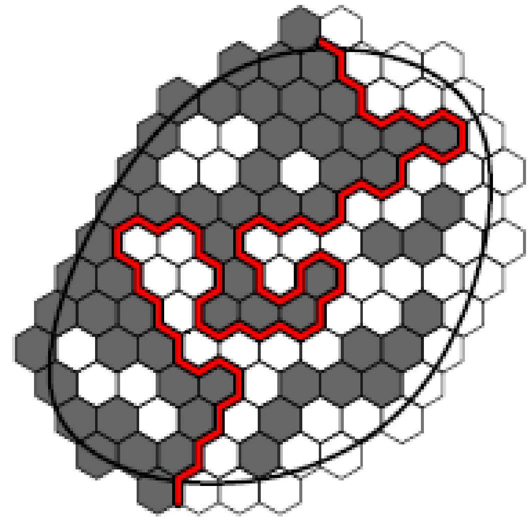
$$\boxed{d_t Z_t(z) = \frac{2}{Z_t(z)} - d_t w(t)}$$



- Bijection:
  $\{\text{nice slits } \gamma\} \leftrightarrow \{\text{continuous } w\}$
- ODE for $Z_t(z)$ involves $w(t)$ only!
- Random $\gamma \longleftrightarrow$ Random $w(t)$

Schramm's approach

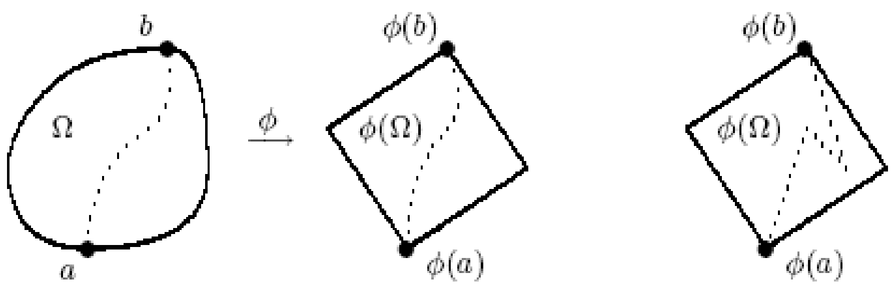
Assume that an interface has a conformally invariant scaling limit, describe it by LE with a random $w(t)$



Conformal invariance

same law

$$\mu(\Omega,a,b) \quad \overset{\phi}{\longmapsto} \quad \phi\left(\mu(\Omega,a,b)\right) \qquad \mu\left(\phi(\Omega),\phi(a),\phi(b)\right)$$



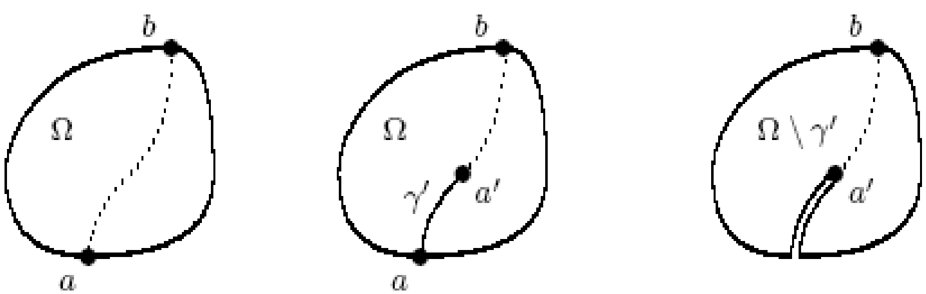
← holds in the limit

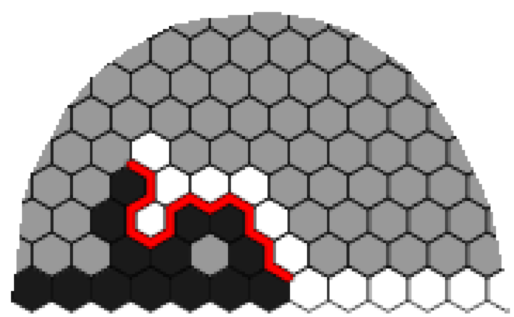
Domain Markov

same law

condition

$$\mu(\Omega,a,b) \quad \longmapsto \quad \mu(\Omega,a,b)\,|\,\gamma' \qquad \mu\left(\Omega\setminus\gamma',a',b\right)$$



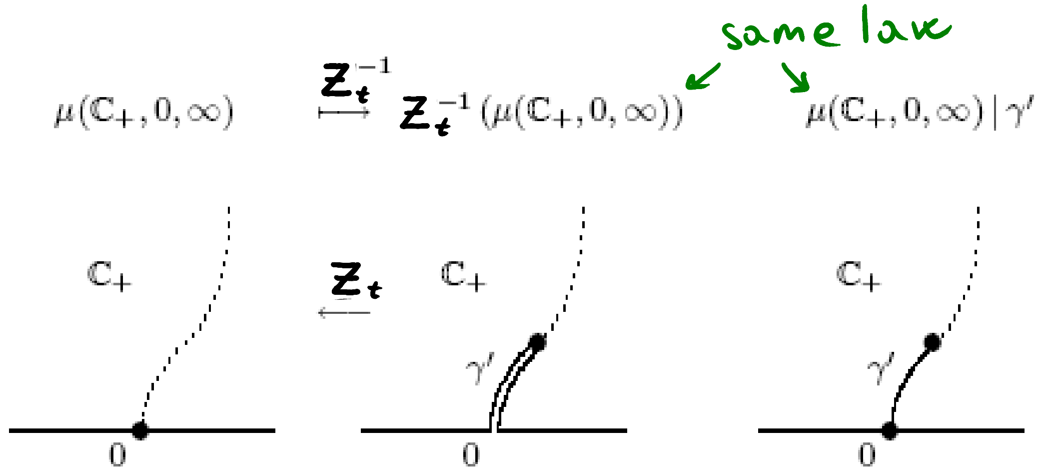


← holds on the lattice (nearest neighbour interaction)

Conformal invariance + domain Markov $\Rightarrow$

Conformal Markov Property

$$\boxed{Z_{t+s} \mid Z_t = Z_t(Z_s)}$$



Expanding at ∞: $z - w(t+s) + \ldots \mid Z_t =$

$= (z - w(t) + \ldots) \circ (z - w(s) + \ldots) = z - (w(t) + w(s)) + \ldots$

So $w(t+s) - w(t) \mid Z_t = w(s)$, $w(t)$ a.s. continuous $\Big\} \Rightarrow$ $w(t) = \sqrt{\varkappa}\, B_t + \alpha t$, $\varkappa \in \mathbb{R}_+$, $\alpha \in \mathbb{R}$

- $\alpha = 0$ by symmetry or scaling
- The principle is very nice, but hard to apply: need to establish conformal invariance first, then evaluate some observable to find $\varkappa$...

Martingale principle: if some "martingale" observable F has a conformally covariant limit, then interface converges to $SLE(\varkappa_F)$

Martingale property of F:

$$F(z,\Omega^\epsilon,a,b) = \mathbb{E}_{\gamma^\epsilon[0,t]} F(z,\Omega^\epsilon\setminus\gamma^\epsilon[0,t],\gamma^\epsilon(t),b)$$

"Interface does not distinguish its past from the boundary of the domain."

Was used to establish convergence to SLE for

percolation [Smirnov 2001]

UST & LERW [Lawler Schramm Werner 2003]

discrete GFF [Schramm-Sheffield, 2005, 2008]

Ising & FK Ising [Smirnov 2006, 2008]

Afterwards SLE can be used to derive exponents.

$SLE(\kappa)$

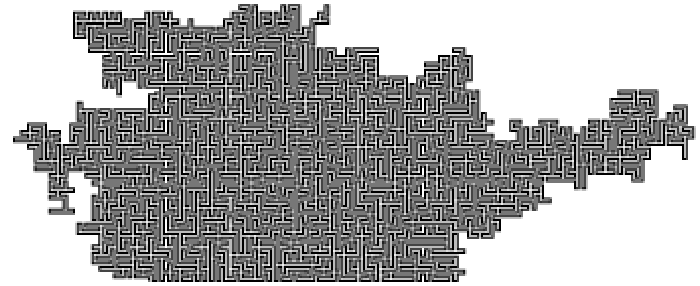
LERW

2

Lawler-Schramm-Werner

UST

8

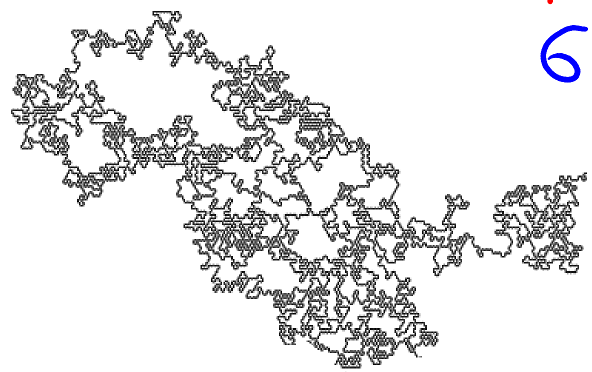


ISING

3

Percolation interface

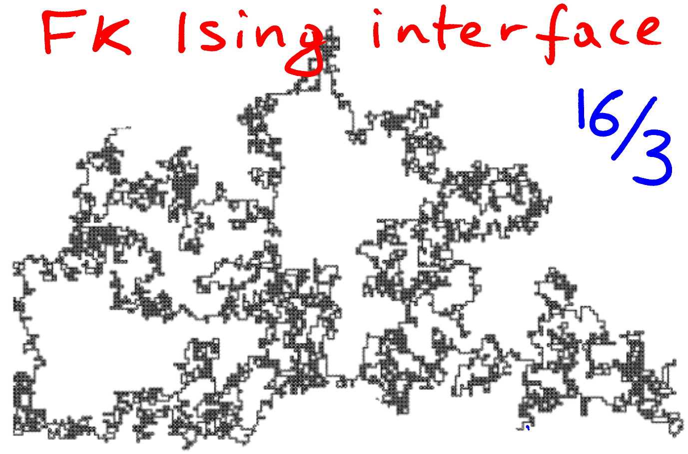
6



Smirnov

FK Ising interface

16/3



4

Schramm-Sheffield

discrete GFF level curve

MASSIVE LIMITS . understood less well
cf. [Itzykson Drouffe]

ISING MODEL take $t = t_c + \varepsilon$.

If mesh $\delta \to 0$, the limit is trivial.
Take double limit $\delta, \varepsilon \to 0$ instead

PERCOLATION take $p = p_c + \varepsilon$ send $\delta, \varepsilon \to 0$

LOOP ERASED RW, **HARMONIC EXPLORER**

start with a RW dying at rate ε

DISCRETE FREE FIELD

replace discrete Δ by $\Delta - \varepsilon^2 \mathrm{Id}$

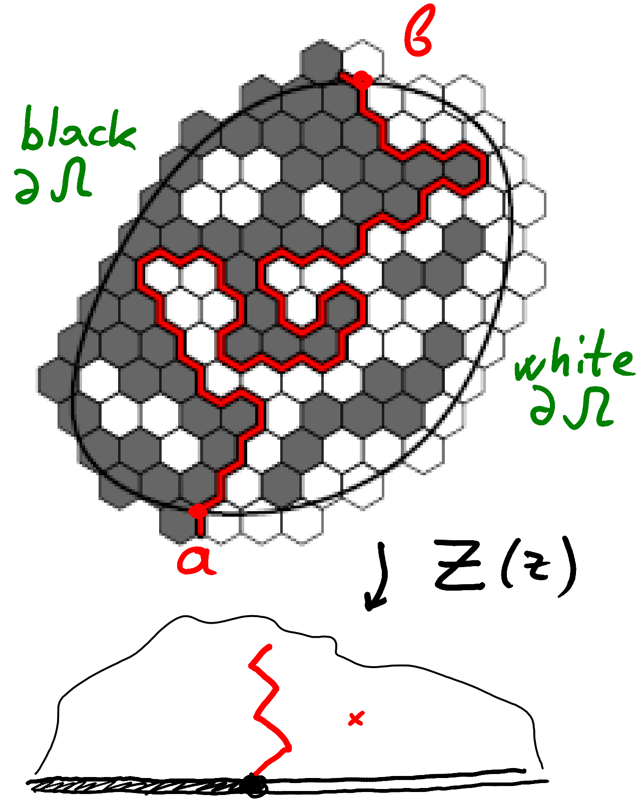
- One model has many massive limits:
  e.g. perturb magnetization for Ising
- Mass arises as a ratio of powers of δ and ε.

- So far all discrete observables are discrete harmonic or holomorphic solutions of Dirichlet, Neumann or Riemann B.V.P. They imply convergence of interfaces to SLEs.
- Corresponding SLE martingales are easy to classify, e.g. the Dirichlet BVP for $SLE(\varkappa)$ gives $M(z) = \log Z_t(z) + (1-\frac{\varkappa}{4}) \log Z_t'(z)$
- Massive limits lead to massive harmonic $\Delta M - m^2 M = 0$ and massive holomorphic $\bar\partial M - im\overline{M} = 0$ observables
- How to describe the massive limits?

Bosonic Observables

Assume, that in some discrete model the interface γ satisfies

- Markov property
- $\mathbb{P}(z \text{ right of } \gamma) \in \text{Harm}$



Then in the scaling limit $\mathbb{P}(z) = \frac{1}{\pi} \log Z$

and $M_t(z) = \log Z_t(z)$ is a martingale wrt γ

Examples [Schramm - Sheffield]

- level lines of the discrete Free Field
- harmonic explorer

  color hexagons inductively while drawing γ:

  $\mathbb{P}(\gamma_n \text{ white}) = \omega_{\gamma_n, \Omega_n \setminus \gamma[0,n-1]}(\text{white } \partial)$

CONCLUSION: SCALING LIMIT IS SLE(4)

PROOF Describe by LE with unknown $w(t)$

Martingale property:

$$\log z = M(z,\Omega) = \mathbb{E}_{\gamma[0,t]} M(z,\Omega_t) = \log Z_t(z)$$

Recall $Z_t(z) = z - w(t) + \frac{2t}{z} + \ldots$, plug in, expand:

Note: absolute bounds for "…", moments of $w(t)$.

$$\log z = \log z - \frac{1}{z}\mathbb{E}_{Z_t} w(t) - \frac{2}{z^2}\mathbb{E}_{Z_t}(w(t)^2 - 4t) + \ldots$$

Equate coefficients: $0 = \mathbb{E}_{Z_t} w(t)$, $0 = \mathbb{E}_{Z_t} w(t)^2 - 4t$ ⊛

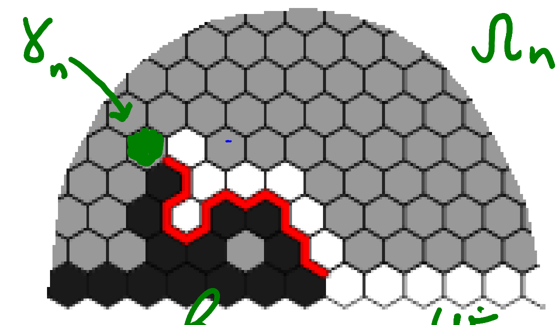
Do the same for domains $\mathbb{C}_+ \setminus \gamma[0,s]$, $\mathbb{C}_+ \setminus \gamma[0,t]$, get ⊛ for increments, so $w(t)$ is a.s. continuous, $w(t)$ and $w(t)^2 - 4t$ are martingales

$$\underset{\text{Lèvy}}{\Longrightarrow} w(t) = \sqrt{4}\, B_t, \text{ so } \gamma \text{ is } SLE(4) \quad \blacksquare$$

What about the massive case?

Once we construct (or assume) scaling limit, can reason in the same way, get conformal invariance wrt metric $m|dz|$.

MASSIVE SLE(4)



Massive version of

- free field $\Delta - m^2 I$
- harmonic explorer color hexagons inductively while drawing γ: $\mathbb{P}(\gamma_n \text{ white}) = \left(1 + \omega^m_{\gamma_n,\Omega_n}(w) - \omega^m_{\gamma_n,\Omega_n}(b)\right)/2$

Described by LE: $dW(t) = \sqrt{4}\, dB_t + m \left(\iint_{\Omega_t} M \cdot P_m \right) dt$

Here: M non-massive observable

P_m massive Poisson kernel $\Delta P_m = m^2 P_m$

Rem Same BM, covariant drift!

PROOF (works for all $\varkappa$) $\qquad \alpha = 1 - \frac{\varkappa}{4}$

Bosonic observable $M(z) = \operatorname{Im}\left(\log Z + \alpha \log Z'\right)$

$\Delta M = 0$, Dirichlet B.C.

SLE $\quad dZ_t = \frac{2}{Z_t}dt - \sqrt{\varkappa}\, dB_t \qquad dZ_t' = -\frac{2Z_t'}{Z_t^2}dt$

$\text{drift}_{SLE}\, M = 0$

Massive martingale observable $M_m(z)$

$\Delta_m M_m = \Delta M_m - m^2 M_m = 0$, same B.C.

mSLE $\quad dZ_t = \frac{2}{Z_t}dt - \sqrt{\varkappa}\, dB_t + d\lambda \qquad dZ_t' = -\frac{2Z_t'}{Z_t^2}dt$

$$\text{drift}_{mSLE}\, M = \cancel{\text{drift}_{SLE} M} + \operatorname{Im}\frac{1}{Z}\, d\lambda = P\, d\lambda$$

Green's function

$$\Delta_m (M_m - M) = m^2 M \Rightarrow M_m = M + m^2 M * G_m$$

Same for massive Poisson: $P_m = P + m^2 P * G_m$

From the previous slide: ② $M_m = M + m^2 M * G_m$

① $\mathrm{drift}_{mSLE} M = P\, d\lambda$ ③ $P_m = P + m^2 P * G_m$

Calculate the massive drift λ:

$$0 = \mathrm{drift}_{mSLE} M_m(z) \overset{②}{=} \mathrm{drift}_{mSLE}\left(M + m^2 M * G_m\right)$$

$$= \underbrace{d_{mSLE} M}_{\| ①} + \underbrace{m^2 d_{mSLE} M * G_m}_{// ①} + m^2 M(\cdot) * \underbrace{d_{mSLE} G_m(\cdot, z)}_{// \text{ Hadamard}}$$

$$= \underbrace{P\, d\lambda + m^2 P\, d\lambda * G_m}_{\| ③} - m^2 M(\cdot) * P_m(\cdot) P_m(z)\, dt$$

$$= P_m\, d\lambda - P_m \cdot m^2 \left(\iint_\Omega M \cdot P_m\right) \cdot dt$$

Conclusion:

$$d\lambda = m^2 \left(\iint_\Omega M \cdot P_m\right) dt$$

$$\underline{Rem} = m^2 \left(\iint_\Omega M_m P\right) dt$$

■

CONCLUSION

$\varkappa \leq 4$, $M(z) = \mathrm{Im}\left(\log Z + (1-\frac{\varkappa}{4})\log Z'\right)$

If there is a massive limit, it is described by LE driven by the diffusion

$$dw(t) = \sqrt{\varkappa}\, dB_t + m^2 \left(\iint_{\Omega_t} M P_m \right) dt \qquad (\star)$$

<u>Rem</u> In particular, diffusion $(\star)$ is well defined, which is not easy: definition involves L.E.

<u>Rem</u> M = non-massive martingale
P_m = massive Poisson kernel

- What can we say about solutions to $(\star)$?
- If we do not know yet existence of the massive limit, can we show that diffusion $(\star)$ is well defined?

$\varkappa = 4$ observable $M(z) \in HARM$ with B.V. 0 and π

mSLE(4) driven by $dw(t) = \sqrt{4}\, dB_t + \lambda(t)\, dt$

with drift $\lambda(t)\, dt = m^2 \left(\iint_{\Omega_t} M_m\, P \right) dt$

- $M_m \in [0, \pi] \Rightarrow |\lambda(t)| \leq \pi m^2 \iint P$
- $\int_0^\infty \lambda^2(t)\, dt \lesssim \int_0^\infty dt \quad \iint P(z) \iint P(w)$

Hadamard

$$= -\int_0^\infty dt \iint d^2z \iint d^2w \; \partial_t G_{\Omega_t}(z,w)$$

$$= \iint \iint G_{\Omega_0}(z,w) - G_{\Omega_\infty}(z,w)\, d^2z\, d^2w$$

$$\leq \iint G_{\Omega_0}(z,w)\, d^2z\, d^2w < \infty$$

for finite Ω_0

$\Rightarrow \mathbb{E}\, e^{\frac{1}{2}\int_0^\infty \lambda^2} < \infty$ (Novikov criterion)

So $w(t)$ is well-defined and absolutely continuous wrt $\sqrt{4}\, B_t$ $\blacksquare$

What we can do (with N. MAKAROV)

- Construct massive versions of LERW, Harmonic Explorer/DFF, UST
  for bosonic observables
- Write drifts in the corresponding massive SLE(2), SLE(4), SLE(8)
- Show for $\varkappa \leq 4$ diffusion is well-defined and abs. cont. wrt BM
  $\Rightarrow$ same scaling exponents
- Show for $\varkappa = 8$ diffusion is singular, but scaling exponents are the same
- Partial results for massive ISING and percolation, (para)fermionic observables
  cf. [Garban, Pete, Schramm] for percolation

QUESTIONS

- Write drifts for all observables
  (each $SLE(\varkappa)$ has a one-parameter family)
- Show abs. continuity for $\varkappa \leq 4$ (dilute)
  singularity for $\varkappa > 4$ (dense)
- Show that massive models are in the same universality class
- How many massive limits can a model have?
  (there is a one-parameter family of martingales)

THANKS!!!