The Ising model of a ferromagnet from 1920 to 2020

Stanislav Smirnov

UNIVERSITÉ DE GENÈVE

Санкт-Петербургский государственный университет

Magnetism

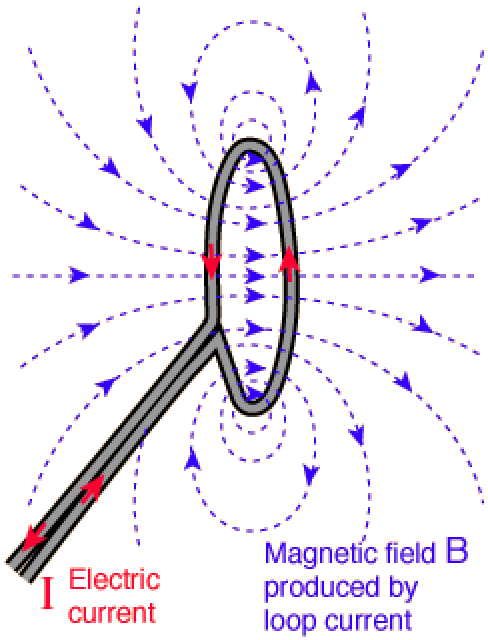
Natural magnetism in lodestone (magnetite) known for millennia

6th century BC: first “scientific” discussions by **Thales** of **Miletus** and **Shushruta** of **Varansi**.

Name derived from **Μαγνησία** – a Greek province rich in iron ore

Several practical uses, but poor understanding of its nature

Relation with electricity



1820 Hans Christian Ørsted discovers by accident that electric current induces magnetic field

1820-30 André-Marie Ampère; Carl Friedrich Gauss; Jean-Baptiste Biot & Félix Savart – a formula

1831 Michael Faraday – varying magnetic flux induces an electric current

1855-1873 James Clerk Maxwell synthesizes theory of electricity, magnetism and light, writes down Maxwell's equations

Ferromagnetism

1895 Pierre Curie in his doctoral thesis studies types of magnetism, discovers

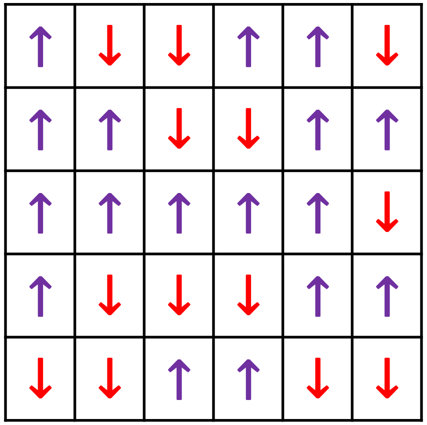
- the effect of temperature
- a phase transition at the Curie point

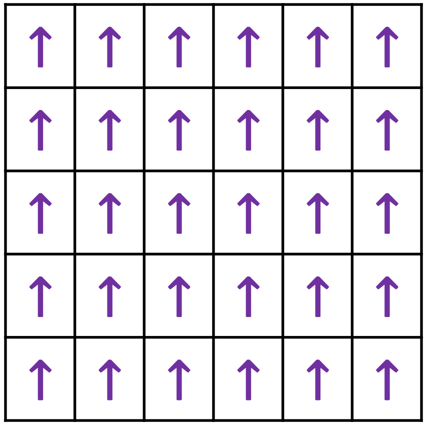
The phenomenon occurs at the atomic scale

1920 Wilhelm Lenz introduces a lattice model for ferromagnetism

Lenz argued that "**atoms are dipoles which turn over between two positions**":

- their free rotatability was incompatible with Born's theory of crystal structure;
- but they can perform turnovers as suggested by experiments on ferromagnetic materials;
- in a quantum-theoretical treatment they would by and large occupy two distinct positions.





Quantum mechanics

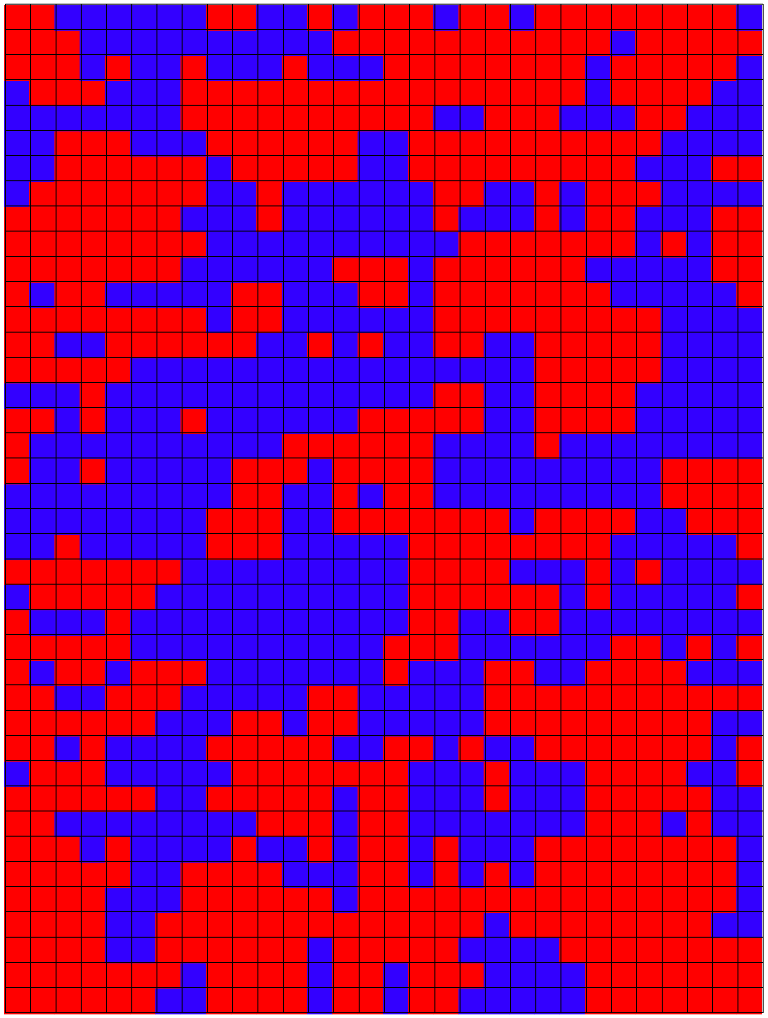
There is a good reason why no explanation for magnetism was found before 20th century: **the Bohr–van Leeuwen theorem** shows that ***magnetism cannot occur in purely classical solids***

The Lenz argument is flawed, but the model is basically correct – the property of **ferromagnetism** is due to two quantum effects:

- **the spin of an electron**
  (hence it has a magnetic dipole moment)
- **the Pauli exclusion principle**
  (nearby electrons tend to have parallel spins)

1920-24: The Lenz-Ising model

Lenz suggested the model to his student **Ernst Ising**, who proposed a specific form of interaction



Squares of two colors, representing spins $s=\pm 1$

Nearby spins want to be the same, parameter $\mathbf{x}$:

$\mathbf{Prob} \asymp x^{\#\{+-\text{neighbors}\}}$

$\asymp \exp(-\beta \sum_{\text{neighbors}} s(u)s(v))$

(in magnetic field multiply by $\exp(-\mu \sum_u s(u))$)

1924: Ernst Ising thesis

"no phase transition in dimension 1"

0 1 .... n n+1

+	–	+	+	+	–	–	+	–	–	+	–	+	+	?

Length n+1 chain, the leftmost spin is +

$$Z = \Sigma_{conf.} x^{\#\{(+)(-) neighbors\}} = (1+x)^n$$

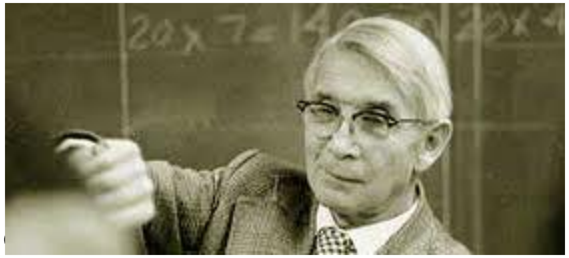
The rightmost spin is + for even powers of x

$$Z_+ = \{ (1+x)^n + (1-x)^n \} / 2$$

So the probability

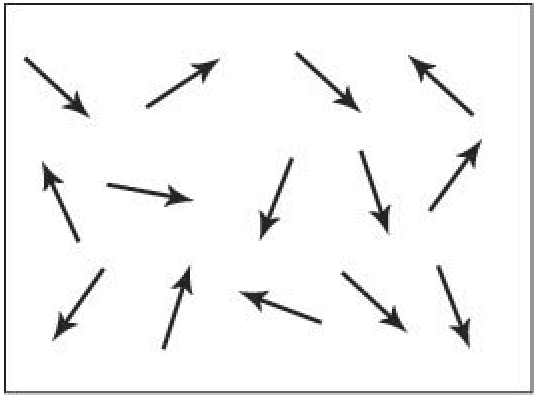
$$P(\sigma(n)=+) = Z_+/Z = \frac{1}{2} + \frac{1}{2}\left(\frac{1-x}{1+x}\right)^n,$$

which tends exponentially to ½

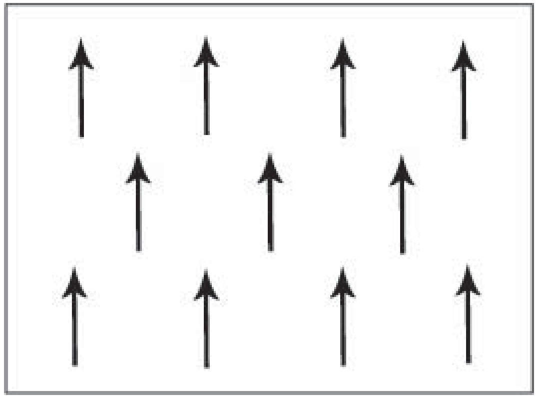


Ising wrongly concluded that there is **no phase transition in all dimensions.**

Never returned to research, teaching physics at a college



The paper was widely discussed (Pauli, Heisenberg, Dirac, …) with consensus that it is oversimplified

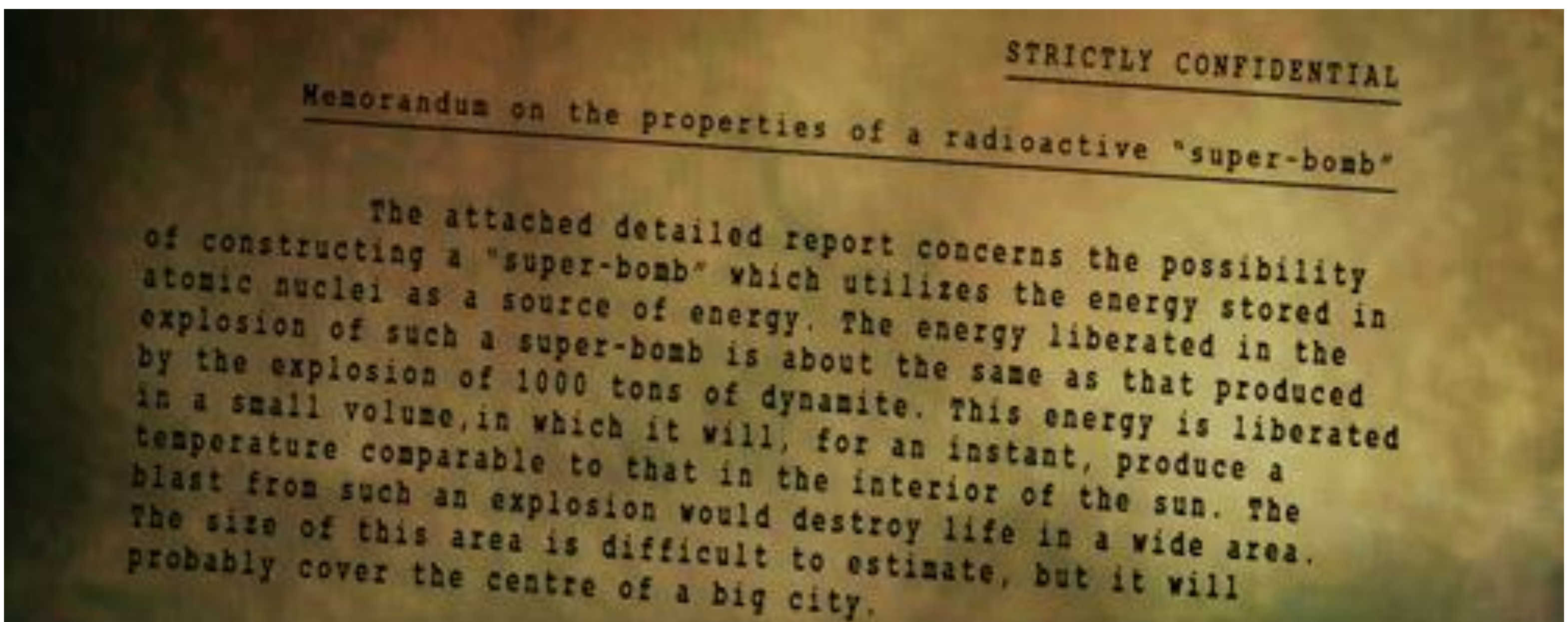


***Heisenberg* introduced an XY model after writing** *"Other difficulties are discussed in detail by Lenz, and Ising succeeded in showing that also the assumption of aligning sufficiently great forces between each of two neighboring atoms of a chain is not sufficient to create ferromagnetism"*

1936 Rudolf Peierls

Unexpectedly proves that in dimension 2 the Lenz-Ising model undergoes a phase transition, which reignites the interest

His perhaps more influential work:
1940 memorandum with Otto Frisch often credited with starting the Manhattan project



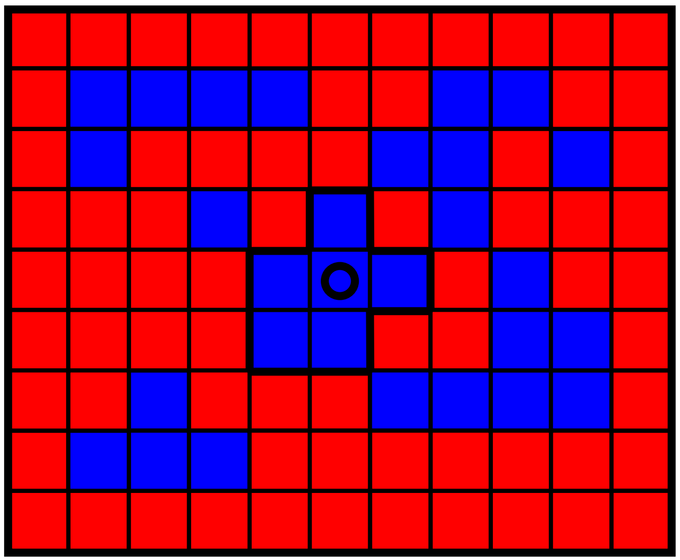
STRICTLY CONFIDENTIAL

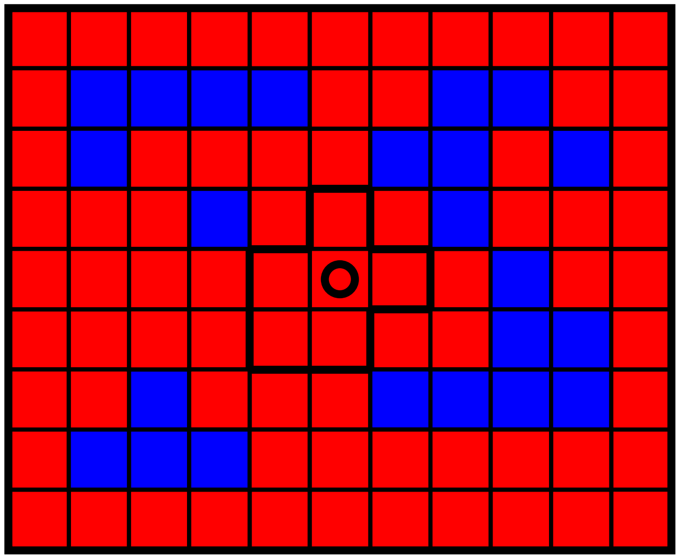
Memorandum on the properties of a radioactive "super-bomb"

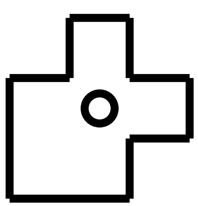
The attached detailed report concerns the possibility of constructing a "super-bomb" which utilizes the energy stored in atomic nuclei as a source of energy. The energy liberated in the explosion of such a super-bomb is about the same as that produced by the explosion of 1000 tons of dynamite. This energy is liberated in a small volume, in which it will, for an instant, produce a temperature comparable to that in the interior of the sun. The blast from such an explosion would destroy life in a wide area. The size of this area is difficult to estimate, but it will probably cover the centre of a big city.

Peierls' argument

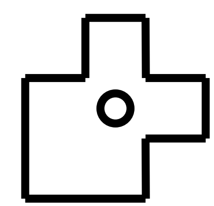
$(2n+1)\times(2n+1)$, boundary "+", $\mathbf{P}[\sigma(0)=\text{"+"}]=?$







$\mathbf{P}[\;\cdot\;] \leq x^l/(1+x^l) \leq x^l$, l=length of

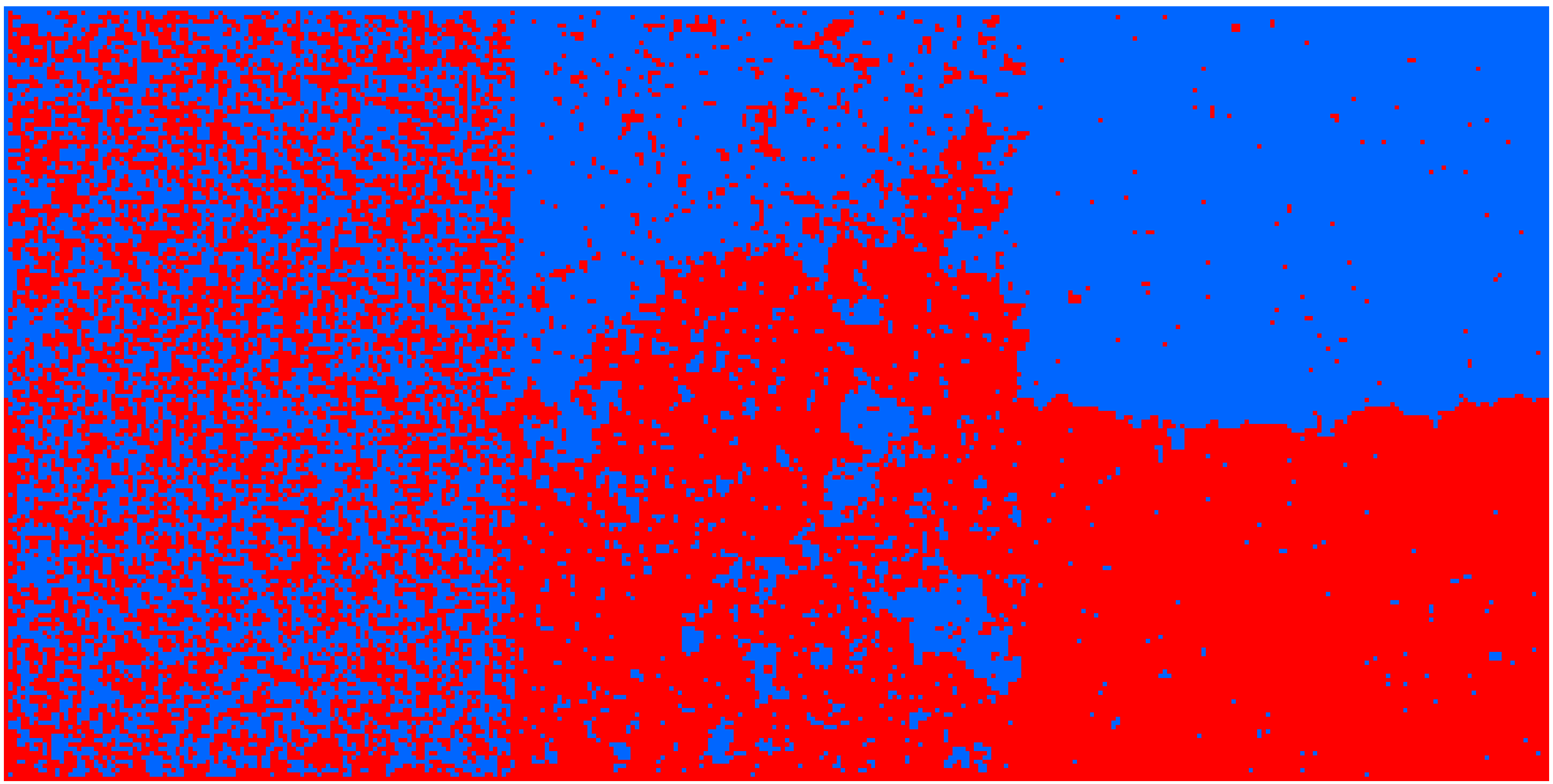


$$\mathbf{P}[\sigma\,(0)=\text{"–"}] \leq \Sigma_{j=1,..,n}\,\Sigma_{l\geq 2j+2}\, 3^l x^l$$

$$\leq (3x)^4/(1\text{-}(3x)^2)(1\text{-}3x) \leq 1/6,\ \text{if } x\leq 1/6.$$

Ising model: the phase transition

$x > x_{crit}$ $x = x_{crit}$ $x < x_{crit}$



$Prob \asymp x^{\#\{+\text{-neighbors}\}}$

1941 Kramers-Wannier

Derive the critical temperature $x_{crit} = 1/(1+\sqrt{2})$

AUGUST 1, 1941 PHYSICAL REVIEW VOLUME 60

Statistics of the Two-Dimensional Ferromagnet. Part I

H. A. KRAMERS, *University of Leiden, Leiden, Holland*

AND

G. H. WANNIER, *University of Texas, Austin, Texas*[1]

(Received April 7, 1941)

In an effort to make statistical methods available for the treatment of cooperational phenomena, the Ising model of ferromagnetism is treated by rigorous Boltzmann statistics. A method is developed which yields the partition function as the largest eigenvalue of some finite matrix, as long as the manifold is only one dimensionally infinite. The method is carried out fully for the linear chain of spins which has no ferromagnetic properties. Then a sequence of finite matrices is found whose largest eigenvalue approaches the partition function of the two-dimensional square net as the matrix order gets large. It is shown that these matrices possess a symmetry property which permits location of the Curie temperature if it exists and is unique. It lies at

$$J/kT_c = 0.8814$$

if we denote by J the coupling energy between neighboring spins. The symmetry relation also excludes certain forms of singularities at T_c, as, e.g., a jump in the specific heat. However, the information thus gathered by rigorous analytic methods remains incomplete.

---

[1] Owing to communication difficulties, one of the authors (G. H. W.) is entirely responsible for the printed text.

1941 Kramers-Wannier

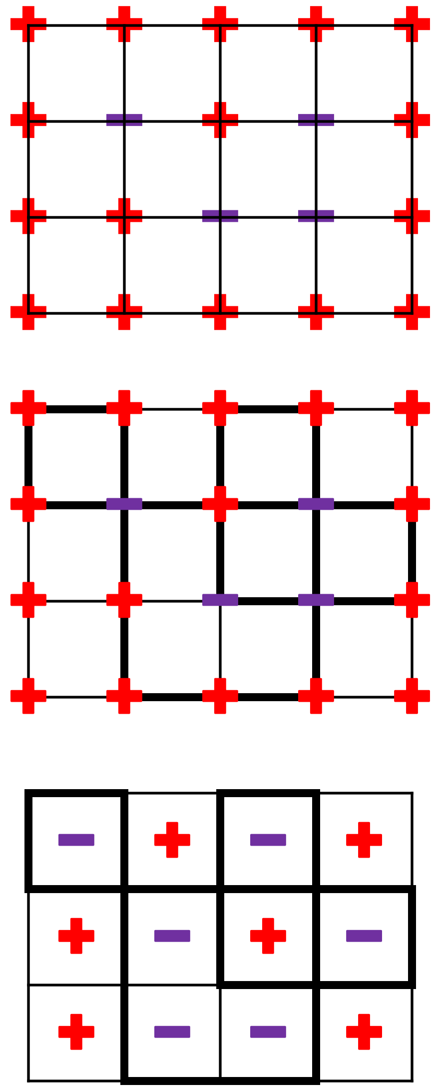
High-low temperature duality

x-Ising $\leftrightarrow$ dual lattice y-Ising, $\frac{x}{1} = \frac{1-y}{1+y}$

$$Z = \Sigma_{spin\ conf.} x^{\#\{(+)(-)neighbors\}}$$
$$\asymp \Sigma_{spin\ conf.} \Pi_{edge<ij>} (1+ys(i)s(j))$$
$$= \Sigma_{spin\ conf.}$$
$$\Sigma_{edge\ conf.} \Pi_{<ij>\ in\ conf.} ys(i)s(j)$$
$$= \Sigma_{edge\ conf.}$$
$$\Sigma_{spin\ conf.} \Pi_{<ij>\ in\ conf.} ys(i)s(j)$$
$$= \Sigma_{even\ edge\ conf.} y^{\#\{edges\}}$$

Self-dual if $x=y$, i.e.

$$x_{crit} = 1/(1+\sqrt{2})$$



1944 Lars Onsager

A series of papers 1944-1950, some with **Bruria Kaufman.** Partition function, magnetization and other quantities derived. **It took a chemist!**

PHYSICAL REVIEW VOLUME 65, NUMBERS 3 AND 4 FEBRUARY 1 AND 15, 1944

Crystal Statistics. I. A Two-Dimensional Model with an Order-Disorder Transition

LARS ONSAGER
Sterling Chemistry Laboratory, Yale University, New Haven, Connecticut
(Received October 4, 1943)

PHYSICAL REVIEW VOLUME 76, NUMBER 8 OCTOBER 15, 1949

Crystal Statistics. II. Partition Function Evaluated by Spinor Analysis

BRURIA KAUFMAN*
Columbia University, New York City, New York
(Received May 11, 1949)

2D Ising is "exactly solvable"

From 1944 widely studied in mathematical physics, with many results by different methods:

Kaufman, Onsager, Yang, Kac, Ward, Potts, Montroll, Hurst, Green, Kasteleyn, McCoy, Wu, Tracy, Widom, Vdovichenko, Fisher, Baxter, …

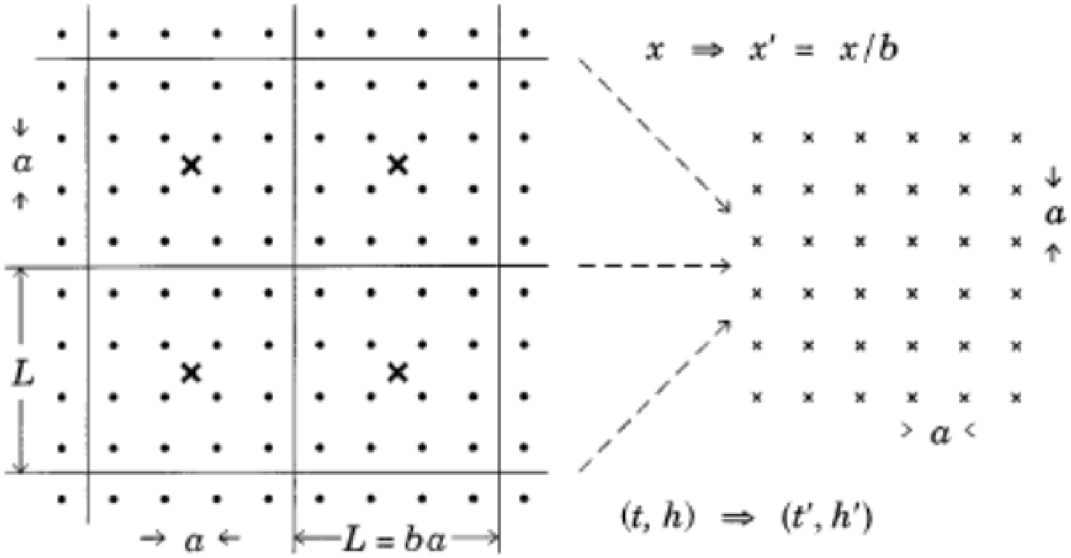
- **Only some results rigorous**
- **Limited applicability to other models, but still motivated much research**

Eventually regained prominence in physics, used in biology, economics, computer science…

1951Renormalization Group

Petermann-Stueckelberg 1951, …
Kadanoff, Fisher, Wilson, 1963-1966, …

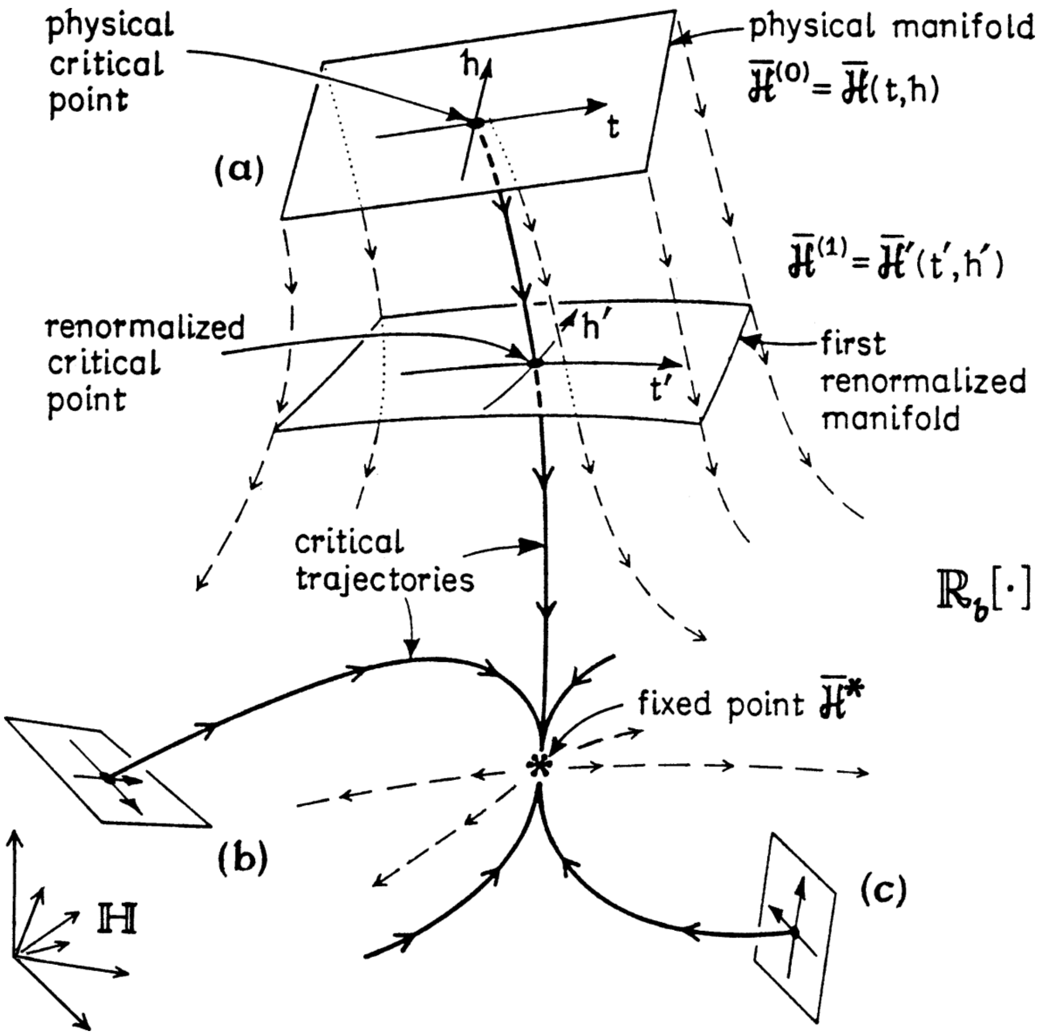
Block-spin renormalization
≈ rescaling +
change of *x*



Conclusion:
At criticality
the scaling limit
is described by a "massless field theory"
The **Curie critical point** is **universal** and hence **translation, scale** and **rotation invariant**

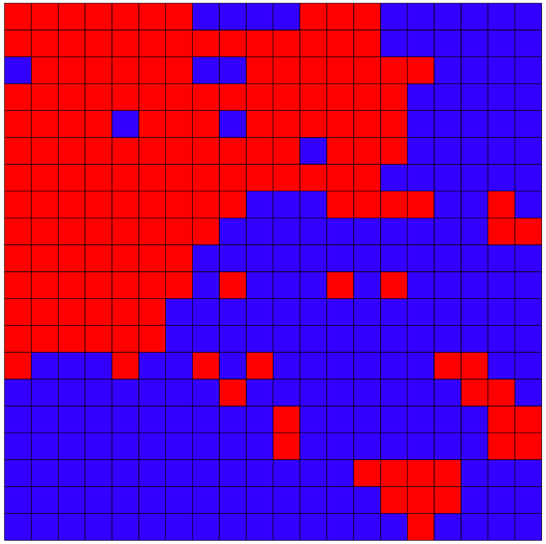
Renormalization Group

From [Michael Fisher,1983]



A depiction of the space of Hamiltonians H showing initial or physical manifolds and the flows induced by repeated application of a discrete RG transformation R*b* with a spatial rescaling factor *b* (or induced by a corresponding continuous or differential RG). Critical trajectories are shown bold: they all terminate, in the region of H shown here, at a fixed point H*. The full space contains, in general, other nontrivial (and trivial) critical fixed points,…

1985 Conformal Field Theory



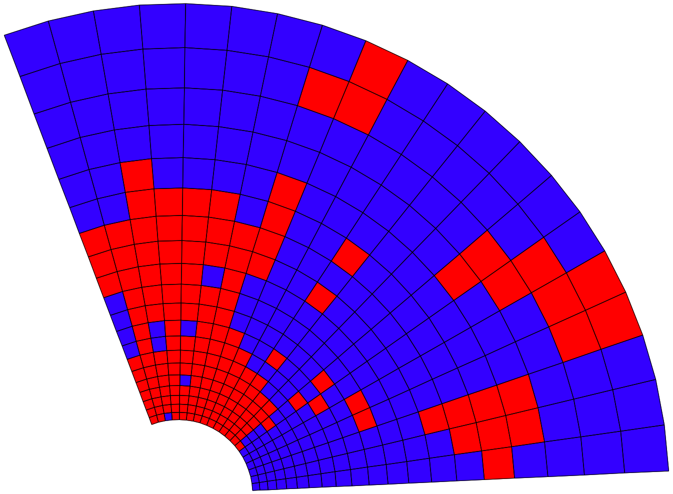
Belavin, Polyakov, Zamolodchikov 1985

Conformal transformations
= those preserving angles
= analytic maps

Locally **translation** +
+ **rotation** + **rescaling**

So it is logical to suppose **conformal invariance** bin the scaling limit.

Allows to derive many quantities (unrigorously)



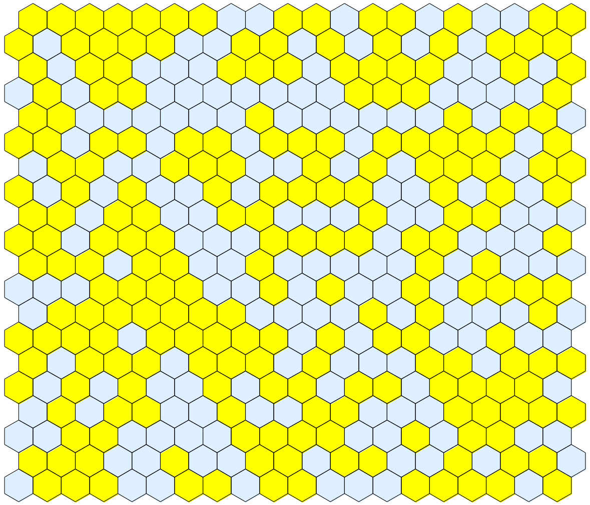
2D CFT

Beautiful algebra, but analytic and geometric parts missing or nonrigorous

Spectacular predictions, e.g. by **Den Nijs** and **Cardy**:

Percolation (Ising at infinite T or x=1): hexagons are coloured white or yellow independently with probability ½.
Is there a top-bottom crossing of white hexagons? Difficult to see! Why?

HDim (percolation cluster)= ?



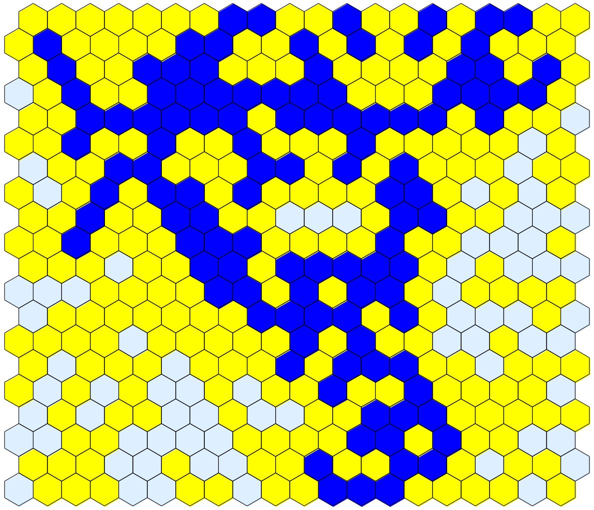
2D CFT

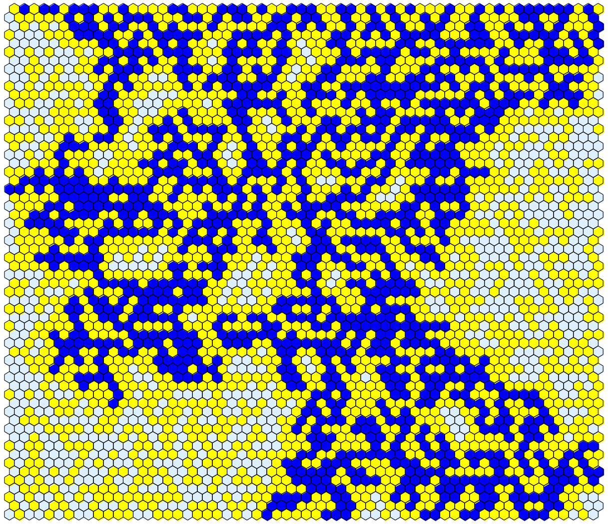
Beautiful algebra, but analytic and geometric parts missing or nonrigorous

Spectacular predictions, e.g. by **Den Nijs** and **Cardy**:

Percolation (Ising at infinite T or x=1): hexagons are coloured white or yellow independently with probability ½. **Connected white cluster touching the upper side is coloured in blue, it has**

HDim (percolation cluster)= 91/48





$$\mathbb{P}\,(\text{crossing}) = \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{4}{3}\right)}\, m^{1/3}\, {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{4}{3}; m\right)$$

Last decade

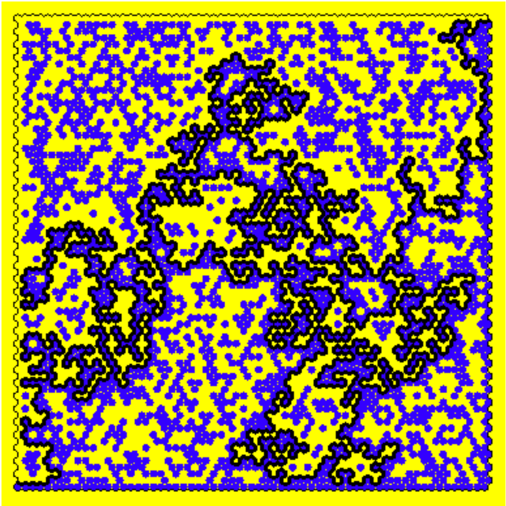
Two analytic and geometric approaches

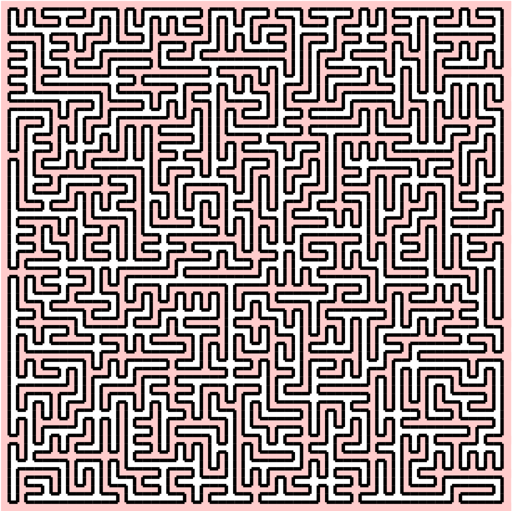
1) Schramm-Loewner Evolution: a geometric description of the scaling limits at criticality
2) Discrete analyticity: a way to rigorously establish existence and conformal invariance of the scaling limit

- **New physical approaches and results**
- **Rigorous proofs**
- **Cross-fertilization with CFT**

Schramm-Loewner Evolution

A way to construct **random conformally invariant fractal curves**, introduced in 1999 by **Oded Schramm** (1961-2008)

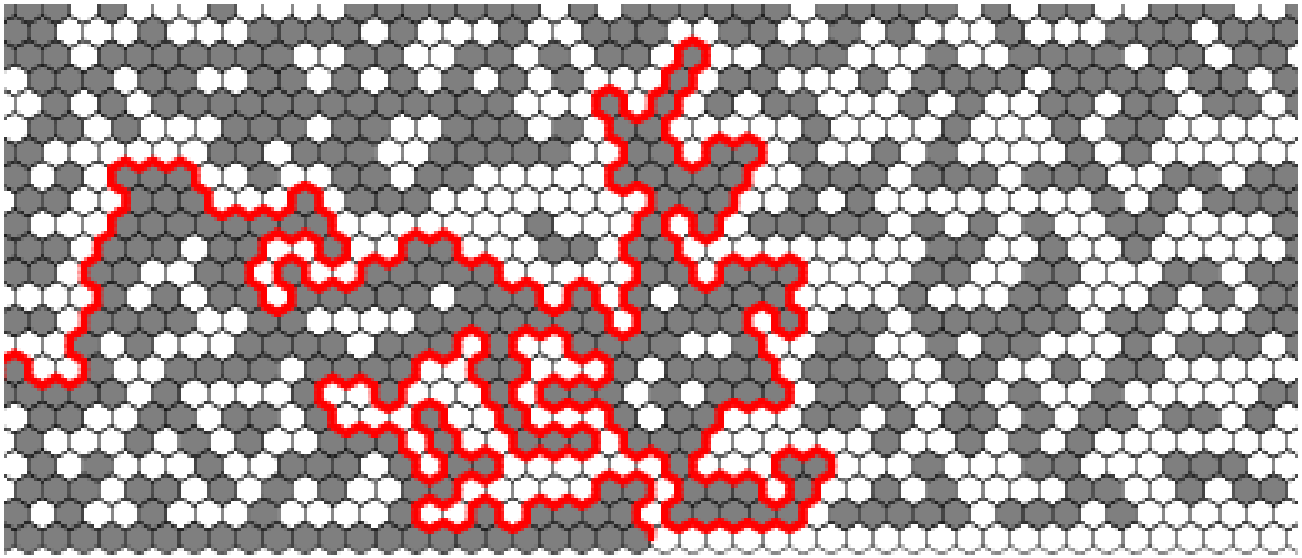




Percolation→SLE(6) [Smirnov, 2001]

Uniform Spanning Tree →SLE(8) [Lawler-Schramm-Werner, 2001]

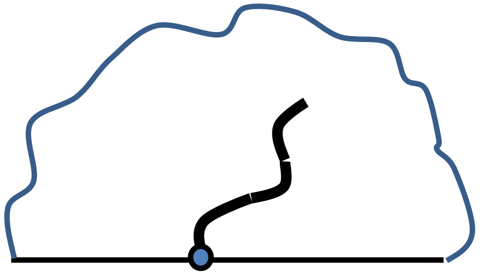
from Oded Schramm's talk 1999

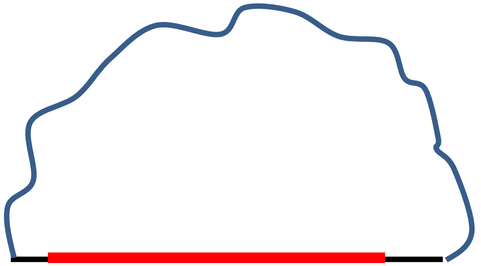


In the figure, each of the hexagons is colored black with probability 1/2, independently, except that the hexagons intersecting the positive real ray are all white, and the hexagons intersecting the negative real ray are all black. There is a boundary path β, passing through 0 and separating the black and the white connected components adjacent to 0. The curve β is a random path in the upper half-plane $\mathbb{H}$ connecting the boundary points 0 and ∞.

Schramm-Loewner Evolution

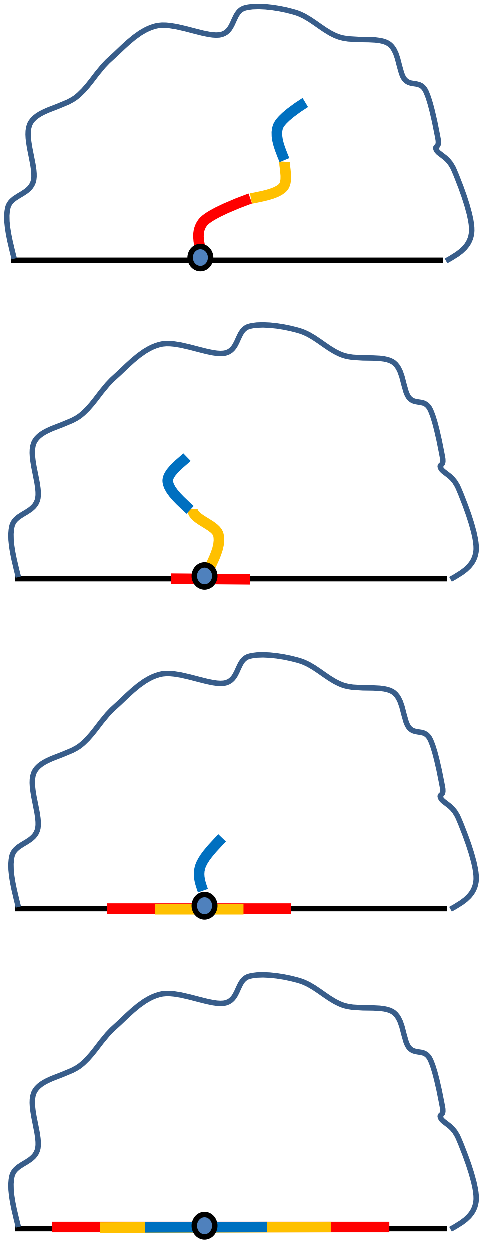
- Draw the slit



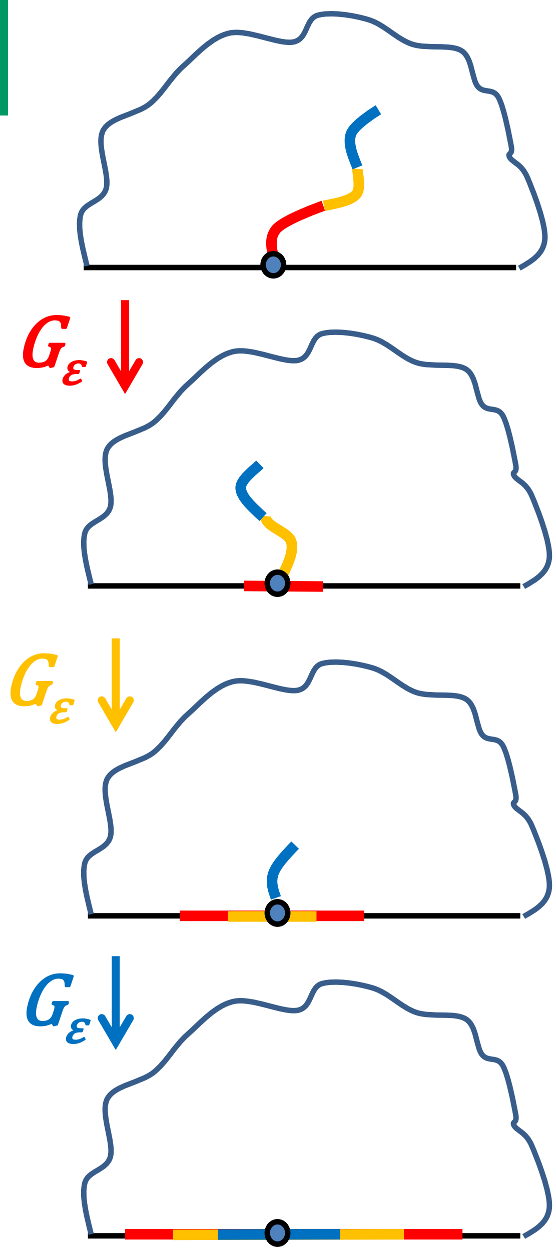


Schramm-Loewner Evolution

- Draw the slit
- Stop at **ε** capacity increments

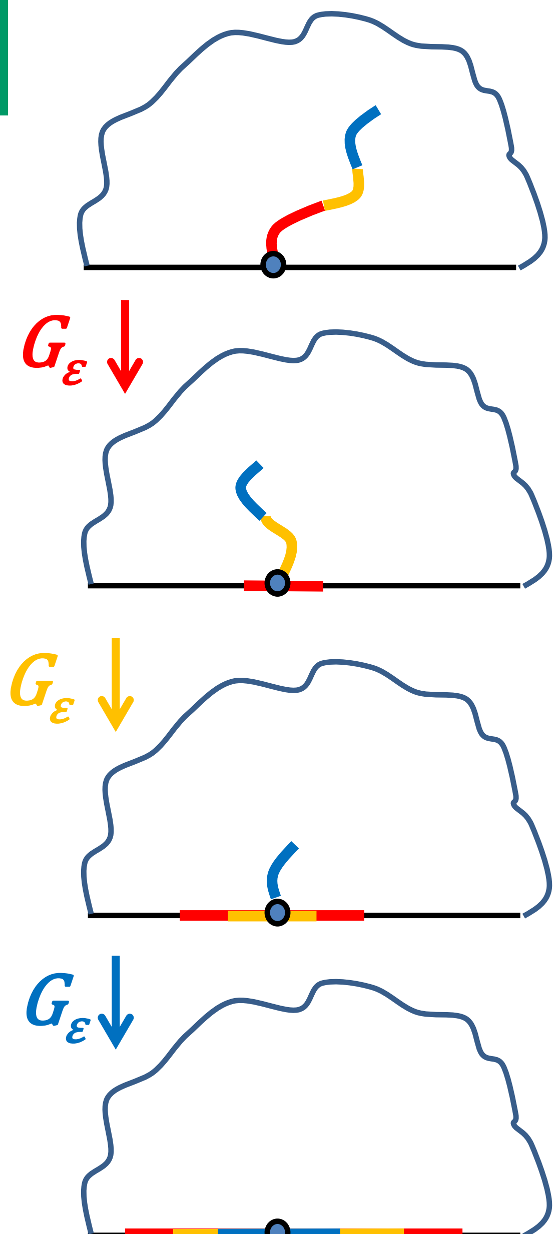


Schramm-Loewner Evolution



- Draw the slit
- Stop at **ε** capacity increments
- Open it up by a conformal map $\boldsymbol{G_\varepsilon = z + w_\varepsilon + \frac{2\varepsilon}{z} + \ldots}$

Schramm-Loewner Evolution



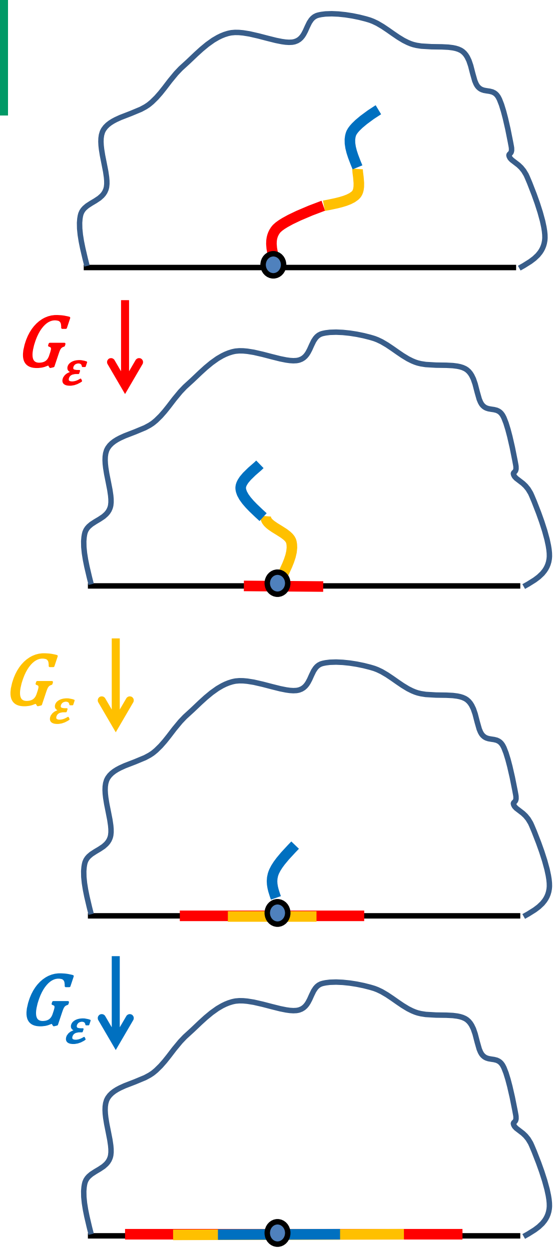
- Draw the slit
- Stop at **ε** capacity increments
- Open it up by a conformal map $G_\varepsilon = z + w_\varepsilon + \frac{2\varepsilon}{z} + \dots$
- Composition of iid maps

$$G_{n\varepsilon} = z + w_{n\varepsilon} + \frac{2n\varepsilon}{z} + \dots =$$

$$= G_\varepsilon(G_\varepsilon(G_\varepsilon(\dots))) =$$

$$= z + (w_\varepsilon + \dots + w_\varepsilon) + \frac{2n\varepsilon}{z} + \dots$$

Schramm-Loewner Evolution



- Draw the slit
- Stop at **ε** capacity increments
- Open it up by a conformal map $G_\varepsilon = z + w_\varepsilon + \frac{2\varepsilon}{z} + \ldots$
- Composition of iid maps

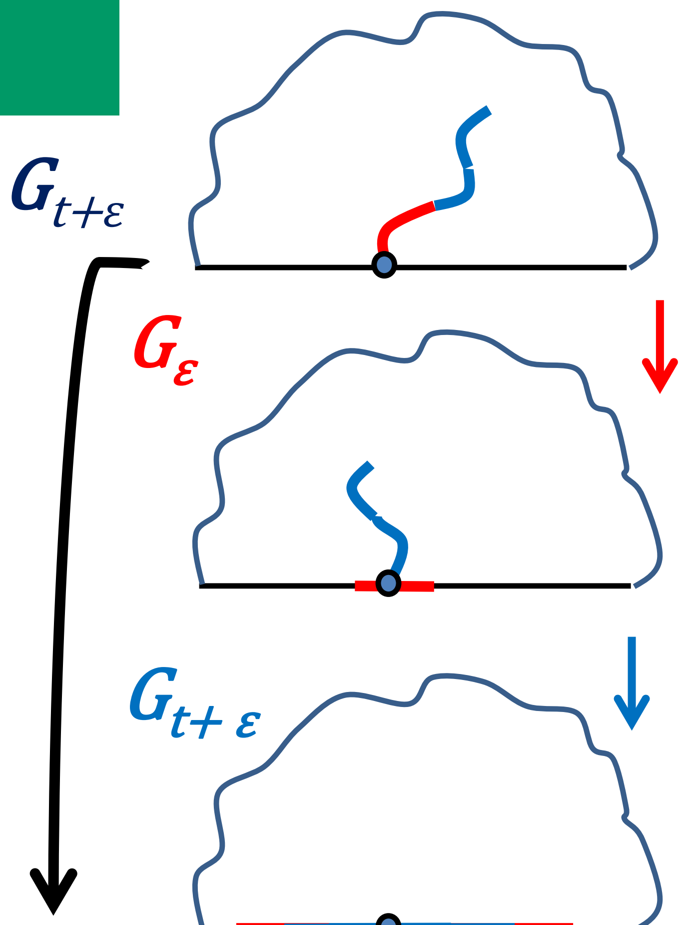
$$G_{n\varepsilon} = z + w_{n\varepsilon} + \frac{2n\varepsilon}{z} + \ldots =$$

$$= G_\varepsilon(G_\varepsilon(G_\varepsilon(\ldots))) =$$

$$= z + (w_\varepsilon + \ldots + w_\varepsilon) + \frac{2n\varepsilon}{z} + \ldots$$

- w_t is a Brownian motion!
- “A random walk on the moduli space”

Schramm-Loewner Evolution



Differentiate the slit map

$$G_t = z + w_t + \frac{2t}{z} + \dots$$

here $2t$ is the slit capacity

$$\partial_t (G_t - w_t) =$$

$$= \lim \frac{1}{\varepsilon} (G_{t+\varepsilon} - w_{t+\varepsilon} - G_t + w_t)$$

$$= \lim \frac{1}{\varepsilon} \left((G_\varepsilon - \mathrm{Id}) \circ G_t - (w_{t+\varepsilon} - w_t) \right)$$

$$= \lim \frac{1}{\varepsilon} \left(\left(w_\varepsilon + \frac{2\varepsilon}{z} + \dots \right) \circ G_t - (w_\varepsilon) \right)$$

$$= \lim \frac{1}{\varepsilon} \left(\frac{2\varepsilon}{z} \circ G_t + \dots \right) = \frac{2}{G_t}$$

Loewner Equation $\partial_t (G_t - w_t) = \frac{2}{G_t}$

Schramm LE: $w_t = \sqrt{\kappa} B_t$, **a Brownian motion**

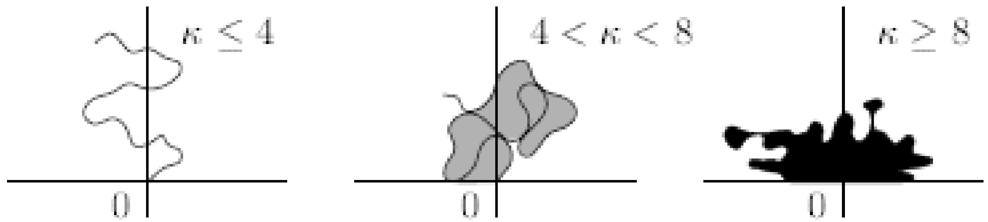
Leads to a random fractal curve

Schramm-Loewner Evolution

SLE=BM on the moduli space. Calculations reduce to Itô calculus, interesting fractal properties

Lemma [Schramm] If an interface has a conformally invariant scaling limit, it is ***SLE(κ)***

Theorem [Schramm-Rohde] SLE phases:



Theorem [Beffara]

$$\mathrm{HDim}\big(SLE(\kappa)\big) = 1 + \frac{\kappa}{8}, \qquad \kappa < 8$$

Theorem [Zhan, Dubedat]

$$SLE(\kappa) = \partial\big(SLE(16/\kappa)\big), \qquad \kappa < 4$$

Discrete holomorphic functions

New approach to 2D integrable models

- **Find an observable *F* (edge density, spin correlation, exit probability,. . .) which is discrete analytic (holomorphic) or harmonic and solves some BVP.**
- **Then in the scaling limit *F* converges to a holomorphic solution *f* of the same BVP.**

We conclude that

- ***F* has a conformally invariant scaling limit**
- **Interfaces converge to Schramm's SLEs**
- **Calculate dimensions and exponents with or without SLE**

Discrete holomorphic functions

Preholomorphic or **discrete holomorphic functions** appeared implicitly already in the work of **Kirchhoff** in **1847**.

- A **graph** models an **electric network**.
- Assume all edges have **unit resistance**.
- Let $F(\vec{uv}) = -F(\vec{vu})$ be the **current** flowing from u to v

Then the **first** and the **second Kirchhoff laws** state that
the sum of currents flowing from a vertex is zero:

$$\sum_{u:\ \text{neighbor of } v} F(\vec{uv}) = 0\,, \tag{1}$$

the sum of the currents around any oriented closed contour γ is zero:

$$\sum_{\vec{uv}\in\gamma} F(\vec{uv}) = 0\,. \tag{2}$$

Rem For planar graphs contours around faces are sufficient

Discrete holomorphic functions

The **second** and the **first Kirchhoff laws** are equivalent to

F being given by the **gradient** of a **potential function** H:

$$F(\vec{uv}) \;=\; H(v) - H(u)\;, \qquad (2')$$

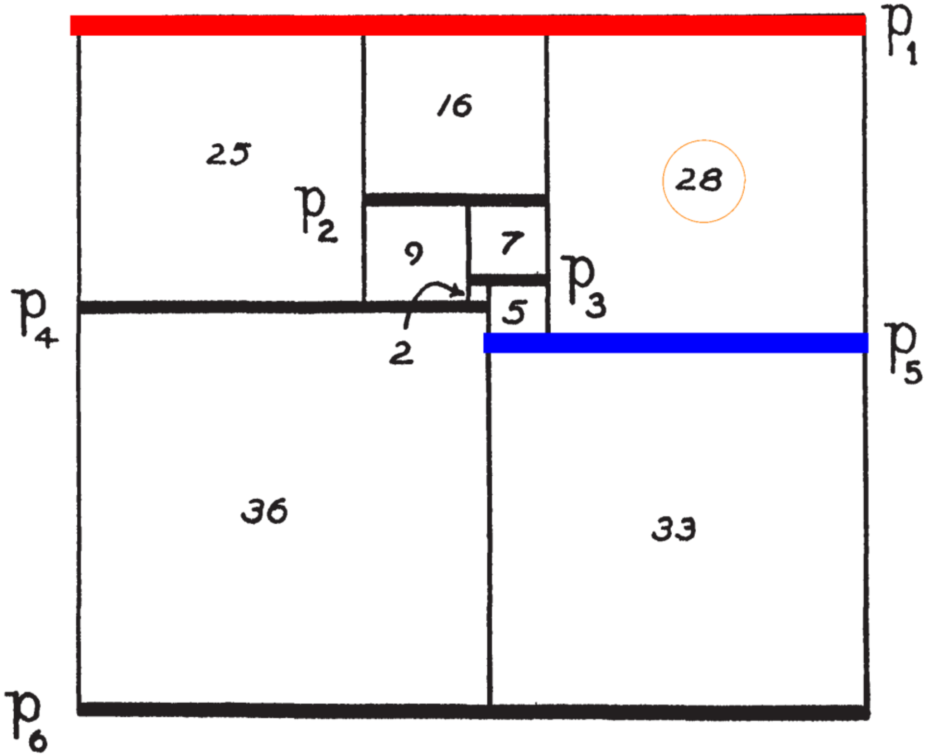
and the latter being **preharmonic**:

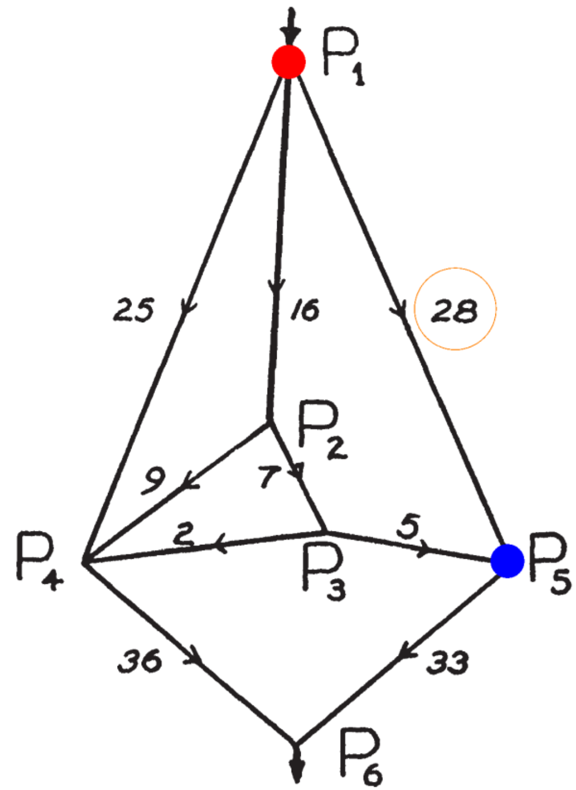
$$0 \;=\; \Delta H(u) \;:=\; \sum_{v:\ \text{neighbor of } u} (H(v) - H(u))\;. \qquad (1')$$

- Different resistances amount to putting weights into **(1')**.
- Preharmonic functions can be defined on any graph, and have been very well studied.
- On planar graphs preharmonic gradients are **preholomorphic**, similarly to harmonic gradients being holomorphic.

Discrete holomorphic functions

Besides the original work of Kirchhoff, the first notable application was perhaps the famous article **[Brooks, Smith, Stone & Tutte, 1940]** *"The dissection of rectangles into squares"* which used preholomorphic functions to construct tilings of rectangles by squares.

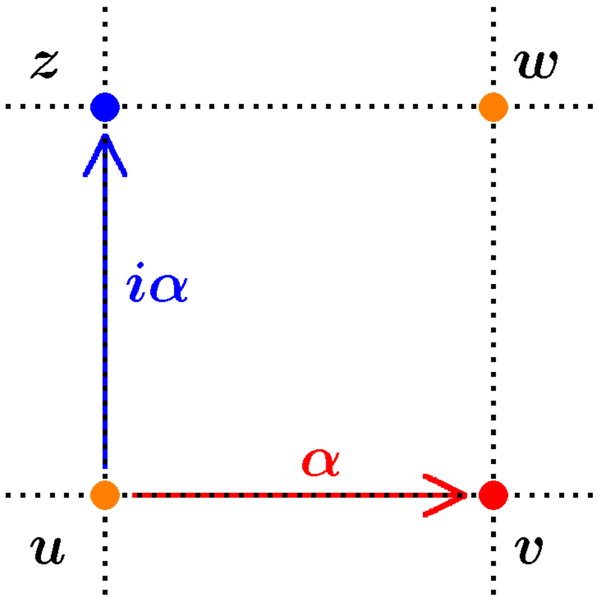


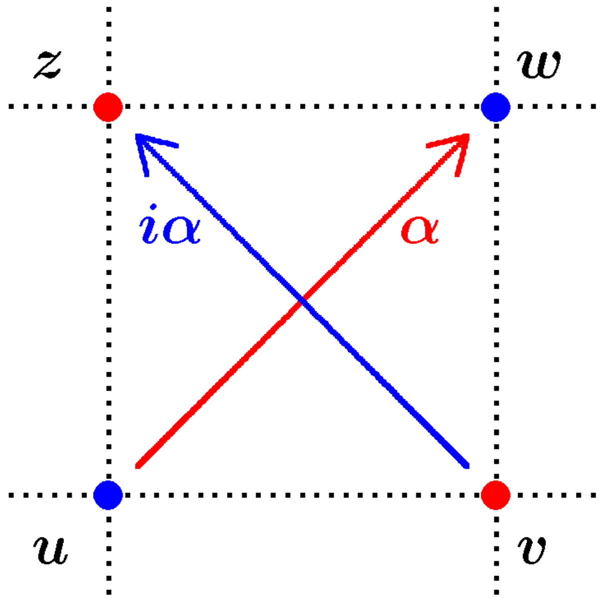


tilings by squares ↔ **preholomorphic functions on planar graphs**

Discrete holomorphic functions

Preholomorphic functions were explicitly studied in **[Isaacs, 1941]** under the name "monodiffric". Isaacs proposed two ways to discretize the **Cauchy-Riemann equations** $\partial_{i\alpha} F = i \partial_{\alpha} F$ on the square lattice:





$$\frac{F(z)-F(u)}{z-u} = \frac{F(v)-F(u)}{v-u} \quad (1^{st})$$

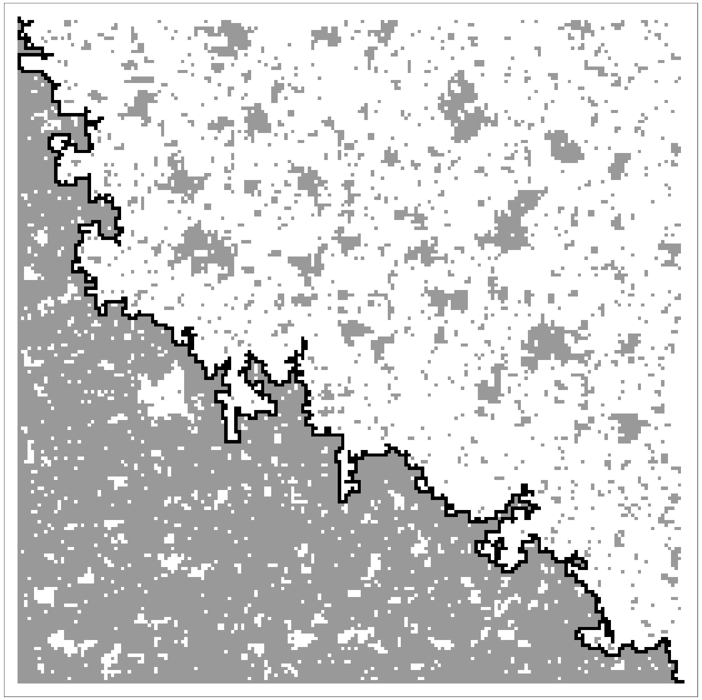
$$F(z) - F(u) = i\,(F(v) - F(u))$$

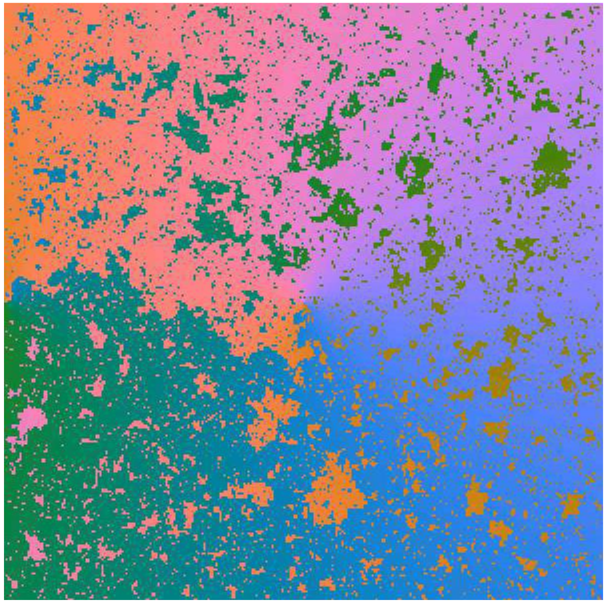
$$\frac{F(z)-F(v)}{z-v} = \frac{F(w)-F(u)}{w-u} \quad (2^{nd})$$

$$F(z) - F(v) = i\,(F(w) - F(u))$$

Preholomorphicity in Ising

[Chelkak, Smirnov 2008-10] Partition function of the critical Ising model with a disorder operator is discrete holomorphic solution of the Riemann-Hilbert boundary value problem. Interface weakly converges to Schramm's SLE(3) curve. *Strong convergence follows with some work* ***[Chelkak, Duminil-Copin, Hongler, Kemppainen, S]***





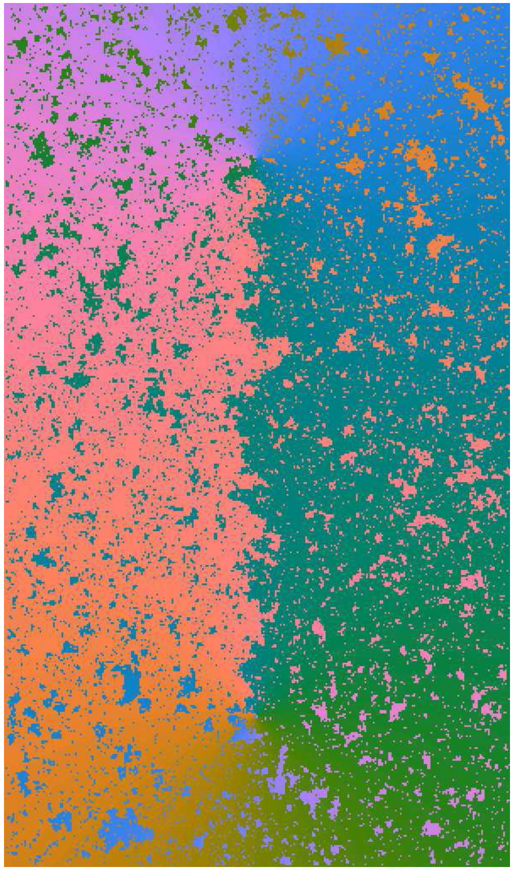
HDim = 11/8

Energy field in the Ising model

Combination of 2 disorder operators is a discrete analytic Green's function solving a Riemann-Hilbert BVP, when fused gives the energy corellation:

Theorem [Hongler – Smirnov, 2013]

At x_c the correlation of neighboring spins satisfies (ε is the lattice mesh; ρ is the hyperbolic metric element; the sign ± depends on BC: + or free):



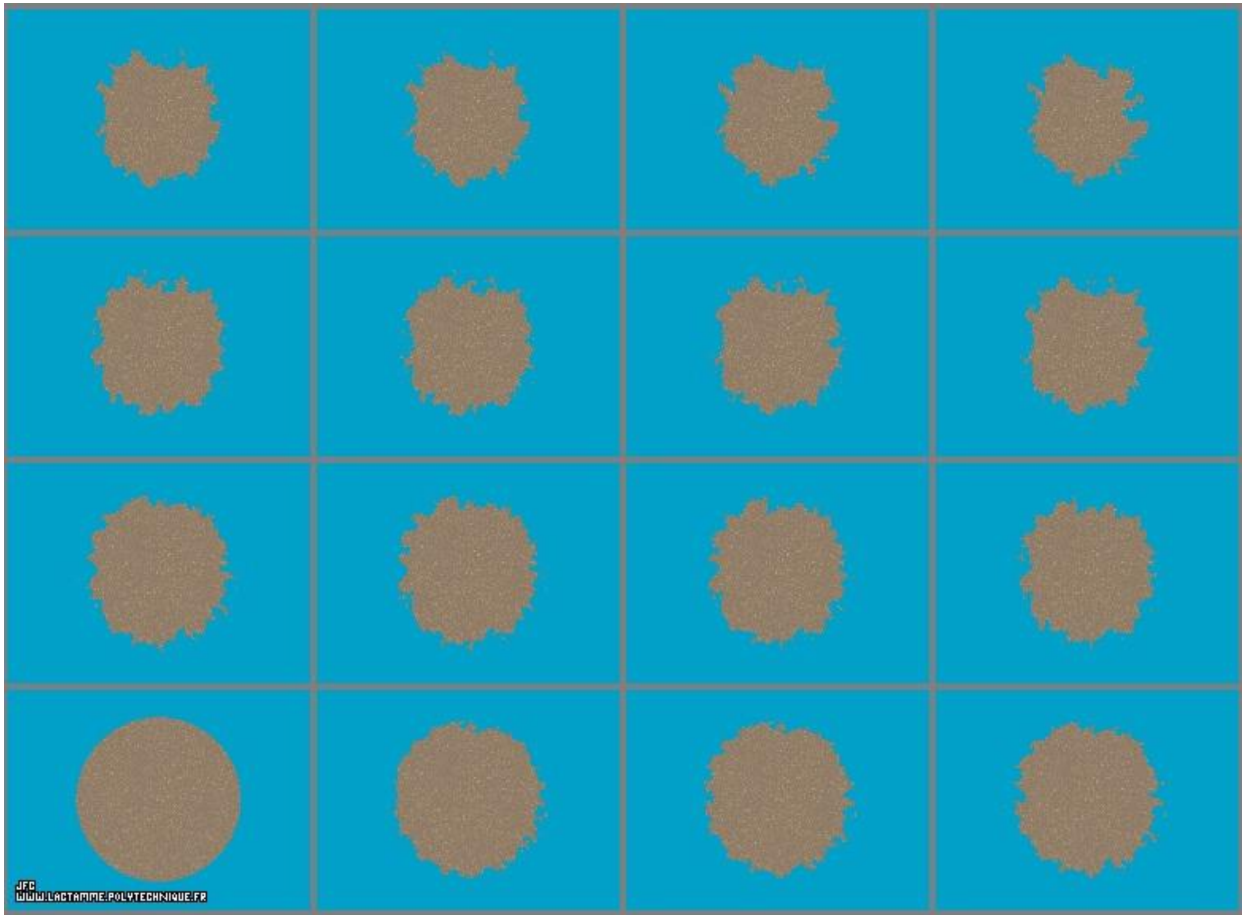
$$\mathbb{E}\ s(u)\,s(v) = \frac{1}{\sqrt{2}} \pm \frac{1}{\pi}\rho_\Omega(u)\,\varepsilon + O(\varepsilon^2)$$

Generalizations to multi spin and energy corellations:

[Chelkak, Hongler, Izyurov]

2D statistical physics:

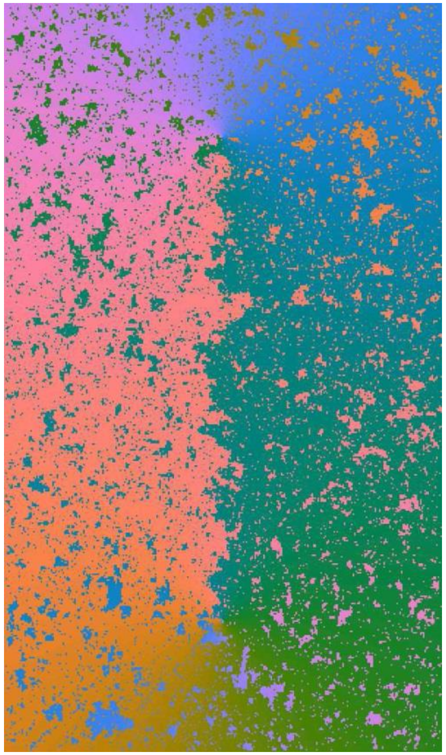
macroscopic effects of microscopic interactions



erosion simulation © J.-F. Colonna

DNA by atomic force microscopy © Lawrence Livermore National Laboratory

The Lenz-Ising model



Proposed as a model for a long unexplained phenomenon

Deemed physically inaccurate and mathematically trivial

Breakthrough by Onsager leads to much theoretical study

Eventually retook its place in physics, biology, computer science.

Much fascinating mathematics, expect more:

- [Zamolodchikov, JETP 1987]: E8 symmetry in 2D Ising. [Coldea et al., Science 2011]: experimental evidence. Time for a proof?
- [Aizenman Duminil-Copin Sidoravicius 2013] In 3D no magnetization at criticality. Other results?

Thank you for your attention!